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A technique for a posteriori error estimation of $h$-$p$ approximations of the Stokes equations

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This paper deals with a posteriori error estimation for finite element approximations of the Stokes problem. The ultimate objective is to analyze how given error quantities of interest are influenced by the residuals, which are viewed as the sources of the numerical error. First, a global error estimator is proposed in terms of the norms of the residuals. Then, techniques to efficiently estimate these norms are advocated. This is followed by the investigation of a general approach to evaluate the numerical error in pointwise, local or global quantities of interest.

1. INTRODUCTION

Error estimation in computational processes has been a subject of interest for more than two decades since the pioneering work of Babuska and Rheinboldt [3]. We refer to Ainsworth and Oden [2] and Verfürth [19] for an extended account of the subject. In the particular case of the Stokes problem, several approaches have been investigated in [18,8,6,16,2].

The method we propose here belongs to the family of Implicit Error Residual Methods. The errors in the velocity and pressure variables are driven by the residuals $R^m_h$ and $R^p_h$ in the momentum equation and the continuity equation respectively. The residuals represent the sources of error in the finite element approximations, and as such, are post-processed to provide meaningful error estimates in global “energy” norms. The evaluation of $R^p_h$ is shown to be exact, local and cheap. The calculation of the error measure $R^m_h$ is however more demanding. A new technique has been developed which provides accurate approximations of $R^m_h$ through a global but inexpensive iterative process. First, error estimates are obtained using spaces of low-order bubble functions as perturbations. These are then corrected by using enriched spaces constructed through an adaptive procedure. This approach avoids the major difficulty of prescribing proper boundary conditions for each subproblem in the Element Residual method [11,7,2]. The effectiveness of the methodology is demonstrated on various test cases of the Stokes problem with smooth and unsmooth solutions.

In recent years, the success of a posteriori error estimation has prompted users to demand error estimates in quantities of interest other than the classical energy norm.

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Works in this field have been undertaken by Babuška and Strouboulis [4] and Becker and Rannacher [10,9] in order to estimate and control the error by adapting the mesh parameters with respect to these quantities of interest. We extend the ideas to the Stokes problem and perform a preliminary investigation of the performance of such techniques to evaluate the error in pointwise, local or global quantities of interest.

2. PRELIMINARIES

Let $\Omega$ denote an open bounded Lipschitz domain in $\mathbb{R}^n$, $n = 2$ or 3, with boundary $\partial \Omega$. We consider the Stokes equations:

$$
-\Delta u + \nabla p = f \quad \text{in } \Omega \\
\nabla \cdot u = 0 \quad \text{in } \Omega
$$

with Dirichlet boundary condition:

$$
u = g, \quad \text{on } \partial \Omega,
$$

where $u = u(x)$ and $p = p(x)$ are respectively a vector-valued and a scalar-valued function defined at point $x = (x_1, x_2, \ldots, x_n)$ in $\Omega$. Since the Stokes equations can be derived as a linearization of the steady-state Navier-Stokes equations, the variable $u$ and $p$ are referred to as the velocity and pressure. The source term $f = f(x)$ is a prescribed body force and $g$ is a function defined on $\partial \Omega$ which must satisfy the compatibility condition,

$$
\int_{\partial \Omega} g \cdot n \, ds = 0.
$$

In what follows, we restrict ourselves to the case of homogeneous boundary conditions, i.e. $g = 0$ on $\partial \Omega$. This simplifies somewhat the theoretical analysis of the Stokes equations, while retaining all their interesting features. We shall use standard notations for various Sobolev spaces of functions defined on $\Omega$. We begin by introducing the trial spaces of velocities $V$ and pressures $Q$ defined by:

$$
V = H_0^1(\Omega) = (H_0^1(\Omega))^n, \quad Q = \{q \in L^2(\Omega) : \int_{\Omega} q \, dx = 0\},
$$

with corresponding norms:

$$
|v|^2_1 = \int_{\Omega} \nabla v : \nabla v \, dx, \quad \|q\|_0^2 = \int_{\Omega} q^2 \, dx.
$$

We also introduce the bilinear forms $a$ and $b$,

$$
a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}; \quad a(u, v) = \int_{\Omega} \nabla u : \nabla v \, dx$$

$$
b : H^1(\Omega) \times L^2(\Omega) \to \mathbb{R}; \quad b(v, q) = -\int_{\Omega} q \nabla \cdot v \, dx.
$$

The bilinear form $a$ is the inner product associated to the norm $|.|_1$ in $V$, so that $a(v, v) = |v|^2_1$, $\forall v \in V$. 
**Lemma 1** The bilinear form $b$ is continuous on $V \times Q$, and in particular, there exists a positive constant $M_b$ such that

$$b(v, q) \leq M_b |v_1|_1 \|q\|_0, \quad \forall v \in V, \forall q \in Q,$$

where $M_b = \sqrt{n}$, where $n$ is the geometrical dimension of the problem.

**Proof** Let $v$ and $q$ be arbitrary functions in $V$ and $Q$. Since $\nabla \cdot v \in Q$, we have:

$$b(v, q) \leq \|\nabla \cdot v\|_0 \|q\|_0.$$

It is now sufficient to show there exists $M_b$ such that $\|\nabla \cdot v\|_0 \leq M_b |v_1|_1$. Indeed,

$$\|\nabla \cdot v\|_0^2 = \int_{\Omega} \left( \sum_{i=1}^{n} \frac{\partial v_i}{\partial x_i} \right)^2 \, dx \leq \int_{\Omega} n \sum_{i=1}^{n} \left( \frac{\partial v_i}{\partial x_i} \right)^2 \, dx \leq n \int_{\Omega} \nabla v : \nabla v \, dx = n |v_1|^2. \quad (4)$$

so that $\|\nabla \cdot v\|_0 \leq \sqrt{n} |v_1|_1$ and $M_b = \sqrt{n}$.

Moreover, it can be shown that $b$ satisfies the standard LBB condition (see Girault and Raviart [12]), i.e. there exists a constant $\beta > 0$ such that

$$\sup_{v \in V \backslash \{0\}} \frac{|b(v, q)|}{|v_1|} \geq \beta \|q\|_0, \quad \forall q \in Q. \quad (5)$$

The Stokes problem is now reformulated in the equivalent weak form:

For $f \in V'$ given, find $(u, p) \in V \times Q$, such that

\[
\begin{align*}
a(u, v) + b(v, p) &= \langle f, v \rangle, \quad \forall v \in V, \\
b(u, q) &= 0, \quad \forall q \in Q
\end{align*}
\]  \quad (6)

Extensive results concerning the existence, uniqueness and regularity of solutions $(u, p)$ in $V \times Q$ of the Stokes problem can be found in Girault and Raviart [12].

