Locking and boundary layer in hierarchical models for thin elastic structures

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Abstract

In the analysis of plate- and shell-like structures using the finite element method, we may encounter difficulties owing to locking and boundary layer effects, which lead to numerical simulations with a large error. There have been considerable advances in finding out the causes and remedies of locking. However, a posteriori detection of this phenomenon in the numerical results without comparing with other reference data is another issue with a great importance. Here, one simple but reliable detection method is introduced. On the other hand, boundary layer is a non-smooth singular part in the solution of the singularly perturbed boundary value problems and restricted within very thin region in the neighborhood of boundaries. If this rapidly varying behavior could not have been captured enough, approximated solution greatly deviates from the exact solution. In this study, a guideline for the optimal mesh design is provided together with mathematical analysis of this effect.

1. Introduction

In the numerical analysis of thin elastic structures such as beam-, arch-, plate- and shell-like bodies using standard finite element schemes, there may occur approximation quality deterioration owing to locking and boundary layer effects. It is widely known that locking is caused by the prevalence of two well-known sets of constraints, shear and membrane constraints. In the poor finite element spaces (low order and coarse meshes), these constraints produce almost null spaces of approximated numerical values.

There have been numerous publications on this topic making considerable advances in overcoming this disaster, notwithstanding further study is still in need. Recently, Pitkäranta [13] reported that locking occurs only in bending-dominated structures, and Suri et al. [22] found that, using hierarchical plate models, only low-order polynomials in the thickness direction of structures cause locking. That is, higher-order theories have the same characteristics of locking as the Reissner–Mindlin theory.

Even though advances have been made on locking, a posteriori locking detection, another significant matter was never mentioned. In practice, it is not so easy for the analyst to check whether the computed numerical results are influenced by locking, and if so, how much the influence is, without comparing with other available reference numerical data. This paper proposes one cheap but reliable method for a posteriori estimation of

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locking based on the recent study mentioned before, which requires only a simple post computation and does not need any comparison with reference data.

As the thickness decreases, governing equation for such structures becomes a singularly perturbed elliptic boundary value problem with respect to the thickness. Accordingly, in narrow thin regions in the neighborhood of boundaries of thin structures, the solution exhibits singular behavior, which exponentially decays, called boundary layer, in the normal direction to the boundary. If this singular behavior is not described enough, then the finite element analysis leads to poor numerical simulations.

Recently, Arnold and Falk [3] mathematically analyzed this problem, and Schwab and Suri [20] reported a quantitative guide for constructing optimal meshes for hp-version finite element methods. In this paper, a theoretical study on this phenomenon is presented together with numerical results showing that size of the first element from the boundary has a significant role to capture this singularity.

2. Hierarchical models using higher-order theories

Let us consider a plate-like structure \( \Omega = \omega \times [-d/2, d/2] \) shown in Fig. 1, where mid-surface, denoted by \( \omega \in \mathbb{R}^2 \), is bounded by a piecewise smooth boundary \( \partial \omega \), and \( d \) is the thickness. Owing to the small thickness compared to the other dimensions in \( x_1 \) and \( x_2 \) directions, variations in displacement fields through the thickness may be approximated using low order polynomials (e.g. the Kirchhoff and the Reissner-Mindlin theories).

However, for the regions of complex state of stresses, these low order theories are insufficient to describe the behavior of bodies. Instead, we should use higher-order theories, or even full three-dimensional linear elasticity theory. In early 1990, new modeling concept called the hierarchical modeling was introduced by Szabó and Babuška et al. [5,24], which can provide suitable models to meet the desired analysis accuracy. Recently, adaptive selection of suitable models was developed by Oden and Cho [7,9,12]. By a hierarchical model in linear elasticity is meant a family of higher-order theories, which are parametrically connected by an order of thickness polynomials in displacement fields. Displacement components \( \mathbf{u}^q \) in higher-order theories are expressed as (see Reddy [14–16] and Reddy and Robbins [17,18]).

\[
\mathbf{u}^q_i(x_1, x_2) = \sum_{l=0}^{q_i} U^q_i(x_1, x_2) L_l(x_3), \quad i = 1, 2, 3
\]

where \( q_i \) is the degree of thickness polynomials \( L_l \) of \( u_i \) (often, we take \( q_1 = q_2 \)). Therefore, the choice of a set of \( q_i \) produces a typical model level \( \mathbf{q} = (q_1, q_2, q_3) \).

We will formulate a hierarchical model starting from the three-dimensional linear elasticity theory.
\[
\begin{align*}
\sigma_i(u_i) + f_i &= 0 \quad \text{in } \omega \times [-d/2, d/2] \\
u_i &= 0 \quad \text{on } \partial \omega_b \times [-d/2, d/2], \quad 1 \leq i, j \leq 3 \\
\sigma_j(u_j) &= g_j \quad \text{on } \Gamma^s
\end{align*}
\]

where \(f_i \in L^2(\Omega)\) are body force components and \(\partial \omega_b \times [-d/2, d/2]\) is Dirichlet boundary regions which are usually lateral sides of structures. Furthermore, traction components \(g_i \in L^2(\Gamma^s)\) are usually applied on the top and bottom surfaces \(\partial \omega^t = \{x \in \mathbb{R}^3 | (x_1, x_2) \in \omega, x_3 = \pm d/2\}\), Neumann boundary \(\Gamma^t\) can be confined to the top and bottom surfaces.

