Fall 2015:
Computational and Variational Methods for Inverse Problems
CSE 397/GEO 391/ME 397/ORI 397
Assignment 5 (due December 14, 2015)

I. An inverse problem for Burgers' equation

Consider the inverse problem for the viscosity field m in the one-dimensional Burgers' equation (this equation is often taken as a one-dimensional surrogate for the Navier-Stokes equations). Taking $[0, L]$ as the spatial domain and $[0, T]$ as the temporal interval, the solution $u = u(x, t) : [0, L] \times [0, T]$ satisfies

$$u_t + uu_x - (mu_x)_x = f \quad \text{in } (0, L) \times (0, T), \tag{1a}$$

$$u(0, t) = u(L, t) = 0 \quad \text{for all } t \in [0, T], \tag{1b}$$

$$u(x, 0) = 0 \quad \text{for all } x \in [0, L]. \tag{1c}$$

Here, $m = m(x) : [0, L] \to \mathbb{R}$ is the spatially-dependent viscosity field we wish to invert for, $f = f(x, t)$ is a given source term, and subscripts t and x indicate partial derivatives with respect to time and space coordinates. The conditions (1b) and (1c) are the boundary and initial conditions, respectively.[1] We are given observations $u^{\text{obs}} = u^{\text{obs}}(x, t)$ for a latter portion of the time interval, i.e., for $t \in [T_1, T]$, where $T_1 > 0$. To invert for the viscosity field m, we thus minimize the functional

$$\mathcal{F}(m) := \frac{1}{2} \int_{T_1}^{T} \int_0^L (u - u^{\text{obs}})^2 \, dx \, dt + \frac{\beta}{2} \int_0^L \frac{dm}{dx} \frac{dm}{dx} \, dx \tag{2}$$

with regularization parameter $\beta > 0$. An efficient optimization method for (2) requires the gradient of $\mathcal{F}$ with respect to m.

1. Derive a weak form of (1) by multiplying (1a) with a test function $p(x, t) : [0, L] \times [0, T]$ that satisfies Dirichlet boundary conditions analogous to (1b), and integrating over space and time. There is no need to impose the initial condition explicitly via a Lagrange multiplier (as we did in class for the initial condition inversion problem), since the inversion parameter (m) does not appear in the initial condition. Instead, you should build the satisfaction of the initial condition into the definition of the solution space (just as you would do with a Dirichlet boundary condition). Use integration by parts on just the viscous term to derive the weak form of the Burgers' equation.

2. Using the Lagrangian approach, derive expressions for the adjoint equation and for the gradient of $\mathcal{F}$ with respect to m. Give weak and strong forms of these expressions. Note that m as well as its variation $\hat{m}$ are functions of space only, while u and the adjoint p are functions of space and time.

II. Inverse advection-diffusion inverse problem, continued from Assignment 4.

Here, we continue solution of the advection-diffusion inverse problem begun in Assignment 4 (with a slight change in the definition of the diffusivity inversion field and value of the source term). There we employed a steepest descent method to minimize the cost functional. Here, we will extend a state-of-the-art inexact Newton-CG method, the source code for which is available on the class webpage

[1] Note that the boundary $\partial\Omega$ of the one-dimensional interval $\Omega = (0, L)$ is simply the points $x = 0$ and $x = L$, i.e., $\partial\Omega = \{0, L\}$

(http://users.ices.utexas.edu/~omar/inverse_problems). Please hand in printouts of your implementations (at least of the lines you modified) together with the results.

We wish to solve the inverse problem for the advection-diffusion equation on $\Omega = [0,1] \times [0,1]$:

$$\min_m \mathcal{F}(m) := \frac{1}{2}\int_\Omega (u - u^{obs})^2 \, dx + \frac{\beta}{2}\int_\Omega \nabla m \cdot \nabla m \, dx, \tag{3}$$

where $u(\boldsymbol{x})$ depends on the log diffusivity $m(\boldsymbol{x})$ through

$$\begin{aligned} -\nabla \cdot (\exp(m)\nabla u) + \boldsymbol{v} \cdot \nabla u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{4}$$

with the advective velocity $\boldsymbol{v}(\boldsymbol{x}) = (v_1, v_2)$, regularization parameter $\beta > 0$, source term $f = 10$, and measurement data $u^{obs}(\boldsymbol{x})$. The latter are synthesized by solving the state equation with $m(x, y) = 2$ for $(x - 0.5)^2 + (y - 0.5)^2 \leq 0.04$ and $m(x, y) = 8$ otherwise (and adding noise). Notice that here we have redefined m to be the natural log of the diffusivity field; this enforces positivity on the diffusivity field.

1. Derive weak and strong forms of the gradient and Hessian action for this problem.

2. Extend `Poisson-InexactNewton.py` (the FEniCS implementation of the inexact Newton and Gauss-Newton methods[2]) by introducing an advection term with velocity $\boldsymbol{v} = (30, 0)$. Report the number of Gauss-Newton and of total CG iterations for a discretization of the domain with 10×10, 20×20, 40×40 and 80×80 finite elements and give the number of unknowns used to discretize the log diffusivity field m (which is called a in the implementation) for each of these meshes. Note that the state and adjoint variables and their incremental variants are discretized with quadratic finite elements, while the inversion parameter (log diffusivity) field is discretized with linear elements. Discuss how the number of iterations changes as the inversion parameter mesh is refined, i.e., as the parameter dimension increases. In these experiments, the noise level should be fixed to the default value (0.01) while the mesh is refined. The "optimal" regularization parameter can be found manually (i.e., by experimenting with a few different values and finding the one that results in a reconstruction that best matches the "true" log diffusivity field), or else by the discrepancy principle, if you are so inclined.

3. Fix the mesh size to 40×40 and consider two noise levels: low noise (0.01) and high noise (0.1). Determine the "optimal" regularization parameter in each case. For each noise level, solve the inverse problem by both Newton and Gauss-Newton methods. Report the number of nonlinear (Newton) and total linear (CG) iterations. Draw conclusions on the relative performance of Newton and Gauss-Newton as a function of the noise level.

4. *Optional:* Replace Tikhonov regularization with total variation regularization and repeat sub-problem 2 above (i.e., report number of Gauss-Newton and CG iterations as the mesh is refined).

[2]Recall that the Gauss-Newton approximation of the Hessian drops all terms that depend on the adjoint variable p.