Let $V^h \subset V$ and $Q^h \subset Q$ denote finite element spaces, possibly $h$-$p$ finite element spaces [14], of the spaces $V$ and $Q$. Approximate solutions to problem (6) are then obtained by solving the following system of discrete equations:

For $f \in V'$ given, find $(u_h, p_h) \in V^h \times Q^h$, such that

\[
\begin{align*}
a(u_h, v) + b(v, p_h) &= \langle f, v \rangle, \quad \forall v \in V^h, \\
b(u_h, q) &= 0, \quad \forall q \in Q^h
\end{align*}
\]  \quad (7)

Let us consider a pair $(u_h, p_h) \in V^h \times Q^h$, not necessarily a solution of (7). The numerical error $(e, E) \in V \times Q$ in $(u_h, p_h)$ is defined as

$$(e, E) = (u, p) - (u_h, p_h). \quad (8)$$

Then, substituting $u$ and $p$ in (6) by $u_h + e$ and $p_h + E$ respectively, we show that the distribution of $(e, E)$ is governed by the system of equations:

\[
\begin{align*}
a(e, v) + b(v, E) &= \mathcal{R}_h^n(v), \quad \forall v \in V, \\
b(e, q) &= \mathcal{R}_h^n(q), \quad \forall q \in Q
\end{align*}
\]  \quad (9)
where the linear functionals $\mathcal{R}_h^m : V \rightarrow \mathbb{R}$ and $\mathcal{R}_h^c : Q \rightarrow \mathbb{R}$

$$\mathcal{R}_h^m(v) = (\mathcal{R}_h^m, v) \equiv (f, v) - a(u_h, v) - b(v, p_h)$$

$$\mathcal{R}_h^c(q) = (\mathcal{R}_h^c, q) \equiv -b(u_h, q)$$

are respectively the residual in the momentum equation and the residual in the continuity equation. The residuals $\mathcal{R}_h^m$ and $\mathcal{R}_h^c$ are viewed as the sources of error. Indeed, whenever they are zero, the numerical error is zero as well. We show in the next section how the residuals can provide reliable global error estimates in specific norms.

### 3. GLOBAL ERROR ESTIMATION

The objective here is to relate the residuals to the numerical errors expressed in some global norms. We recall (see [12, p.22]) that the space $V = H^1_0(\Omega)$ can be decomposed into the direct sum $V = J \oplus J^\perp$, where the spaces $J$ and $J^\perp$ are defined as:

$$J = \{ v \in V; \ b(v, q) = 0, \ \forall q \in Q \},$$

$$J^\perp = \{ v \in V; \ a(v, w) = 0, \ \forall w \in J \}.$$

The space $J^\perp$ is the orthogonal complement of $J$ with respect to $H^1_0(\Omega)$ for the inner product $a(\cdot, \cdot)$. It follows that the error in the velocity variable $e \in V$ can be uniquely decomposed into a sum of two vectors, $e_d \in J$ and $e_\perp \in J^\perp$, such that $e = e_d + e_\perp$.

We naturally measure the error $(e, E)$ in the usual global norm

$$\|(e, E)\|^2 = |e|^2 + |E|^2 = |e_d|^2 + |e_\perp|^2 + |E|^2.$$

and consider the following norms for the residuals $\mathcal{R}_h^m$ and $\mathcal{R}_h^c$

$$\|\mathcal{R}_h^m\|_\infty = \sup_{v \in V \setminus \{0\}} \frac{|\mathcal{R}_h^m(v)|}{|v|_1},$$

$$\|\mathcal{R}_h^c\|_\infty = \sup_{q \in Q \setminus \{0\}} \frac{|\mathcal{R}_h^c(q)|}{\|q\|_0}.$$

**Lemma 2** With the above definitions and assumptions:

$$\beta |e_\perp|_1 \leq \|\mathcal{R}_h^c\|_\infty \leq M_b |e_\perp|_1.$$

**Proof** From the definition of the residual $\mathcal{R}_h^c$, we have, for all $q \in Q$

$$\mathcal{R}_h^c(q) = b(e, q) = b(e_\perp, q) \leq M_b |e_\perp|_1 \|q\|_0.$$

The upper bound in (12) follows as

$$\|\mathcal{R}_h^c\|_\infty = \sup_{q \in Q} \frac{\mathcal{R}_h^c(q)}{\|q\|_0} \leq \sup_{q \in Q} \frac{M_b |e_\perp|_1 \|q\|_0}{\|q\|_0} \leq M_b |e_\perp|_1.$$

The divergence operator is an isomorphism of $J^\perp$ onto $Q$, so the function $\nabla \cdot e_\perp$ belongs to $Q$ and therefore

$$\|\nabla \cdot e_\perp\|^2 = -\mathcal{R}_h^c(\nabla \cdot e_\perp) \leq \|\mathcal{R}_h^c\|_\infty \|\nabla \cdot e_\perp\|_0$$

which yields $\|\nabla \cdot e_\perp\|_0 \leq \|\mathcal{R}_h^c\|_\infty$. From Girault and Raviart [12, p.24, p.81], we obtain the lower bound in (12) as

$$\beta |e_\perp|_1 \leq \|\nabla \cdot e_\perp\|_0 \leq \|\mathcal{R}_h^c\|_\infty.$$

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Lemma 3 With the above definitions and assumptions, we have:

(1) \[ |e_d| \leq \| \mathcal{R}_h^n \|_* \]  
(2) \[ \| \mathcal{R}_h^n \|_* \leq |e|_1 + M_b \| E \| _0 \]  
(3) \[ \beta \| E \|_0 \leq \| \mathcal{R}_h^n \|_* + |e_\perp|_1 \]

Proof

1. By simple algebra, we get:

\[ |e_d|^2 = a(e_d, e_d) = a(e, e_d) = \mathcal{R}_h^n(e_d) - b(e_d, E) = \mathcal{R}_h^n(e_d) \leq \| \mathcal{R}_h^n \|_* |e_d|_1 \]

which shows (13).

2. We also have

\[ \| \mathcal{R}_h^n \|_* = \sup_{v \in V} \frac{\mathcal{R}_h^n(v)}{|v|_1} \leq \sup_{v \in V} \frac{a(e, v) + b(v, E)}{|v|_1} \]

\[ \leq \sup_{v \in V} \frac{a(e, v)}{|v|_1} + \sup_{v \in V} \frac{b(v, E)}{|v|_1} \leq |e|_1 + M_b \| E \|_0 \]

3. The upper bound for the error in the pressure variable follows from the inf-sup condition on the bilinear form \( b \), i.e.

\[ \beta \| E \|_0 \leq \sup_{v \in V} \frac{b(v, E)}{|v|_1} \leq \sup_{v \in J^\perp} \frac{b(v, E)}{|v|_1} \leq \sup_{v \in J^\perp} \frac{\mathcal{R}_h^n(v) - a(e, v)}{|v|_1} \]

\[ \leq \sup_{v \in J^\perp} \frac{\mathcal{R}_h^n(v)}{|v|_1} + \sup_{v \in J^\perp} \frac{a(e, v)}{|v|_1} \leq \sup_{v \in V} \frac{\mathcal{R}_h^n(v)}{|v|_1} + \sup_{v \in J^\perp} \frac{a(e_\perp, v)}{|v|_1} \]

\[ \leq \| \mathcal{R}_h^n \|_* + |e_\perp|_1 \]

Theorem 1 Let (e, E) be the numerical error in an approximation (uh, ph) of the Stokes problem. Then there exist positive constants \( C_1 \) and \( C_2 \) such that

\[ C_1 \|(e, E)\|^2 \leq \| \mathcal{R}_h^n \|^2 + \| \mathcal{R}_h^n \|^2 \leq C_2 \|(e, E)\|^2 \]  

where \( C_1 \) and \( C_2 \) depend only on the constants \( \beta \) and \( M_b \) respectively:

\[ C_1 = \frac{1}{2} \min(1, \beta^4), \quad C_2 = \max(2 + M_b^2, 2M_b^4). \]  

\[ (16) \]  

\[ (17) \]
Proof The proof of this theorem directly follows from Lemma 2 and Lemma 3. In particular, the lower bound is obtained from inequalities (12), (13) and (15), while the upper bound is derived from the inequalities (12) and (14).