For the variational formulation for the three-dimensional linear elasticity problem (2), admissible displacement space \(V(\Omega)\) for finite strain energy is

\[
V(\Omega) = \{v \in [H^1(\omega \times [-d/2, d/2])]^3 : v = 0 \quad \text{on } \partial \omega_b \times [-d/2, d/2]\}
\]

In order for \(u^0\) in Eq. (1) to belong to the space \(V(\Omega)\), the following condition is sufficient.

\[
U_i'(x_1, x_2) \in H^1(\omega), \quad U_i' = 0 \quad \text{on } \partial \omega_b, \quad 0 \leq \ell \leq q_i
\]

Further mathematical detail can be found in [12]. Now we define the admissible displacement space \(V^q(\Omega) \subset V(\Omega)\) for the hierarchical models:

\[
V^q(\Omega) = \left\{ \begin{array}{l}
V_i^q = \sum_{\ell=0}^q V_i'(x_1, x_2)L_\ell(x_3) | V_i'(x_1, x_2) \in H^1(\omega) \\
V_i' = 0 \quad \text{on } \partial \omega_b, \quad i = 1, 2, 3
\end{array} \right\}
\]

Here, we comment that although it does not make sense to specify \(V_i' = 0\) along the line boundary \(\partial \omega_b\), but \(V_i' = 0 \quad \text{on } \partial \omega_b \times [-d/2, d/2]\) acceptable in \(H^1(\Omega)\) spaces. Defining the space \(V^q(\Omega)\), we have the following weak formulation

\[
\text{Find } u^q \in V^q(\Omega) \quad \text{such that } \forall v^q \in V^q(\Omega)
\]

\[
a(u^q, v^q) = \ell(v^q)
\]

Where, the bilinear functional \(a : V^q(\Omega) \times V^q(\Omega) \rightarrow \mathbb{R}\) and the linear functional \(\ell : V^q(\Omega) \rightarrow \mathbb{R}\) are expressed by the following dimensionally-reduced forms, in which integration through the thickness is done analytically:

\[
\begin{align*}
a(u^q, v^q) &= \int_\omega \left[ \int_{-d/2}^{d/2} e(v^q) : \sigma(u^q) \, dx_3 \right] \, d\omega \\
\ell(v^q) &= \int_\omega \left[ \int_{-d/2}^{d/2} v^q \cdot f \, dx_3 \right] \, d\omega + \int_{\partial \Omega_+} v^q_n(x_1, x_2, -d/2) g^- \, d\omega + \int_{\partial \Omega_-} v^q_n(x_1, x_2, +d/2) g^+ \, d\omega
\end{align*}
\]

Here, symbols (+, -) represent top and bottom surfaces \(\partial \Omega^+\) and \(\partial \Omega^-\), respectively.

The solution \(u^q\) of (6) is defined as the energy projection of the three-dimensional elasticity theory \(u\) onto the subspace \(V^q(\Omega)\) (i.e. \(a(u - u^q, v^q) = 0, \forall v^q \in V^q(\Omega)\)). We define the \((q_1, q_2, q_3)\) hierarchical model as a member of set consisting of parametrically connected approximate solutions \((u_1^q, u_2^q, u_3^q)\) with model level \(q = (q_1, q_2, q_3)\)

Among the characteristics of hierarchical models [23], for the reader, we record the results of limit analysis \((d \rightarrow 0)\) of hierarchical models. As the thickness tends to zero, every hierarchical model except for the \((1, 1, 0)\) model approaches the lowest model, the Kirchhoff theory. According to the decrease in the thickness, structures become plane stress problem, while the \((1, 1, 0)\) model obtained using \([E]\), three-dimensional elasticity moduli, describes plane strain problem. This contradiction leads to a different limit for the \((1, 1, 0)\) model. However, if we replace material moduli with \([E_{RM}]\) used for the Reissner–Mindlin theory, the \((1, 1, 0)\) model approaches correct limit. We will denote the \((1, 1, 0)\) model obtained using the modified material moduli \([E_{RM}]\) as the \((1, 1, 0)^*\) model hereafter.
Here, \( p = E/2(1 + \nu) \) and \( K \) is shear correction factor (5/6 \( \leq \kappa \leq 1 \), we use \( \kappa = 1 \)).

Next, we briefly describe finite element approximations of hierarchical models with level \( q = (q_1, q_2, q_3) \). A restriction of \( v^q \in V^q(\Omega) \) to \( \Omega_k = \omega_k \times [-d/2, d/2] \) is denoted by

\[
v^q|_{\Omega_k} = v^q_k \in V^q(\Omega_k)
\]

Then, the local finite element approximation \( v^q_{k,h} = v^q_k|_{\Omega_k} \) belongs to the product space of two spaces; \( V^{q,h}(\Omega_k) = P(\omega_k) \otimes \mathcal{D}_k(-d/2, d/2) \), in which \( P(\omega_k) \) is the two-dimensional polynomial space over \( \omega_k \) with approximate degree \( p_k \) (in this work, \( p_\alpha \) for each variable \( x_\alpha \approx p_\alpha, \alpha = 1, 2 \), and \( 2_\kappa(-d/2, d/2) \) is the thickness polynomial space of degree \( q_\kappa \).

