Final exam for Numerical Analysis: Linear Algebra
December 14, 2019. Closed books.

Hand in solutions on separate sheets. Motivate all answers (except Question 1). Write your name!

Question 1: (25p) For this question, please provide *only the answer*, no motivation.

(a) Consider the matrix $\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$, where $\mathbf{A}_{11}$ is a square and non-singular. Then $\mathbf{A}$ admits the factorization $\mathbf{A} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{X} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{Y} \end{bmatrix}$. Specify the matrices $\mathbf{X}$ and $\mathbf{Y}$.

(b) Let $\mathbf{A}$ be a real square matrix for which $\|\mathbf{Ax}\| = \|\mathbf{x}\|$ for every vector $\mathbf{x}$. What can you say about the singular values of $\mathbf{A}$? What about the eigenvalues?

(c) Let $\mathbf{A}$ be a 5×5 square matrix with the characteristic polynomial $p(x) = \det(x\mathbf{I} - \mathbf{A})$. Suppose that $\mathbf{A}^3 = \mathbf{0}$. What does this tell you about p?

(d) Let $\mathbf{A} \in \mathbb{C}^{5\times5}$ and $\mathbf{b} \in \mathbb{C}^{5\times1}$. Suppose that the Krylov spaces $\mathcal{K}_p = \text{span}(\mathbf{b}, \mathbf{Ab}, \dots, \mathbf{A}^{p-1}\mathbf{b})$ have dimension precisely p for $p = 1, 2, \dots, 5$. Let $\{\mathbf{q}_j\}_{j=1}^5$ be a sequence of vectors such that $\{\mathbf{q}_j\}_{j=1}^p$ forms an orthonormal basis for $\mathcal{K}_p$ for $p = 1, 2, \dots, 5$. Which of the following statements are true:

(i) If $i < j$, then $\mathbf{q}_i^*\mathbf{A}\mathbf{q}_j = 0$.

(ii) If $i > j+1$, then $\mathbf{q}_i^*\mathbf{A}\mathbf{q}_j = 0$.

(iii) If $\mathbf{A}$ is symmetric and $i < j-1$, then $\mathbf{q}_i^*\mathbf{A}\mathbf{q}_j = 0$.

(iv) The set $\{\mathbf{A}^2\mathbf{b}, \mathbf{A}^3\mathbf{b}, \mathbf{A}^4\mathbf{b}\}$ is linearly independent.

(e) Let $\mathbf{A}$ be a 3×3 matrix of rank 2. Let $\mathbf{b} = [1, 2, 1]^*$, and let $\mathbf{x}_\star$ denote the least squares solution to $\mathbf{Ax} = \mathbf{b}$. You know that $\mathbf{u}_1 \cdot \mathbf{b} = 1$ and $\mathbf{u}_2 \cdot \mathbf{b} = \sqrt{2}$, where $\mathbf{u}_1$ and $\mathbf{u}_2$ are the two left singular vectors of $\mathbf{A}$ associated with non-zero singular values. Evaluate $\|\mathbf{Ax}_\star - \mathbf{b}\|$.

Question 2: (15p) Consider the two vectors $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$.

(a) (5p) Apply the Gram-Schmidt process to the vectors $\mathbf{u}$ and $\mathbf{v}$, in that order, to produce two orthogonal vectors $\mathbf{x}$ and $\mathbf{y}$. Specify $\mathbf{x}$ and $\mathbf{y}$.

(b) (5p) Set $\mathbf{A} = \begin{bmatrix}\mathbf{u}\ \mathbf{v}\end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 2 & -2 \end{bmatrix}$. Compute the QR factorization of $\mathbf{A}$.

(c) (5p) Set $\mathbf{B} = \begin{bmatrix} 3 & 0 \\ -6 & -1 \\ 6 & 2 \end{bmatrix}$. Compute the QR factorization of $\mathbf{B}$.

Hint: Compare $\mathbf{A}$ and $\mathbf{B}$ carefully. Problem (c) can be solved with almost no effort!

Question 3: (15p) Consider the matrices $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 3 & 7 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 4 & 3 \\ 4 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

(a) (7p) Find a unitary matrix $\mathbf{Q}$ and a tridiagonal matrix $\mathbf{H}$ such that $\mathbf{A} = \mathbf{QHQ}^*$.

(b) (8p) Find a unitary matrix $\mathbf{Q}$ and a tridiagonal matrix $\mathbf{H}$ such that $\mathbf{B} = \mathbf{QHQ}^*$.