This theorem is similar to the Theorem 6.1 introduced by Ainsworth and Oden in [2]. It shows that the global quantity \( \| \mathcal{R}_h^\varepsilon \|^2 + \| \mathcal{R}_h^\varepsilon \|^2 \) is equivalent to the global error measure \( \|(e, E)\|^2 \). We emphasize that it represents a meaningful quantitative global estimate as long as the constants \( C_1 \) and \( C_2 \) do not take values far from one. Here the constant \( C_2 \) takes the value 4 in two-dimensions and 6 in three-dimensions since \( M_2 = \sqrt{n} \). Therefore, the accuracy and robustness of this error estimate essentially depends on the value of \( C_1 \), a fortiori, \( \beta \), which depends itself on the problem in consideration. Moreover, we emphasize that such a global result does not provide any reliable information about the local (elementwise or pointwise) error, as we know that the error can propagate far away from their sources \( \mathcal{R}_h^\varepsilon \) and \( \mathcal{R}_h^\varepsilon \). This actually motivated the work undertaken by Babuška, Strouboulis and co-workers [4] and Oden and Feng [18] on pollution error in order to estimate the contributions of the residuals to the error outside the region of interest. However, the quantity \( \| \mathcal{R}_h^\varepsilon \|^2 + \| \mathcal{R}_h^\varepsilon \|^2 \) provides information about the location and intensity of the sources of errors, and as such, should be used to determine local refinement indicators.

4. EVALUATING NORMS OF RESIDUALS

This section is devoted to the evaluation of \( \| \mathcal{R}_h^\varepsilon \| \) and \( \| \mathcal{R}_h^\varepsilon \| \). The objectives are, on one hand, to minimize the cost of the computations while retaining a certain accuracy, and, on the other hand, to obtain elementwise contributions of these quantities.

4.1. Residual in the continuity equation

The residual \( \mathcal{R}_h^\varepsilon \) gives us information about whether the discrete velocity \( u_h \) does or does not satisfy the incompressibility constraint. This is stated in the next lemma, where the norm of the residual \( \mathcal{R}_h^\varepsilon \) is expressed in terms of the divergence of \( u_h \).

Lemma 4 Let \( u_h \in V^h \) be the discrete velocity of the Stokes problem. Then,

\[
\| \mathcal{R}_h^\varepsilon \| = \| \nabla \cdot u_h \|_0. \tag{18}
\]

Proof From the definition of the residual \( \mathcal{R}_h^\varepsilon \), we have, for all \( q \) in \( Q \):

\[
\mathcal{R}_h^\varepsilon (q) = -b(u_h, q) = \int_Q q \nabla \cdot u_h \, dx \leq \| \nabla \cdot u_h \|_0 \| q \|_0.
\]

It follows that

\[
\| \mathcal{R}_h^\varepsilon \| = \sup_{q \in Q} \| \mathcal{R}_h^\varepsilon (q) \| \leq \sup_{q \in Q} \frac{\| \nabla \cdot u_h \|_0 \| q \|_0}{\| q \|_0} \leq \| \nabla \cdot u_h \|_0. \tag{19}
\]

We also have

\[
\| \nabla \cdot u_h \|_0^2 = \mathcal{R}_h^\varepsilon (\nabla \cdot u_h) \leq \| \mathcal{R}_h^\varepsilon \| \| \nabla \cdot u_h \|_0,
\]

that is, \( \| \nabla \cdot u_h \|_0 \leq \| \mathcal{R}_h^\varepsilon \| \), and the equality (18) has just been proven.
The result of Lemma 4 is remarkable in the sense that the evaluation of $\|R_h^c\|_*$ is both exact and cheap, as it is equivalent to compute the $L^2(\Omega)$-norm of the function $\nabla \cdot u_h$. In addition, it is straightforward to decompose $\|R_h^c\|_*$ into a sum of elementwise contributions as:

$$\|R_h^c\|^2 = \|\nabla \cdot u_h\|^2 = \sum_K \|\nabla \cdot u_h\|^2_{0,K} = \sum_K \int_{\Omega_K} |\nabla \cdot u_h|^2 \, dx$$

where $\sum_K$ represents the sum over all the elements in a finite element partition.

### 4.2. Residual in the momentum equation

Simple considerations reveal that the evaluation of $\|R_h^m\|_*$ is neither cheap nor exact. At best, one obtains approximations of it. There exist basically two approaches to compute such approximations. The first one always delivers lower bounds on $\|R_h^m\|_*$, while the other provides upper bounds. Here we only consider the former, which is referred to as the \textit{conforming method}.

#### 4.2.1. Exact evaluation

Since $V$ is a Hilbert space, the \textit{Riesz Representation Theorem} tells us there exists a unique element $\varphi$ in $V$ which satisfies:

$$a(\varphi, v) = R_h^m(v), \quad \forall v \in V,$$

as well as:

$$|\varphi|_1 = \sqrt{a(\varphi, \varphi)} = \|R_h^m\|_*.$$

In other words, the residual $R_h^m \in V'$ is identified with an element $\varphi \in V$ of equal norm. Moreover, the function $\varphi$ gives information about the local intensity of the residual $R_h^m$, such as:

$$\|R_h^m\|^2_* = |\varphi|^2_1 = \sum_K |\varphi|^2_{1,K} = \sum_K \int_{\Omega_K} \nabla \varphi : \nabla \varphi \, dx.$$

However, the problem (20) cannot be solved numerically due to the infinite dimension of the space $V$.

#### 4.2.2. Conforming finite element approximations

The objective here is to construct a finite element subspace $\bar{V}^h \subset V$, hence the label \textit{conforming method}, in which to approximate problem (20), that is, to find $\varphi_h \in \bar{V}^h$ such that:

$$a(\varphi_h, v) = R_h^m(v), \quad \forall v \in \bar{V}^h.$$  

We show that the norm of $\varphi_h$ is always smaller than the norm of $R_h^m$ and that both quantities are indeed equivalent. First, we observe from (20) and (22) that:

$$a(\varphi - \varphi_h, v) = 0, \quad \forall v \in \bar{V}^h.$$  

(23)
In consequence, $\varphi_h$ is simply the orthogonal projection of $\varphi$ onto the space $\overline{V}^h$ with respect to the inner product $a(\cdot, \cdot)$ associated to $|\cdot|_1$. It follows that:

$$|\varphi|_1^2 + |\varphi - \varphi_h|_1^2 = |\varphi_h|_1^2,$$

which yields the inequality $|\varphi - \varphi_h|_1 \leq |\varphi|_1$. Such a result can be sharpened, as stated in the next lemma.