In order that the global finite element approximation \( v^{q,h} \in V^{q,h}(\Omega) \) possess finite strain energies, the global finite element approximation must be continuous at the interelement boundaries (i.e. \( v^{q,h} \in (\mathcal{V}^{q,h}(\Omega))^{3} \)). Then the finite element approximation of (6) is

\[
\text{Find } u^{q,h} \in V^{q,h}(\Omega) \text{ such that } V u^{q,h} \in V^{q,h}(\Omega)
\]

\[
a(u^{q,h}, v^{q,h}) = \ell(v^{q,h})
\]

3. Locking phenomenon

3.1. Locking analysis using hierarchical models

In this section, we briefly review shear and membrane lockings using hierarchical models for cylindrical shell-like structures. We will see that locking occurs only for bending-dominated structures and it is related to only low order model levels \( (q \leq 1) \) implying that higher-order models exhibit the same features as the lower-order models such as the Reissner–Mindlin theory and the Koiter shell theory. Based on these results, we will introduce a cheap but reliable method for a posteriori detection in the next section.

Here, for our purpose, we consider a cylindrical shell of radius \( R \) with uniform normal tractions \( g_N \) applied equally on the top and bottom surfaces. A polar coordinate system is used, in which \( x_1 \) directs the axial direction while \( x_2 \) and \( x_3 \) direct the circumferential and radial directions, respectively. In order to analyze the numerical locking phenomenon, let us define membrane, bending and transverse shear strains by \( \varepsilon^m, \varepsilon^b \) and \( \varepsilon^t \).

Then, for the \( (q_1, q_2, q_3) \) hierarchical model, these strains in polar coordinate system are as follows \( (\alpha, \beta = 1, 2 \text{ and } k = 1, 2, 3 \text{ and } L_\ell \text{ are assumed to be monomials } (x_3)') \).

\[
(i) \text{ membrane strains: } \{\beta^{\ell}_{a\beta}, \beta^{\ell}_{33}\}
\]

\[
\varepsilon^m_{a\beta} = \frac{1}{2} (u^q_{a,\beta} + u^q_{\beta,a})_{\text{even}} + \delta_{a2} \delta_{b2} \left( \frac{u^q_{3}}{R} \right)_{\text{even}}
\]

\[
= \frac{1}{2} \left( \sum_{\ell = 0, 2, \ldots}^{q_3} U^\ell_{a,\beta} + \sum_{\ell = 0, 2, \ldots}^{q_3} U^\ell_{\beta,a} \right) L_\ell + \delta_{a2} \delta_{b2} \sum_{\ell = 0, 2, \ldots}^{q_3} \frac{U^\ell_{3}}{R} L_\ell
\]
(ii) bending strains: \( \{ \kappa_{\alpha \beta}^r, \kappa_{33}^r \} \)

\[
\bar{e}_{\alpha \beta}^o = \frac{1}{2} \left( \frac{u_{\alpha \beta}^o + u_{\beta \alpha}^o}{2} \right)_{\text{odd}} + \frac{u_{\alpha \beta}^o}{R} \left( \frac{u_{\alpha \beta}^o}{R} \right)_{\text{odd}}
\]

\[
= \frac{1}{2} \left( \sum_{r=1,3, \ldots}^{q_r} U_{\alpha \beta}^r L_r + \sum_{r=1,3, \ldots}^{q_r} U_{\beta \alpha}^r L_r \right) + \delta_{\alpha \beta} \sum_{r=1,3, \ldots}^{q_r} \frac{U_r^2}{R} L_r
\]

\[
= \sum_{r=1,3, \ldots}^{q_r} \kappa_{\alpha \beta}^r L_r
\]

\[
\bar{e}_{33} = \sum_{r=1,3, \ldots}^{q_r} \kappa_{33}^r L_r , \quad \kappa_{33}^r = U_{33}^{r+1} \text{coef}(L_{r+1})
\]

(iii) transverse shear strains: \( \{ \rho_a^r \} \)

\[
\bar{e}_{3a}^o = \frac{1}{2} \left( \frac{u_{3a}^o + u_{a3}^o}{2} \right) + \delta_{2a} \frac{u_{3a}^o}{R}
\]

\[
= \frac{1}{2} \left( \sum_{r=0,1,2, \ldots}^{q_r} U_{3a}^r L_r' + \sum_{r=0,1,2, \ldots}^{q_r} U_{a3}^r L_r \right) + \delta_{2a} \sum_{r=0,1,2, \ldots}^{q_r} \frac{U_r^2}{R} L_r
\]

\[
= \sum_{r=0,2, \ldots}^{q_r} \rho_a^r L_r , \quad \rho_a^o = U_{3a}^{r+1} \text{coef}(L_{r+1}) + U_{a3}^{r+1} + \delta_{2a} \frac{U_r^2}{R}
\]

