

**Fall 2015:**  
**Computational and Variational Methods for Inverse Problems**  
**CSE 397/GEO 391/ME 397/ORI 397**  
**Assignment 1 (due Sept 23, 2015)**

In this assignment we will generalize the deblurring inverse problem considered in class in two ways: first, we will consider a different 1D blurring operator (this one compactly supported), and second, we will extend to deblurring of 2D images. In both cases, you will compare the two methods for selection of the regularization parameter (L-curve and Morozov discrepancy) to the optimal choice of the regularization parameter (since you will know the true image).

MATLAB code files required for this assignment can be downloaded from the course webpage, [http://users.ices.utexas.edu/~omar/inverse\\_problems](http://users.ices.utexas.edu/~omar/inverse_problems).

**Problem 1:** Extend the MATLAB code `deconv1D.m` to handle a different blurring operator than the Gaussian we studied in class. In particular, use the blurring operator

$$d(x) = \int_0^1 k(x-x') m(x') dx' \quad \text{for } 0 < x < 1,$$

with  $k(x) = c^{-2} \max(0, c - |x|)$  and  $c = 0.2$ . Use 200 discretization points and the resulting matrix  $K$  to blur the (discrete version of the) true image

$$m_{\text{true}}(x) = \begin{cases} 0.75 & 0.1 < x < 0.25 \\ 0.25 & 0.3 < x < 0.32 \\ \sin^4(2\pi x) & 0.5 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Add normally distributed noise<sup>1</sup>  $\mathbf{n}$  with mean zero and variance  $\sigma^2 = 0.1$ . The resulting blurred and noisy image data is  $\mathbf{d} = \mathbf{K}\mathbf{m}_{\text{true}} + \mathbf{n}$ .

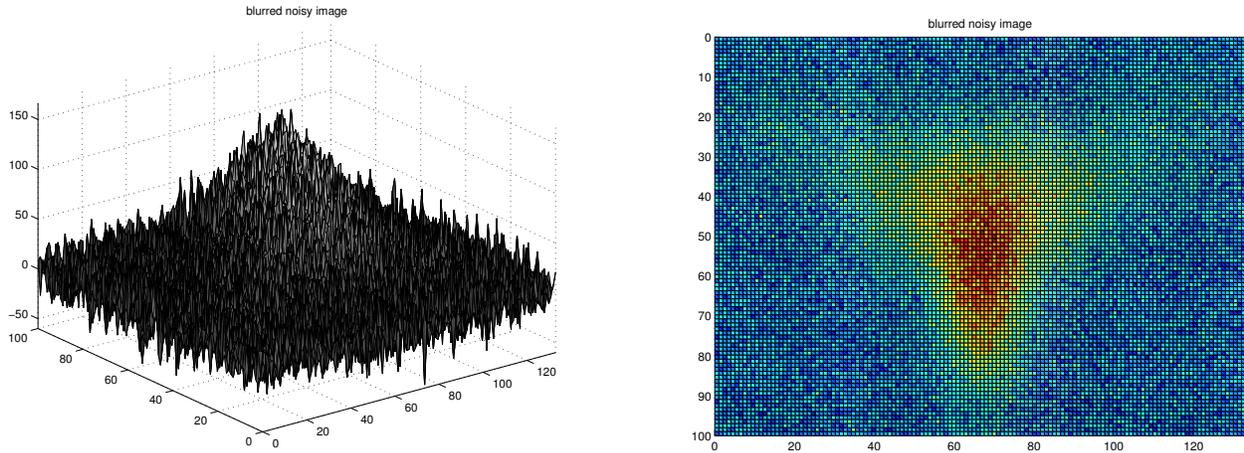
- a) Use the truncated singular value decomposition<sup>2</sup> filter (TSVD) with  $\alpha = 0.0001, 0.001, 0.01, 0.1, 1$  to compute the regularized reconstructions  $\mathbf{m}_{\alpha}^{\text{TSVD}}$ .
- b) Use the Tikhonov filter with the same values for  $\alpha$  for the reconstructions  $\mathbf{m}_{\alpha}^{\text{Tikh}}$ .
- c) Determine the (approximate) optimal value of the regularization parameter  $\alpha$  in the Tikhonov regularization using the L-curve criterion.
- d) Determine the (approximate) optimal value of the regularization parameter  $\alpha$  in the Tikhonov regularization using Morozov's discrepancy criterion, i.e., find the largest value of  $\alpha$  such that

$$\|\mathbf{K}\mathbf{m}_{\alpha} - \mathbf{d}\| \leq \delta$$

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<sup>1</sup>Use the MATLAB function `randn`.

<sup>2</sup>MATLAB provides the function `svd` to compute the singular value decomposition of a matrix.



**Figure 1:** Noisy and blurred gray value image plotted as surface seen from side (left) and from top (right, in pseudo color). In MATLAB, images are represented as matrices, in which each entry corresponds to the gray value of a pixel.

where  $\delta = \|\mathbf{n}\|$  and  $\mathbf{m}_\alpha$  is the solution of the Tikhonov-regularized inverse problem with regularization parameter  $\alpha$ .

- e) Plot the  $L_2$  norm error in the reconstruction,  $\|\mathbf{m}_{\text{true}} - \mathbf{m}_\alpha\|$ , as a function of  $\alpha$ , where  $\mathbf{m}_\alpha$  is the Tikhonov regularized solution. Which value of  $\alpha$  (approximately) minimizes this error? Compare the “optimal” values of  $\alpha$  obtained in parts c, d, and e of this problem and comment on any differences.

**Problem 2:** We wish to use Tikhonov regularization to reconstruct the blurred and noisy 2D image shown in Figure 1. A MATLAB code to create the noisy data from the image `longhorn.png` and compute the Tikhonov reconstruction is provided in function `deconv2D.m`.<sup>3</sup> Note that the Tikhonov-regularized inverse problem is solved iteratively with the conjugate gradient (CG) method<sup>4</sup> (rather than with a direct solver) since it is prohibitive to explicitly construct the 2D blurring operator matrix  $\mathbf{K}$ . Instead, we can apply it to a vector using 1D blurring operators in the  $x$  and  $y$  directions.<sup>5</sup>

- a) How much memory would be required to store the matrix  $\mathbf{K}$  if we followed the same approach as in the 1D deblurring problem?
- b) Determine the (approximately) optimal value of the regularization parameter  $\alpha$  in the Tikhonov-regularized solution using the L-curve criterion.

<sup>3</sup>Before running this function, an image has to be imported as a matrix into MATLAB’s workspace: choose File→Import Data and select the image `longhorn.png`.

<sup>4</sup>If you have not heard of this method, don’t worry—we’ll cover it later in class. The CG method is simply an iterative way to solve linear systems without explicitly constructing the coefficient matrix; all we need to provide is the product of the matrix with a given vector.

<sup>5</sup>A similar situation occurs typically in the solution of inverse problems with PDEs; we cannot explicitly form the inverse operator, but we can form its action on a vector.

- c) Which value of  $\alpha$  minimizes the  $L_2$  norm of the difference between the true image and the Tikhonov-regularized reconstruction? How does this value compare to the  $\alpha$  yielded by the L-curve?