**Lemma 5** Let $\varphi \in V$ and $\varphi_h \in \overline{V}^h$ satisfy (20) and (22) respectively. If $\varphi \neq 0$, let $\overline{V}^h$ be such that $\varphi_h \neq 0$. Then, there exists a constant $\sigma$, $0 \leq \sigma < 1$ such that:

$$|\varphi - \varphi_h|_1 \leq \sigma |\varphi|_1.$$  

**Proof** Since $|\varphi_h|_1 \neq 0$, $|\varphi - \varphi_h|_1$ is strictly less than $|\varphi|_1$ by (24).

**Theorem 2** Let $\varphi \in V$, $\varphi_h \in \overline{V}^h$ and $\overline{V}^h$ satisfy the conditions of Lemma 5. Then, there exists a constant $\sigma$, $0 \leq \sigma < 1$, such that:

$$\sqrt{1 - \sigma^2} \|\mathcal{R}_h^n\|_* \leq |\varphi_h|_1 \leq \|\mathcal{R}_h^n\|_*.$$  

**Proof** The upper bound follows from equality (24). Indeed, $|\varphi_h|_1 \leq |\varphi|_1 = \|\mathcal{R}_h^n\|_*$. Now, making use of (24) and (25), we have:

$$|\varphi|_1^2 - |\varphi_h|_1^2 = |\varphi - \varphi_h|_1^2 \leq \sigma^2 |\varphi|_1^2,$$

that is

$$(1 - \sigma^2) |\varphi|_1^2 \leq |\varphi_h|_1^2,$$

from which the lower bound follows.

Theorem 2 shows that the quantities $|\varphi_h|_1$ and $\|\mathcal{R}_h^n\|_*$ are equivalent. However, $|\varphi_h|_1$ provides a valuable approximation of $\|\mathcal{R}_h^n\|_*$ as long as $\sqrt{1 - \sigma^2}$ is close to one. The constant $\sigma$ clearly depends on how rich the finite element space $\overline{V}^h$ is, but, at the same time, we point out that even a crude approximation $\varphi_h$ of $\varphi$ may be sufficient to obtain an acceptable estimate of $\|\mathcal{R}_h^n\|_*$. For instance, by committing a relative error $\sigma = 40\%$ in approximating $\varphi$, we still approximate $\|\mathcal{R}_h^n\|_*$ with an effectivity of 91.6%.

The finite element space $\overline{V}^h$ has to be chosen so that the action of the residual $\mathcal{R}_h^n$ is different from zero for at least one element of $\overline{V}^h$. In the case where the pair $(u_h, p_h)$ is the solution of problem (7), we observe that the residual vanishes on $V^h$, that is:

$$\mathcal{R}_h^n(v) = 0, \quad \forall v \in V^h.$$  

It follows that the space $\overline{V}^h$ should be larger than $V^h$ itself. It is then constructed by enriching $V^h$ with elements of a space $W^h$, called space of perturbations, which satisfies:

$W^h \neq \{0\}, \quad W^h \subset V, \quad V^h \cap W^h = \{0\}$.  

(29)
so that $V^h \subset \tilde{V}^h = V^h + W^h \subset V$. Since $V^h \subset \tilde{V}^h$, problem (22) is more expensive to solve than the original one. At this point, we want to reduce the cost of the computations by taking advantage of the fact that the residual vanishes on the space $V^h$. In other words, we would like to approximate $\|R_h^m\|$ by the norm of the function $\psi^*_h \in W^h$ satisfying

$$a(\psi^*_h, v) = R_h^m(v), \quad \forall v \in W^h,$$

which implies, using Young's inequality, that

$$|v^h + w^h|_1^2 \geq |v^h|_1^2 + 2a(v^h, w^h) + |w^h|_1^2 \geq |v^h|_1^2 - 2\gamma |v^h|_1 |w^h|_1 + |w^h|_1^2 \geq (1 - \gamma^2) |w^h|_1^2.$$

Thus,

$$|v^h + w^h|_1 \geq \frac{1 - \gamma^2}{\gamma} |w^h|_1.$$

The strengthened Cauchy-Schwartz Inequality allows us to relate the accuracy of the approximation $|\psi^*_h|_1 \approx |\varphi^*_h|_1$ to $\gamma$, the cosine of the angle between the spaces $V^h$ and $W^h$:

**Theorem 3** Let $\varphi \in V$, $\varphi^*_h \in \tilde{V}^h$ and $\tilde{V}^h$ satisfy the conditions of Lemma 5. Let the strengthened Cauchy-Schwartz inequality (31) hold for the spaces $V^h$ and $W^h$, where $\tilde{V}^h = V^h + W^h$ and $R_h^m = 0$ on $V^h$. Let $\psi^*_h \in W^h$ satisfy (30). Then

$$\sqrt{(1 - \sigma^2)(1 - \gamma^2)} \|R_h^m\|_* \leq |\psi^*_h|_1 \leq \|R_h^m\|_*.$$

**Proof** The upper bound is readily obtained as $W^h \subset V$. In order to prove the lower bound, we write $\varphi^*_h = \varphi^*_h + \psi^*_h$, where $\varphi^*_h \in V^h$ and $\psi^*_h \in W^h$. Then, from equation (22), we have, since $\varphi^*_h \in V^h$:

$$a(\varphi^*_h, \varphi^*_h) = R_h^m(\varphi^*_h) = 0,$$

and from (22) and (30), we get:

$$a(\varphi^*_h, \psi^*_h) = R_h^m(\psi^*_h) = a(\psi^*_h, \psi^*_h).$$

Hence,

$$|\varphi^*_h|_1^2 = a(\varphi^*_h, \varphi^*_h) = a(\varphi^*_h, \varphi^*_h + \psi^*_h) = a(\varphi^*_h, \psi^*_h) = a(\psi^*_h, \psi^*_h) \leq |\psi^*_h|_1 |\psi^*_h|_1.$$
Applying the strengthened Cauchy-schwartz inequality to solution \( \varphi_h \) yields:
\[
\sqrt{1 - \gamma^2} |\psi_h^*|_1 \leq |\varphi_h^* + \varphi_h^*|_1 = |\varphi_h|_1,
\]
which, combined to (36), gives
\[
|\varphi_h|_1^2 \leq |\psi_h|_1 |\psi_h^*|_1 = \frac{1}{\sqrt{1 - \gamma^2}} |\psi_h|_1 |\varphi_h|_1,
\]
that is,
\[
|\varphi_h|_1 \leq \frac{1}{\sqrt{1 - \gamma^2}} |\psi_h|_1.
\]

Then, using inequality (27) shown in Theorem 2, allows us to write:
\[
\sqrt{1 - \sigma^2} |\varphi|_1 \leq |\varphi_h|_1 \leq \frac{1}{\sqrt{1 - \gamma^2}} |\psi_h|_1.
\]

The lower bound is proved.