Scripts even and odd mean, respectively, terms corresponding to even and odd thickness polynomials. Neglecting CT, we express total strain energy \( U(\nu) \):

\[
U = U_m + U_b + U_s
\]

\[
= \sum_{r=0,2, \ldots}^{q_r} \sum_{i=0,2, \ldots}^{7} \left( \frac{1}{i+j+1} \right) \left( \frac{d}{2} \right)^{i+j+1} \int_\omega \left\{ 2 \mu \kappa_{\alpha \beta}^r \kappa_{\alpha \beta}^r + \lambda \beta_{\alpha \beta}^r \beta_{\alpha \beta}^r \right\} d\omega
\]

\[
+ \sum_{r=1,3, \ldots}^{q_r} \sum_{i=1,3, \ldots}^{7} \left( \frac{1}{i+j+1} \right) \left( \frac{d}{2} \right)^{i+j+1} \int_\omega \left\{ 2 \mu \kappa_{\alpha \beta}^r \kappa_{\alpha \beta}^r + \lambda \beta_{\alpha \beta}^r \beta_{\alpha \beta}^r \right\} d\omega
\]

\[
+ \sum_{r=0,1, \ldots}^{q_r} \sum_{i=0,1, \ldots}^{7} \left( \frac{1}{i+j+1} \right) \left( \frac{d}{2} \right)^{i+j+1} \int_\omega 2 \mu \beta_{\alpha \beta}^r \beta_{\alpha \beta}^r d\omega, \quad i+j = \text{even}
\]

Scaling the external traction \( g_n \) by a factor of \( d^3 / 12 \), we have the following variational formulation in the form of thickness order:

Find \( u^o \in V^0(\Omega) \) such that \( \forall \ u^o \in V^0(\Omega) \)

\[
\mathcal{A}(u^o, v^o) + \frac{1}{d^3} \mathcal{B}(u^o, v^o) + \text{H.O.T.} = \ell(v^o)
\]

(15)

Here,

\[
\mathcal{A}(u^o, v^o) = \int_\omega \left\{ 2 \mu \kappa_{\alpha \beta}^o (u^o) \kappa_{\alpha \beta}^o (v^o) + \lambda \kappa_{\alpha \beta}^o (u^o) \kappa_{\alpha \beta}^o (v^o) \right\} d\omega
\]

\[
\mathcal{B}(u^o, v^o) = 12 \int_\omega \left\{ 2 \mu \beta_{\alpha \beta}^o (u^o) \beta_{\alpha \beta}^o (v^o) + \lambda \beta_{\alpha \beta}^o (u^o) \beta_{\alpha \beta}^o (v^o) \right\} d\omega + 12 \int_\omega 2 \mu \beta_{\alpha \beta}^o (u^o) \beta_{\alpha \beta}^o (v^o) d\omega
\]

\[
\ell(v^o) = \sum_{r=0,2,4, \ldots}^{q_r} \left( \frac{d}{2} \right)^{r-3} \int_\omega g_n V_r^o d\omega
\]
and

$$H.O.T. = A^{H.O.T.} (\sim \mathcal{O}(d^2)) + B^{H.O.T.} (\sim \mathcal{O}(1))$$

(17)

As the thickness $d$ tends to zero, $A(u^a, v^a)$ should be zero in order to have finite strain energies (i.e. $d^{-2} B(u^a, v^a) < +\infty$) for finite external work $\ell(v^a)$. As a result, two types of constraints prevail in order that $A(u^a, v^a)$ vanishes for all $v^a$:

- **Shear constraints**

$$\rho^0_x = 0 ; \quad U^1_x + \delta_{xx} \frac{U^0_x}{R} = -U^0_x$$

(18)

- **Membrane constraints**

$$\beta_{\alpha\beta}^0 = 0 , \quad (U^0_{,\alpha} + U^0_{,\beta} + \delta_{\alpha\beta} \frac{U^0_x}{R} = 0$$

(19)

The first constraints are well known as the Kirchhoff constraints for plate-like structures and the Kirchhoff–Love constraints for shell-like structures.

In poor finite element spaces (low order and coarse meshes), these constraints may be satisfied by restricting low order terms $U^a (\ell \leq 1)$ to be almost null spaces. Here, between two functionals $A(u^a, v^a)$, $B(u^a, v^a)$ expressed in terms of these low order terms, bending strain energy is proportional to $d^3$ while the other by $d$. Accordingly, owing to locking, values of $A(u^a, v^a)$ are getting much smaller when compared to the other, as $d \to 0$.

In addition, from the above variational form (15) and Eqs. (18) and (19), we can find out two major results.

First, for membrane-dominated structures (i.e. $A(u^a, v^a) \approx 0$) which have scaling factor $d$ for external tractions, these constraints do not prevail. Furthermore, the prevalence of the constraints is limited to low order model levels ($q \leq 1$), and which implies there is no effect on locking by increasing the model level. Second, for plate-like structures, membrane locking does not occur because $1/R = U^0_a = 0$. Therefore, locking in shell-like structures is stronger than for plate-like structures.

3.2. A posteriori estimation of locking

From the limit analysis ($d \to 0$), all regular hierarchical models approach the lowest model, the Kirchhoff theory (the Kirchhoff–Love theory in shells). As a result, total strain energy approximated by the $(q_1, q_2, q_3)$ model approaches that of the Kirchhoff theory as the thickness tends to zero. But, from the numerical aspect, the Kirchhoff theory requires $C^1$-finite elements. So, in practice, the Reissner–Mindlin theory $\nu^{RM}$ and the degenerated 3D shell model $u^{\text{deg}}$ are substituted, which are equivalent to the $(1, 1, 0)^*$ model as discussed in the previous section.