Hint: Problem (a) can be solved with almost no computation required. Problem (b) may require some more arithmetic. Make sure that you describe your steps clearly to maximize your chances for partial credit in case you get some arithmetic errors, or in case you do not have time to finish.

Question 4: (15p) Let $\mathbf{A} \in \mathbb{R}^{m\times m}$. Recall that we say that $\mathbf{A}$ is *symmetric positive definite (spd)* if $\mathbf{A}$ is symmmetric, and for every nonzero vector $\mathbf{v} \in \mathbb{R}^{m\times 1}$ it is the case that $\mathbf{v}^*\mathbf{Av} > 0$.

(a) (5p) Suppose that $\mathbf{A} = \mathbf{X}^*\mathbf{X}$ for some nonsingular square matrix $\mathbf{X}$. Prove that $\mathbf{A}$ is spd.

(b) (4p) Suppose that $\mathbf{A} = \mathbf{R}^*\mathbf{R}$ where $\mathbf{R}$ is an upper triangular matrix. What is $\mathbf{R}(1,1)$?

(c) (6p) Compute the Cholesky factorization of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Question 5: (15p) Let $\mathbf{A} \in \mathbb{C}^{m\times m}$ be a non-singular matrix, and let $\mathbf{b} \in \mathbb{C}^m$ be a fixed vector. Suppose that you have an algorithm for solving $\mathbf{Ax} = \mathbf{b}$ that produces an answer $\tilde{\mathbf{x}}$ that satisfies $(\mathbf{A} + \delta\mathbf{A})\tilde{\mathbf{x}} = \mathbf{b}$ for some matrix $\delta\mathbf{A}$ such that

$$\frac{\|\delta\mathbf{A}\|}{\|\mathbf{A}\|} \leq \varepsilon. \tag{1}$$

(a) (10p) Prove that

$$\frac{\|\tilde{\mathbf{x}} - \mathbf{x}_\star\|}{\|\tilde{\mathbf{x}}\|} \leq \kappa(\mathbf{A})\,\varepsilon, \tag{2}$$

where $\mathbf{x}_\star$ is the exact solution to $\mathbf{Ax}_\star = \mathbf{b}$, and where $\kappa(\mathbf{A})$ is the condition number of $\mathbf{A}$. Observe that (2) looks slightly different from the bounds we saw in class, since it has $\|\tilde{\mathbf{x}}\|$ in the denominator instead of $\|\mathbf{x}_\star\|$. This change permits us to get a bound that holds for every positive ε, not just asymptotically as $\varepsilon \to 0$.

(b) (5p) Suppose that (1) holds asymptotically as the rounding precision $\varepsilon \to 0$. Prove that

$$\frac{\|\tilde{\mathbf{x}} - \mathbf{x}_\star\|}{\|\mathbf{x}_\star\|} \leq \kappa(\mathbf{A})\,\varepsilon + O(\varepsilon^2), \qquad \text{as } \varepsilon \to 0. \tag{3}$$

Question 6: (15p) Let $\mathbf{A} \in \mathbb{R}^{m\times m}$ and $\mathbf{b} \in \mathbb{R}^{m\times 1}$. After two steps of the Arnoldi process, starting with $\mathbf{q}_1 = \mathbf{b}/\|\mathbf{b}\|$, as usual, we find that the following factorization holds *exactly*:

$$\mathbf{A}\big[\mathbf{q}_1\ \mathbf{q}_2\big] = \big[\mathbf{q}_1\ \mathbf{q}_2\big]\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

In other words, we ended up with the nontypical (but lucky!) case where $h_{32} = 0$.

(a) (5p) Prove that the linear space $V = \mathrm{span}(\mathbf{q}_1, \mathbf{q}_2)$ is an invariant subspace of $\mathbf{A}$. (Recall that a linear space W is an *invariant subspace* of a matrix $\mathbf{A}$ if $\mathbf{Ax} \in W$ whenever $\mathbf{x} \in W$.)

(b) (5p) Prove that the two distinct eigenvalues of $\mathbf{H}_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ are also eigenvalues of $\mathbf{A}$.

(c) (5p) Specify the eigenvectors of $\mathbf{A}$ associated with the two eigenvalues of $\mathbf{H}_2$. These will be expressed as linear combinations of $\mathbf{q}_1$ and $\mathbf{q}_2$.