The above theorem shows that the accuracy of the approximation \( |\psi_h|_1 \approx |\varphi|_1 \) only depends on the constants \( \sigma \) and \( \gamma \).

We recall that \( \sigma \) depends on the richness of the space \( \overline{V}^h \), or \( W^h \). The main difficulty is to construct \( W^h \) so that the approximations \( \psi_h \in W^h \) deliver reliable estimates of \( \| R_h \| \) at the lowest cost. In \( h-p \) finite element methods, the perturbations can be conveniently constructed from layers of piecewise polynomial basis functions involving monomials of degree between \( p+1 \) and \( p+q, q \geq 1 \), where \( p \) is defined as the maximal degree of the basis functions in \( V^h \). Obviously, in two-dimensional problems, such basis functions consist of edge and/or interior bubble functions. In three-dimensional problems, they would consist of edge, face and/or interior bubble functions. One question one has to answer is "what is the best value for \( q \)?" Actually, this is problem dependent, and one should consider a method in which the value of \( q \) is increased as needed.

We recall that \( \gamma \) represents the angle between \( V^h \) and \( W^h \). Its value is directly related to the choice of the shape functions used to construct the basis functions of \( V^h \) and \( W^h \). Here we use hierarchic shape functions based on integrated Legendre polynomials (see Szabó and Babuška [17, chapt. 6]), which satisfy the orthogonality property with respect to the inner product \( a(\cdot, \cdot) \).

Meanwhile, the cost in solving (30) is controlled by the dimension of \( W^h \). However, the bilinear form \( a(\cdot, \cdot) \) is symmetric and positive definite so that the finite system (30) can be cheaply solved using the iterative Conjugate Gradient method (CG). Moreover, we emphasize that the solution does not have to be highly accurate as shown by Theorem 2, so that it is usually sufficient to perform a few iterations.

Finally, conforming methods always deliver lower bounds on \( R_h \). In order to construct upper bounds, one has to approximate problem (20) in a space larger than \( V \). This leads to the equilibrated residual method developed by Ainsworth and Oden [1,2] or Ladevèze and Leguillon [13]. In this approach, the global problem (20) is decoupled into a collection of local problems, usually constructed on each of the elements in the partition. Our present conforming method avoids the major difficulty of prescribing boundary conditions for each subproblem typical of the equilibrium methods at comparable cost.
5. ESTIMATION AND CONTROL OF ERROR QUANTITIES OF INTEREST

Let $L$ denote a linear functional defined on the product space $V \times Q$. We suppose that we are interested in the error quantity $L(e, E) \in \mathcal{R}$, which we want to estimate and control. For example, $L(\cdot, \cdot)$ may represent something more localized than the global estimate (16), such as the error in $u$ and $p$ or their gradients at a point $x_0 \in \Omega$ or along a curve $\Gamma \subset \Omega$; e.g. $L(e, E) = E(x_0)$ or $L(e, E) = \int_{\Gamma} e \cdot n \, ds$, etc. We cite more examples later. The main objective is then to relate $L(e, E)$ to the sources of error $\mathcal{R}_h^m$ and $\mathcal{R}_h^c$; in other words, we would like to find linear functionals $\omega^m$ and $\omega^c$, if they exist, such that

$$L(e, E) = \omega^m(\mathcal{R}_h^m) + \omega^c(\mathcal{R}_h^c).$$  \hspace{1cm} (40)

These functions are viewed as influence functions as they indicate the influence of the residuals on the quantity $L(e, E)$. They are defined on the bidual of $V$ and $Q$, and since these spaces are reflexive, we have:

$$\omega^m \in V \quad \text{and} \quad \omega^c \in Q. \hspace{1cm} (41)$$

This is understood in the sense that for each $\omega^m \in V''$ and each $\omega^c \in Q''$, there exist $\tilde{\omega}^m \in \tilde{V}$ and $\tilde{\omega}^c \in \tilde{Q}$ respectively such that

$$\langle \omega^m, \mathcal{R}_h^m \rangle_{V'' \times V'} = \langle \tilde{\omega}^m, \mathcal{R}_h^m \rangle_{\tilde{V}'' \times \tilde{V}'} \quad \text{and} \quad \langle \omega^c, \mathcal{R}_h^c \rangle_{Q'' \times Q'} = \langle \tilde{\omega}^c, \mathcal{R}_h^c \rangle_{\tilde{Q}'' \times \tilde{Q}'}.$$

By identifying $\tilde{\omega}^m$ and $\tilde{\omega}^c$ with $\omega^m$ and $\omega^c$ respectively, the relation (40) becomes

$$L(e, E) = \mathcal{R}_h^m(\omega^m) + \mathcal{R}_h^c(\omega^c).$$  \hspace{1cm} (42)

Substituting for the terms $\mathcal{R}_h^m(\omega^m)$ and $\mathcal{R}_h^c(\omega^c)$ in (42) using (9), rearranging and assuming $a$ symmetric, we finally get

$$L(e, E) = a(e, \omega^m) + b(\omega^m, E) + b(e, \omega^c) \hspace{1cm} (43)$$

$$= a(\omega^m, e) + b(e, \omega^c) + b(\omega^m, E). \hspace{1cm} (44)$$

The influence functions can thus be obtained as solutions of the global dual problem:

Find $(\omega^m, \omega^c) \in V \times Q$, such that

$$a(\omega^m, v) + b(v, \omega^c) + b(\omega^m, q) = L(v, q), \forall (v, q) \in V \times Q \hspace{1cm} (45)$$

which is equivalent to the generalized Stokes problem:

Find $(\omega^m, \omega^c) \in V \times Q$, such that

$$a(\omega^m, v) + b(v, \omega^c) = L(v, 0), \forall v \in V \hspace{1cm} (46)$$

$$b(\omega^m, q) = L(0, q), \forall q \in Q$$

It readily follows that the functions $\omega^m$ and $\omega^c$ do exist and are unique as $L(v, 0) \in V'$ and $L(0, q) \in Q'$ (see Girault and Raviart [12]). Obviously, (46) cannot be solved exactly for the functions $\omega^m$ and $\omega^c$ in the general case. At best, one seeks numerical approximations of $\omega^m$ and $\omega^c$. This raises two questions, which should be simultaneously investigated:
1. How do we effectively calculate finite element approximations of $\omega^m$ and $\omega^c$? In particular, is it necessary to solve a global problem?

2. How do we utilize the relation (42) to derive elementwise refinement indicators in order to control the error quantity $L(e, E)$?

From a computational point of view, we observe that the cost involved in approximating $\omega^m$ and $\omega^c$ in the spaces $V^h$ and $Q^h$ is almost negligible, as the resulting finite system has already been factorized once to calculate the solution $(u_h, p_h)$. The cost therefore reduces to perform one backward and one forward substitutions. However, we recall that the residuals are identically zero on $V^h \times Q^h$ whenever the pair $(u_h, p_h)$ is the solution of the finite system (7). Therefore, if we approximate $\omega^m$ and $\omega^c$ by $\omega^m_h \in V^h$ and $\omega^c_h \in Q^h$, we obtain that

$$R^m_h(\omega^m_h) = 0 \quad \text{and} \quad R^c_h(\omega^c_h) = 0,$$

which implies

$$L(e, E) = R^m_h(\omega^m) + R^c_h(\omega^c) \approx R^m_h(\omega^m_h) + R^c_h(\omega^c_h) = 0.$$

In this case, the quantity $L(e, E)$ would be estimated by a quantity which proves to be 0. This reveals that $\omega^m$ and $\omega^c$ should be approximated in spaces of greater dimensions than $V^h$ and $Q^h$.