In the degenerated 3D shell model, displacement components are as follows [11]:

\[
\begin{align*}
    u_1(x_1, x_2, x_3) &= u_1^0(x_1, x_2) + x_3 u_3^1(x_1, x_2), \quad \alpha = 1, 2 \\
    u_3(x_1, x_2, x_3) &= u_3^0(x_1, x_2)
\end{align*}
\]

(20)

Considering cylindrical shell-like bodies and calibrating external load by a factor of $E/(12(1 - \nu^2))$, we have total strain energy for the degenerated 3D shell model

\[
U(u^{\text{deg}}) = 6d \int_\omega \left[ \nu(\beta_{11} + \beta_{22})^2 + (1 - \nu) \sum_{\alpha, \beta = 1}^2 \beta_{\alpha\beta}^2 \right] R d\theta dz + 6d \int_\omega (1 - \nu)(\rho_1^2 + \rho_2^2)R d\theta dz + d \int_\omega \left[ \nu(\kappa_{11} + \kappa_{22})^2 + (1 - \nu) \sum_{\alpha, \beta = 1}^2 \kappa_{\alpha\beta}^2 \right] R d\theta dz
\]

(21)

Where, $d$ is an axial coordinate of cylindrical shell-like bodies, and $\beta_{\alpha\beta}, \kappa_{\alpha\beta}, \rho_\alpha$ being previously defined. It was shown previously that shear and membrane lockings bring poor approximations of bending strain energies. Thus
it is natural to split the total strain energy into two parts, bending strain energy part $U_b$ and residual strain energy part $U_m + U_{sh}$. Then

$$U(u^q) = U_b(u^q) + U_m(u^q)$$

$$= U_b(u^{deg}) + U_m(u^{deg}) \quad \text{as } d \to 0$$  \hspace{1cm} (22)

Here, we define bending dominance $\eta^q$ of $q$-hierarchical model by the ratio of bending strain energy $U_b$ to the total strain energy $U$.

$$\eta^q = \frac{U_b(u^q)}{U(u^q)}$$  \hspace{1cm} (23)

As the thickness tends to zero, from Eq. (22), $\eta^q \to \eta^{deg}$ (for shell-like bodies) or $\eta^{RM}$ (for plate-like bodies). We denote $\eta^{BD} = \eta^{deg}$ or $\eta^{RM}$, then we have

**COROLLARY 3.1.** If numerical locking does not occur, $\eta^{BD}$ has the following properties as $d \to 0$.

(i) $\eta^{BD} \to 1$ for bending-dominated problems

(ii) $\eta^{BD} \to 0$ for membrane-dominated problems  \hspace{1cm} (24)

**LEMMA 3.1.** For the bending-dominated thin structures, $\eta^{BD} \to 0$ implies that the numerical locking prevails.

**Methodology.** From the numerical output of displacement components obtained using $q$-hierarchical models, we compute element-wise strain energies and corresponding bending strain energies due to the components corresponding to the $(1,1,0)^*$ model. Then, algebraic sum of element-wise contributions gives values of $U$ and $U_b$. This computation is implemented by adding a very simple and cheap post-routine into existing finite element programs.

4. Boundary layer and finite element meshes

As a structure becomes thinner, all hierarchical models approach the lowest model, the Kirchhoff theory, which cannot describe a complexity in displacement or stress fields through the thickness near the boundary. However, since this limit analysis is obtained on the assumption of infinite mid-surface domain, it is valid only at the interior of structures when finite structures are considered. Here, for our purpose of the study of boundary layer, let us employ the Reissner-Mindlin plate theory (i.e. the $(1,1,0)^*$ hierarchical plate model).

Defining $\theta = (\theta_1, \theta_2)^T = (u_n', u_0')$, $w = u_0'$, and $D = E/12(1-\nu^2)$, we write governing equations of the Reissner-Mindlin theory ($\Delta = \nabla \cdot \nabla$)

$$\begin{align*}
&\bar{D} \left\{ (1-\nu) \Delta \theta + (1+\nu) \nabla \cdot \theta + 2K\mu d^{-2}(\theta - \nabla w) \right\} = 0 \\
&\kappa \mu d^{-2} \nabla \cdot (\theta - \nabla w) = g_N
\end{align*}$$  \hspace{1cm} (25)

According as the thickness becomes smaller, Eq. (25) becomes a singularly perturbed boundary value problem with respect to a parameter $d$. Then, a singular behavior of the solution $(w, \theta)$ near the boundary prevails.

Taking divergence to the first equation and using the second relation, we have the following biharmonic equation of $w(x, y)$:

$$\Delta^2 w = \frac{g_N}{\bar{D}} - \frac{d^2}{\kappa \mu} \Delta g_N \quad \text{in } \omega$$  \hspace{1cm} (26)

Since solutions of biharmonic equations are smooth functions, $w(x, y)$ does not contain singularity near the boundary. This implies that the boundary layer singularity is included in rotation components $\theta(x, y)$. Using this result, we decompose the rotation $\theta(x, y)$ into two parts; a smooth part and a singular boundary part as follows:

$$\theta(x, y) = \theta^{SM}(x, y) + \theta^{BL}(x, y)$$  \hspace{1cm} (27)
Next, we apply the perturbation method to Eq. (25), for which the solution \((w, \theta)\) is asymptotically expanded in a form of

\[
w(x, y) = \sum_{i=0}^{\infty} d^i w_i(x, y)\]

\[
\theta(x, y) = \sum_{i=0}^{\infty} d^i \{\theta_{i, SM}(x, y) + \chi(p) \Phi_i(p/d, s)\}
\]