One strategy, proposed by Becker and Rannacher [10], consists in using (47) such that

$$L(e, E) = R^m_h(\omega^m) + R^c_h(\omega^c) - R^m_h(\omega^m_h) - R^c_h(\omega^c_h),$$

Then, an upper bound can be derived as:

$$|L(e, E)| = |R^m_h(\omega^m - \omega^m_h) + R^c_h(\omega^c - \omega^c_h)| = |a(\varphi, \omega^m - \omega^m_h) - b(u_h, \omega^c - \omega^c_h)|$$

$$= \left| \sum_K a_K(\varphi, \omega^m - \omega^m_h) - \sum_K b_K(\omega^c - \omega^c_h) \right|$$

$$\leq \sum_K |a_K(\varphi, \omega^m - \omega^m_h)| + \sum_K |b_K(\omega^c - \omega^c_h)|$$

$$\leq \sum_K \left( |\varphi|_{1,K} |\omega^m - \omega^m_h|_{1,K} + ||\nabla \cdot u_h||_{0,K} ||\omega^c - \omega^c_h||_{0,K} \right)$$

$$\leq \sum_K \sqrt{|\varphi|^2_{1,K} + ||\nabla \cdot u_h||^2_{0,K}} \sqrt{|\omega^m - \omega^m_h|^2_{1,K} + ||\omega^c - \omega^c_h||^2_{0,K}},$$

where $\varphi$ is the function in $V$ satisfying (20). Becker and Rannacher [10] use (48) to estimate the quantities $L(e, E)$ using local interpolation properties to evaluate the error quantity $|\omega^m - \omega^m_h|_{1,K}$ and $||\omega^c - \omega^c_h||_{0,K}$. They thus introduce an interpolation constant they arbitrarily choose to be 1. Here, we could estimate $L(e, E)$ by introducing the global
error estimates developed in section 3 into (49). However, those estimates are not local and do not reflect the error propagation in $\omega_h^m$ and $\omega_h^e$. In general, such a procedure overestimates $|L(e, E)|$ by several orders of magnitude. If one is interested in accurate values of $L(e, E)$, one has to consider another approach.

The strategy we propose is to solve a global problem defined in some finite element spaces $\overline{V}^h$ and $\overline{Q}^h$ satisfying

$V^h \subset \overline{V}^h \subset V, \quad Q^h \subset \overline{Q}^h \subset Q$.

Obviously it is more expensive to solve this global problem than the solution problem in $V^h \times Q^h$. In order to optimize the selection of $\overline{V}^h$ and $\overline{Q}^h$, we propose to construct them by adapting $V^h$ and $Q^h$ according to a combination of global error estimates on $(\omega_h^m, \omega_h^e)$ as well as error estimates on $L(e, E)$ using (49). Thus, we can utilize the global error estimates (very cheap) developed in the previous sections. Let $(\tilde{\omega}_h^m, \tilde{\omega}_h^e)$ denote the solution in $\overline{V}^h \times \overline{Q}^h$. Then we have

\begin{equation}
L(e, E) \approx R_h^m(\tilde{\omega}_h^m) + R_h^e(\tilde{\omega}_h^e)
\end{equation}

\begin{equation}
\leq \sum_K \sqrt{\|\nabla \cdot u_h\|^2_{0,K} + \|\nabla \cdot u_h\|^2_{0,K}} + \sqrt{\|\omega_h^m - \omega_h^m\|^2_{1,K} + \|\omega_h^e - \omega_h^e\|^2_{0,K}}.
\end{equation}

Therefore, the quantity $L(e, E)$ is estimated using (50) and controlled by adapting the mesh according to the elementwise indicators produced by (51).

We now provide some examples for which we would be interested in calculating the quantity $L(e, E) \in H$. The error measure $L(e, E)$ obviously represents the numerical error we do in computing $L(u_h, p_h)$ instead of $L(u, p)$. Indeed,

$L(e, E) = L(u, p) - L(u_h, p_h)$.

Linear quantities of potential interest in the variable $(u, p)$ are pointwise values of the velocity component $u$, $L(u, p) = u(x_0)$, $x_0 \in \Omega$, volume flow rates through a surface $\Gamma$, $L(u, p) = \int_{\Gamma} u \cdot n \, d\Gamma$, where $n$ is the unit normal to $\Gamma$, or directional derivatives of the velocity averaged over one element $\Omega_K$, $L(u, p) = \int_{\Omega_K} \nabla u \cdot l \, dx$, where $l$ is a given unit vector. In the case one is interested in nonlinear quantities $N(u, p) \in H$, one performs the expansion:

$N(u, p) = N(u_h, p_h) + N'(u_h, p_h) \cdot (u - u_h, p - p_h) + \ldots$

so that the error quantity $N(u, p) - N(u_h, p_h)$ may be approximated by

$N(u, p) - N(u_h, p_h) \approx N'(u_h, p_h) \cdot (e, E)$.

Then, we denote the linear quantity $N'(u_h, p_h) \cdot (e, E)$ by $L(e, E)$ and all the analysis described above applies. For example, let us suppose that one is interested in the kinetic energy $K_e$ in some subregion $\Omega_K$ of $\Omega$. We employ

$N(u, p) = K_e(u) = \frac{1}{2} \int_{\Omega_K} u^2 \, dx.$

It results that the error in the kinetic energy may be approximated by

$K_e(u) - K_e(u_h) \approx L(e, E) = \int_{\Omega_K} u_h \cdot e \, dx.$
6. NUMERICAL EXPERIMENTS

6.1. Global error estimation

In the first example, the Stokes problem is solved on the unit square $\Omega = [0, 1] \times [0, 1]$ with the data $f$ chosen such that the exact solution is given by the velocity profile $u = (u, v)$ shown in Figure 2 and by the pressure $p = 0$. We consider uniform meshes formed of squared elements with size $h \times h$, $h = 1/2, 1/4, 1/8, 1/16, 1/32$. The polynomial degree $p$ of the basis functions in $V^h$ is uniformly set to 2 or 3.