Referring to [4], the boundary functions \(\Phi_i(p/d, s)\) are expressed by smooth functions \(C_i(s)\) along the boundary and exponentially decaying function \(\exp(-\gamma p a/d)\) in the normal direction to the boundary

\[
\Phi_i(p/d, s) = C_i(s) \exp(-\gamma p a/d)
\]

Here, \(\gamma\) is a constant ranging 0.24–0.29 which depends on the regularity of solutions, and \(a\) is a characteristic length of structures. A cut-off function \(\chi(p)\) is defined by

\[
\chi(p) = \begin{cases} 
1: & p \leq p_0 \\
0: & p > p_0 
\end{cases}
\]

in which \(p_0\) depends on the type of boundary conditions, material constants and the thickness ratio \(d/a\).

According to the decomposition of \(\theta(x, y)\), we can split Eq. (25) into two sets of equations; equations of smooth part \((w_i, \theta_{i, SM})\) and equations of singular boundary part \((\theta_i, \theta_{i, SM})\). First, for \((w_i, \theta_{i, SM})\), we put the expansions (28) into the governing equations (25) and equate powers of \(d\). Then it is not difficult to obtain the following biharmonic equations governing \(w_i(x, y)\) and relations between \(w_i(x, y)\) and \(\theta_{i, SM}(x, y)(i = 0, 1, 2, \ldots)\)

\[
\Delta^2 w_i = \delta_{i,0} \frac{g_N}{D} - \frac{g_N}{\kappa \mu} \Delta w_{i-2} + \delta_{i,4} \left( \frac{D}{\kappa \mu} \right)^2 g_N
\]

Now, as \(d \to 0\), we have \(w = w_0\) and \(\theta = \theta_{0, SM} + \chi \Phi_0\) in which \(w_0\) satisfies the Kirchhoff plate theory, the limit model. Since this theory does not have boundary layer, \(\Phi_0\) vanishes.

Second, using \(w^{BL} = 0\), we get the governing equation for divergence-free \(\theta^{BL}\)

\[
\nabla \cdot \theta^{BL} = 0
\]

Then, introducing a scalar function \(\Theta(x, y)\) satisfying \(\nabla \times \Theta = -2\kappa \mu d^{-2} \theta^{BL}\), we obtain homogeneous partial differential equation for \(\Theta(x, y)\)

\[
- \frac{d^2}{12 \kappa} \Delta \Theta + \Theta = 0 \quad \text{in } \omega
\]

With a change of variables \((x, y) \to (p/d, s)\) fitted to the boundary \(\partial \omega\), we asymptotically expand a scalar function \(\Theta(p/d, s)\) using a sequence of \(\{\psi_i\}_{i=0}^{\infty}\)

\[
\Theta(p/d, s) = \sum_{i=0}^{\infty} d^i \psi_i(p/d, s), \quad \lim_{p/d \to \infty} \psi_i = 0
\]

Taking curl in \((p/d, s)\)-coordinate and using the fact of \(\Phi_0 = 0\) and divergence-free \(\theta^{BL}\), we can relate \(\psi_i\) to \(\Phi_i\) as following \((\tau(s) - \text{curvature of the boundary})\)

\[
\Phi_i = \left\{ \frac{\partial \psi_{i-1}}{\partial (p/d)} - i \left( \frac{\tau(s) p/d}{\partial s} \right) \psi_{i-2} - \frac{\partial \psi_{i-2}}{\partial s} n \right\}, \quad i \geq 1
\]

Substituting the expansion (34) into Eq. (33), taking Laplacian in \((p/d, s)\)-coordinate and equating same powers of \((p/d, s)\), we obtain a set of governing equations for \(\psi_i\).
Table 1
Vanishing terms in the expansion of boundary function according to the type of boundary conditions

<table>
<thead>
<tr>
<th>Boundary conditions</th>
<th>Vanishing terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hard clamped</td>
<td>θ · n - θ · s - w = 0</td>
</tr>
<tr>
<td>Soft clamped</td>
<td>θ · n = M_i = w = 0</td>
</tr>
<tr>
<td>Hard simply supported</td>
<td>M_n = θ · s = w = 0</td>
</tr>
<tr>
<td>Soft simply supported</td>
<td>M_n = M_i = w = 0</td>
</tr>
<tr>
<td>Free</td>
<td>M_n = M_i = Q_x = 0</td>
</tr>
</tbody>
</table>

Consequently, the perturbation method with a decomposition of (w, θ) provides two sets of governing equations of w_0 and θ_0, and two sets of relations between w_0 and θSM and between Ψ_0 and Ψ_1.

To obtain a set of boundary conditions corresponding to each power of d, we need to apply the expansions of w, θ to each boundary condition listed in Table 1. Here, θSM and Ψ are switched to w, and Ψ through Eqs. (31) and (35), respectively. Since a boundary layer singularity is contained solely in θBL, its intensity according to the type of boundary conditions is determined by the nonvanishing lowest order of Ψ functions. The reader may refer to [3] for the detailed mathematical work on this procedure, instead we record his results in Table 1, where the strongest boundary layer is observed for free and soft simply supported cases.