![Figure 1. Exact velocity component $u$.](image1)

![Figure 2. Exact velocity component $v$.](image2)

We consider the global error estimator $\eta$ defined as:

$$\eta^2 = \|R^u_h\|^2 + \|R^p_h\|^2 = |\phi|^2 + \|\nabla \cdot u_h\|^2$$

where $\phi \in V$ is approximated by a function $\phi_h \in \tilde{V}^h$ or a function $\psi_h \in W^h$ satisfying (22) or (30). We characterize the extra degree (with respect to $p$) of the perturbations in $W^h$ by the pair $q = (q_{\text{edge}}, q_{\text{int}})$, $q_{\text{edge}}$ being the extra degree for the edge bubbles and $q_{\text{int}}$ the extra degree for the interior bubbles. We recall that the cost of this error estimator is controlled by the cost in solving (22) or (30), that is, by the size of $W^h$ and the number of iterations performed using the Conjugate Gradient method. What is sought is a trade-off between accuracy in the approximations $\psi_h$ and cost (time) spent to solve for them. Values of approximations of $\phi$ and corresponding times are given in Table 1 and Table 2 for various values of $h$, $p$ and $q$. The approximations are normalized with respect to overkills using $q = (2, 2)$ while the times are normalized with respect to the times spent to solve for the solutions $(u_h, p_h)$. The number of iterations in the CG algorithm are determined according to a preset tolerance in solution accuracy. We first observe that the performances greatly increases as the size of the problem increases. For example, for $h = 1/16$, $p = 2$, the residual is estimated within a 1% accuracy while spending less than 7% of the solution time. We also observe that it is approximated with
Table 1
Normalized values of approximations of $\|\mathcal{R}_h\|$.  

<table>
<thead>
<tr>
<th>$h$</th>
<th>$p$</th>
<th>$q = (0,1)$</th>
<th>$q = (1,0)$</th>
<th>$q = (1,1)$</th>
<th>$q = (1,1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>2</td>
<td>0.7620</td>
<td>0.5235</td>
<td>0.8520</td>
<td>0.8571</td>
</tr>
<tr>
<td>1/4</td>
<td></td>
<td>0.7300</td>
<td>0.8282</td>
<td>0.8983</td>
<td>0.9040</td>
</tr>
<tr>
<td>1/8</td>
<td></td>
<td>0.8154</td>
<td>0.9684</td>
<td>0.9797</td>
<td>0.9805</td>
</tr>
<tr>
<td>1/16</td>
<td></td>
<td>0.8184</td>
<td>0.9941</td>
<td>0.9947</td>
<td>0.9948</td>
</tr>
<tr>
<td>1/4</td>
<td>3</td>
<td>0.8406</td>
<td>0.8952</td>
<td>0.9679</td>
<td>0.9693</td>
</tr>
<tr>
<td>1/8</td>
<td></td>
<td>0.8815</td>
<td>0.9781</td>
<td>0.9895</td>
<td>0.9896</td>
</tr>
</tbody>
</table>

Table 2
Normalized times and number of iterations () performed using the CG method.  

<table>
<thead>
<tr>
<th>$h$</th>
<th>$p$</th>
<th>$q = (0,1)$</th>
<th>$q = (1,0)$</th>
<th>$q = (1,1)$</th>
<th>$q = (1,1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>2</td>
<td>0.169 (2)</td>
<td>0.196 (2)</td>
<td>0.695 (6)</td>
<td>3.454 (8)</td>
</tr>
<tr>
<td>1/4</td>
<td></td>
<td>0.065 (2)</td>
<td>0.117 (4)</td>
<td>0.443 (7)</td>
<td>2.925 (9)</td>
</tr>
<tr>
<td>1/8</td>
<td></td>
<td>0.015 (2)</td>
<td>0.042 (4)</td>
<td>0.241 (8)</td>
<td>1.309 (9)</td>
</tr>
<tr>
<td>1/16</td>
<td></td>
<td>0.006 (2)</td>
<td>0.018 (3)</td>
<td>0.069 (5)</td>
<td>0.474 (8)</td>
</tr>
<tr>
<td>1/4</td>
<td>3</td>
<td>0.071 (2)</td>
<td>0.013 (3)</td>
<td>0.123 (9)</td>
<td>0.921 (9)</td>
</tr>
<tr>
<td>1/8</td>
<td></td>
<td>0.004 (2)</td>
<td>0.004 (4)</td>
<td>0.088 (9)</td>
<td>0.339 (9)</td>
</tr>
</tbody>
</table>

less than one percent of error using $W^h$ instead of $\bar{V}^h$ with $q = (1,1)$. Moreover the number of iterations is kept very small while requiring reasonable accuracy of the order $10^{-2}$. We also compute effectivity indices, defined as:

$$\gamma_u = \frac{\eta}{\|(e, E)\|}$$

as well as elementwise effectivity indices. Results are collected in Table 3 for the case $\psi_h \in W^h$, $q = (1,1)$. We also show in Figure 3 and Figure 4 the elementwise distribution of the exact errors and of the effectivity indices for $h = 1/8$ and $p = 2$. The global effectivity indices are excellent as they are all close to unity. On the other hand, the local effectivity indices, as it is expected for the case of a global error estimator, may behave poorly, and in some cases drop to a value as low as 0.012. However, these are always near unity in elements with large error, and diverge from unity only in elements with rather small error.

6.2. Backward facing step Stokes problem
The second example is the classical backward facing step Stokes problem. The problem is slightly different than in Bank and Welfert [8], as we consider a homogeneous Neumann
Table 3
Global and elementwise effectivity indices.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$p$</th>
<th>$\gamma^{u,p}$</th>
<th>$\min\gamma_{K}^{u,p}$</th>
<th>$\max\gamma_{K}^{u,p}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>2</td>
<td>0.760</td>
<td>0.717</td>
<td>0.921</td>
</tr>
<tr>
<td>1/4</td>
<td>2</td>
<td>0.927</td>
<td>0.274</td>
<td>1.103</td>
</tr>
<tr>
<td>1/8</td>
<td>2</td>
<td>1.038</td>
<td>0.069</td>
<td>1.321</td>
</tr>
<tr>
<td>1/16</td>
<td>2</td>
<td>1.066</td>
<td>0.033</td>
<td>1.388</td>
</tr>
<tr>
<td>1/32</td>
<td>2</td>
<td>1.073</td>
<td>0.012</td>
<td>1.408</td>
</tr>
<tr>
<td>1/4</td>
<td>3</td>
<td>1.028</td>
<td>0.519</td>
<td>1.051</td>
</tr>
<tr>
<td>1/8</td>
<td>3</td>
<td>1.084</td>
<td>0.167</td>
<td>1.339</td>
</tr>
<tr>
<td>1/16</td>
<td>3</td>
<td>1.096</td>
<td>0.025</td>
<td>1.381</td>
</tr>
</tbody>
</table>

Figure 3. Elementwise distribution of the errors.

Figure 4. Elementwise distribution of the effectivity indices.

boundary condition downstream of the domain. The solution develops a discontinuity in the pressure variable due to the discontinuity in the geometry of the boundary. The norm of the residual $\mathcal{R}_h^n$ is then expected to be very large in the near region of the reentrant corner and small in the farfield, as illustrated in Figure 5, where the solution has been computed on a pre-adapted mesh of 120 elements.