Next, let us consider the finite element mesh design which describes well the boundary layer singularity. From Eq. (29), this singular behavior is one-dimensional, i.e. in the normal direction to the boundary, and decaying exponentially. Therefore, mesh design problem is to optimally capture exp(-γpa/d) function defined on normal line segment ∈ [0, ρ_o). Since boundary function becomes steeper as ρ → 0, optimal finite element meshes should be of irregular graded pattern getting denser as ρ → 0.

In the next section, a quantitative mesh gradient especially for stress resultants is suggested through a significant model problem.

5. Numerical experiments

5.1. A posteriori estimation of locking

Three model problems, one plate- and two shell-like problems, are considered. Fig. 2 shows a uniform thickness square plate-like structure clamped at all boundaries with a uniform normal traction g_z on the top surface. From the symmetry of the problem, we consider only a quarter for numerical simulation.

Table 2 contains the computed bending dominancies, where the (1, 1, 2) hierarchical model is approximated using 16 uniform elements (h = l/4). As the thickness decreases, the estimated bending dominancies ηBD approach zero for the approximation order p of 1. This indicates that complete shear locking happens for linear elements. For quadratic and cubic elements, considerable decreases in ηBD are shown from a/d = 100 to 500 even though they are not much compared to linear elements. On the other hand, estimated values for quartic elements monotonously approach 1. Fig. 3 depicts the results of the shear locking, where the calibrated...
Table 2
Estimated bending dominance $\eta_{\text{BD}}$ of the plate-like problem

<table>
<thead>
<tr>
<th>$P$</th>
<th>$a/d = 3$</th>
<th>$a/d = 5$</th>
<th>$a/d = 10$</th>
<th>$a/d = 30$</th>
<th>$a/d = 100$</th>
<th>$a/d = 500$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.24788</td>
<td>0.43845</td>
<td>0.51345</td>
<td>0.17460</td>
<td>0.01965</td>
<td>0.00080</td>
</tr>
<tr>
<td>2</td>
<td>0.27214</td>
<td>0.52789</td>
<td>0.81026</td>
<td>0.94996</td>
<td>0.96505</td>
<td>0.65624</td>
</tr>
<tr>
<td>3</td>
<td>0.27376</td>
<td>0.53148</td>
<td>0.81854</td>
<td>0.97297</td>
<td>0.97632</td>
<td>0.71800</td>
</tr>
<tr>
<td>4</td>
<td>0.27725</td>
<td>0.52990</td>
<td>0.80243</td>
<td>0.98879</td>
<td>0.99900</td>
<td>1.00000</td>
</tr>
</tbody>
</table>

Fig. 3. Shear locking in the (1, 1, 1, 2) hierarchical model (16 uniform elements).

Fig. 4. Quad-cylindrical shell-like structure with uniform normal tractions.

deflection $u_D D/(g a^4)$ ($D \sim$ flexural rigidity) is plotted along the thickness. For approximation order $p$ of 4, locking is not observed at all.

Fig. 4 shows a quad-cylindrical shell-like structure clamped at the bottom side and loaded by uniform normal tractions $g(\theta, z)$. This problem is easily checked to be bending-dominated when the thickness is small. The left half is taken for numerical computation.

The estimated bending dominancies are listed in Table 3, where the $(2, 2, 2)$ hierarchical model is approximated with 16 uniform elements. The estimated bending dominancies for $p \leq 2$ decrease quickly as the body becomes thin. Fig. 5 shows numerical locking in the calibrated radial displacement $u_D/(g R^2)$ of the point $P$. For thin shell-like structures, linear and quadratic elements suffer complete locking on the coarse meshes. Results confirm that locking in shell-like structures is more serious than in plate-like structures.

Fig. 6 is a cylindrical can clamped at both ends subjected to a uniform internal pressure $p_0$. This structure is membrane-dominated, because the major deformation is due to the membrane force in the circumferential direction and the bending deformation near boundaries becomes negligible as the thickness decreases.

Table 4 presents the estimated bending dominancies $\eta_{\text{BD}}$ obtained by applying the $(2, 2, 2)$ hierarchical model with 16 uniform elements to a darkened quarter. Strain energy contributed by the bending deformation is negligible regardless of the thickness ratio. Fig. 7 shows the calibrated strain energies of the $(2, 2, 2)$ hierarchical model with respect to the thickness, where locking is not shown at all.

From the results of three-model problems, we found that estimation of bending dominance shows good indication of locking in actual numerical data. In addition, $\eta_{\text{BD}}$ are so small in the serious deteriorated output.

Table 3
Estimated bending dominance $\eta_{\text{BD}}$ of the shell-like problem

<table>
<thead>
<tr>
<th>$p$</th>
<th>$R/d = 3$</th>
<th>$R/d = 6$</th>
<th>$R/d = 10$</th>
<th>$R/d = 30$</th>
<th>$R/d = 100$</th>
<th>$R/d = 500$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.13848</td>
<td>0.04597</td>
<td>0.01772</td>
<td>0.00204</td>
<td>0.00018</td>
<td>0.00001</td>
</tr>
<tr>
<td>2</td>
<td>0.85369</td>
<td>0.77107</td>
<td>0.86063</td>
<td>0.13892</td>
<td>0.1438</td>
<td>0.0058</td>
</tr>
<tr>
<td>3</td>
<td>0.91622</td>
<td>0.96962</td>
<td>0.97842</td>
<td>0.96143</td>
<td>0.97948</td>
<td>0.99889</td>
</tr>
<tr>
<td>4</td>
<td>0.91842</td>
<td>0.97519</td>
<td>0.98880</td>
<td>0.99821</td>
<td>0.99786</td>
<td>0.99979</td>
</tr>
</tbody>
</table>
that we can easily check the locking phenomenon. Even though it is difficult to determine the critical value of $\eta^{BD}$ for the onset of the locking, the serious numerical locking has remarkably small values, $\eta^{BD} \approx O(10^{-2})$. Consequently, the estimation of $\eta^{BD}$ is a reliable tool to check numerical locking provided the type of the problem is known a priori.