We are then motivated to construct $W^h$ in an adaptive manner in order to minimize its dimension, that is, the number of degrees of freedom associated to problem (30). As a first guess, the residual is approximated by $\psi_h \in W^h$ with $q = (1,0)$. Then the space is successively enriched with interior and edge bubble functions of higher degrees only in the elements for which the contribution to $|\psi_h|_1$ is large. Results are shown in Table 4 where the approximations of $||\mathcal{R}_h^n||_1$, computed either in the complete space $W^h_{\text{cmp}}$, with uniform $q = (q_{\text{edge}}, q_{\text{int}})$, or in the adapted space $W^h_{\text{adp}}$ with $q$ varying between $(0,0)$
Figure 5. Elementwise distribution of the residual $\|R_h^A\|$.

to $(q_{\text{edge}}, q_{\text{int}})$ depending on the element, are compared. It is clear that the number of degrees of freedom, and a fortiori the cost of the error estimator, is greatly reduced for comparable accuracy. The global effectivity index for this problem has been computed as $\gamma^p = 0.884$ and $0.915$ for $p = 2$ and $3$ respectively, noting that the exact solutions are actually approximated by overkills of degree $p + 3$.

Table 4

Approximations of $\|R_h^A\|$ in complete or adapted spaces of perturbations with corresponding numbers of degrees of freedom (1).

<table>
<thead>
<tr>
<th>$q = (1, 0)$</th>
<th>$q = (1, 1)$</th>
<th>$q = (2, 1)$</th>
<th>$q = (2, 2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>$W_h^{\text{cmp}}$</td>
<td>$W_h^{\text{adp}}$</td>
<td>$W_h^{\text{cmp}}$</td>
</tr>
<tr>
<td>2</td>
<td>0.1078</td>
<td>0.1247</td>
<td>0.1275</td>
</tr>
<tr>
<td></td>
<td>(530)</td>
<td>(650)</td>
<td>(1250)</td>
</tr>
<tr>
<td>3</td>
<td>0.0291</td>
<td>0.0527</td>
<td>0.0531</td>
</tr>
<tr>
<td></td>
<td>(530)</td>
<td>(650)</td>
<td>(1730)</td>
</tr>
</tbody>
</table>

In Figures 6 and 7, we respectively show an adapted mesh and the corresponding solution $(u_h,p_h)$. Such a mesh has been automatically adapted from a given coarse mesh in order to achieve a preset tolerance of 1.2% in the global relative error.

6.3. Error estimation in the quantity of interest

The following results constitute a preliminary study of error estimation in local quantities of interest. We consider the test case described in section 6.1 and suppose we are interested in evaluating the quantity

$$L(u,p) = \frac{1}{|\Omega_K|} \int_{\Omega_K} \frac{\partial u}{\partial y} \, dx \in \mathbb{R}$$

where $|\Omega_K|$ represents the area of a given subdomain $\Omega_K \subset \Omega$. Then, the error in the computed quantity $L(u_h,p_h)$, where $(u_h,p_h)$ is the finite element solution in $V^h \times Q^h$, is
Figure 6. Example of an adapted mesh (201 elements).

given by \( L(e, E) \). We consider a uniform mesh with \( h = 1/8 \) and \( p = 2 \) and choose the subdomain \( \Omega_K \) to be either element 1 located in the lower left corner of the domain, or element 55 the next to last element in the upper right corner. Then, \( L(e, E) \) is estimated by three quantities in \( \mathbb{R} \) denoted \( \eta_1, \eta_2 \) and \( \eta_3 \). The first one, \( \eta_1 \), is obtained as:

\[
\eta_1 = \mathcal{R}_h^m(\bar{\omega}_h^m) + \mathcal{R}_h(\bar{\omega}_h)
\]

where the discrete influence functions \( \bar{\omega}_h^m \) and \( \bar{\omega}_h \) belong to the spaces \( \bar{\mathbb{V}}_h \) and \( \bar{\mathbb{Q}}_h \) defined using basis functions up to the degree \( p + 1 \) and are solutions of the generalized Stokes problem (46). The second one, \( \eta_2 \), is calculated as

\[
\eta_2 = \sum_K \sqrt{\| \psi_h \|_{1,K}^2 + \| \nabla \cdot u_h \|_{0,K}^2} \sqrt{\| \lambda^m - \lambda_h^m \|_{1,K}^2 + \| \omega^c - \omega_h^c \|_{0,K}^2}
\]

where \( \| \lambda^m - \lambda_h^m \|_{1,K} \) and \( \| \omega^c - \omega_h^c \|_{0,K} \) are approximated using the global error estimator. We also note that \( \psi_h \in \mathbb{W}_h, q = (1,1) \), is an approximation of \( \varphi \in \mathbb{V} \), that is, of the residual \( \mathcal{R}_h^m \). Finally the last approximation, \( \eta_3 \), is computed as

\[
\eta_3 = \sum_K \sqrt{\| \psi_h \|_{1,K}^2 + \| \nabla \cdot u_h \|_{0,K}^2} \sqrt{\| \bar{\omega}_h^m - \lambda_h^m \|_{1,K}^2 + \| \bar{\omega}_h^c - \omega_h^c \|_{0,K}^2}
\]

where \( \bar{\omega}_h^m \) and \( \bar{\omega}_h^c \) have been defined earlier. The results are displayed in Table 5. The elementwise distribution of \( \eta_2 \) and \( \eta_3 \) are shown in Figures 8-9 and Figures 10-11 for element 1 and element 55 respectively.

Table 5
Estimates \( \eta_1, \eta_2 \) and \( \eta_3 \) of the error quantity \( L(e, E) \) for elements 1 and 55.

<table>
<thead>
<tr>
<th>Element</th>
<th>( L(u, p) )</th>
<th>( L(e, E) )</th>
<th>( \eta_1 )</th>
<th>( \eta_2 )</th>
<th>( \eta_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.00364</td>
<td>0.00088</td>
<td>0.000081</td>
<td>0.003304</td>
<td>0.015205</td>
</tr>
<tr>
<td>55</td>
<td>1.6699</td>
<td>0.0258</td>
<td>0.0271</td>
<td>0.3733</td>
<td>0.4783</td>
</tr>
</tbody>
</table>
Figure 7. Velocity and pressure on adapted mesh with relative estimated error of less than 1.2%.

The conclusions of this preliminary study are twofold. First, we observe that only the quantity $\eta_1$ represents an accurate approximation of $L(e, E)$. The other two quantities $\eta_2$ and $\eta_3$ provide very pessimistic estimates as expected, but provide local refinement indicators as they reflect the intensity of the elementwise contribution to the error quantity $L(e, E)$. Secondly, we remark that the distribution $\eta_2$, relative to the case involving element 1 (see Figures 8 and 9) is very different from the one of $\eta_3$. This can be explained by the fact that the quantities $|\omega^m - \omega_h^m|_{1,K}$ and $|\omega^s - \omega_h^s|_{0,K}$ are approximated using a global error estimator, which does not take in account the propagation of the errors away from their sources. On the other hand, the discrepancy is less noticeable for element 55, because the region of large numerical errors in the influence functions superpose with the regions of large sources of errors $\mathcal{R}_h^m$ and $\mathcal{R}_h^s$.

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REFERENCES


Figure 10. Elementwise contributions of $\eta_2$ for element 55.

Figure 11. Elementwise contributions of $\eta_3$ for element 55.