5.2. Mesh design for boundary layer

As mentioned earlier, boundary layer effect is caused by singular behavior of the solution within a thin region near the boundary for bending-dominated thin structures. Therefore, it is naturally important to capture this singular behavior in order to produce acceptable results, particularly to obtain acceptable stress fields.

Here, we provide one example showing the effect of the mesh, in particular, on stress resultants $N_{ap}$, $Q_a$, and $M_{ap}(\alpha, \beta = x, y)$ defined by

$$N_{ap} = \int_{-d/2}^{d/2} \sigma_{ap} \, dz, \quad Q_a = \int_{-d/2}^{d/2} \sigma_{az} \, dz, \quad M_{ap} = \int_{-d/2}^{d/2} z \sigma_{ap} \, dz$$

$$\text{(36)}$$
A model problem is shown in Fig. 8 and its solution is independent of $x$. Among the numerical results, we record $N_y$, $Q_y$, and $M_y$ to illustrate the effect of the first element (boundary element) size and the approximation order.

Fig. 9 describes three different mesh patterns selected for our purpose. Here, we use the $(5, 5, 5)$ hierarchical model and hierarchical shape function for finite element approximations.

Tables 5 and 6 contain the computed stress resultants at points $A$ and $B$ for different mesh patterns and different approximation orders. Exact values of stress resultants using engineering beam theory are: $Q_y = 0$, $M_y = 1/24$ at $A$, $Q_y = -1/2$, $M_y = -1/12$ at $B$, respectively, and $N_y = \text{Const}$. At point $A$, three types of mesh patterns do not yield any remarkable difference in results; but at point $B$, they show considerable influences on the computational accuracy.

First, from Figs. 10, 12 and 14, we observe that a uniform mesh pattern produces poor stress resultants, particularly for shear stress resultants $Q_y$. On the other hand, numerical results obtained with non-uniform meshes are improved and considerably approach exact values as the boundary element size and the approximation order are refined. Second, Figs. 11, 13 and 15 show the $p$-convergence variations in stress resultants at $B$. Especially for shear stress resultants, mesh 2 and mesh 3 make a big difference. Consequently, numerical results suggest that nonuniform meshes with the thin narrow boundary element of size $0.1(d/a)$ provide accurate stress resultants together with acceptable $p$-convergence rates in localized $L^\infty$ sense.

Stress resultants in $y$-direction are plotted in Figs. 16, 17 and 18 for three different mesh patterns. We see the fluctuations of the computed values using approximation order $p$ of 5 at the region near the clamped boundary.

### Table 5

<table>
<thead>
<tr>
<th>$p$</th>
<th>Mesh 1</th>
<th>Mesh 2</th>
<th>Mesh 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_y$</td>
<td>$\begin{bmatrix} 0.20974E-02 \ 0.21182E-02 \ 0.21379E-02 \ 0.21380E-02 \ 0.21381E-02 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0.21379E-02 \ 0.21380E-02 \ 0.21381E-02 \ 0.21382E-02 \ 0.21383E-02 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0.21379E-02 \ 0.21380E-02 \ 0.21381E-02 \ 0.21382E-02 \ 0.21383E-02 \end{bmatrix}$</td>
</tr>
<tr>
<td>$Q_y$</td>
<td>$\begin{bmatrix} -0.25290E+00 \ -0.36428E+00 \ -0.45865E+00 \ -0.46058E+00 \ -0.46063E+00 \end{bmatrix}$</td>
<td>$\begin{bmatrix} -0.36428E+00 \ -0.45865E+00 \ -0.46058E+00 \ -0.46063E+00 \ -0.46063E+00 \end{bmatrix}$</td>
<td>$\begin{bmatrix} -0.45865E+00 \ -0.46058E+00 \ -0.46063E+00 \ -0.46063E+00 \ -0.46063E+00 \end{bmatrix}$</td>
</tr>
<tr>
<td>$M_y$</td>
<td>$\begin{bmatrix} 0.40432E-01 \ 0.42968E-01 \ 0.39647E-01 \ 0.39647E-01 \ 0.39647E-01 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0.42968E-01 \ 0.39647E-01 \ 0.39647E-01 \ 0.39647E-01 \ 0.39647E-01 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0.39647E-01 \ 0.39647E-01 \ 0.39647E-01 \ 0.39647E-01 \ 0.39647E-01 \end{bmatrix}$</td>
</tr>
</tbody>
</table>
Table 6

<table>
<thead>
<tr>
<th>$p$</th>
<th>Mesh 1</th>
<th>Mesh 2</th>
<th>Mesh 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$0.11240E - 02$</td>
<td>$0.14519E - 02$</td>
<td>$0.20260E - 02$</td>
</tr>
<tr>
<td>3</td>
<td>$0.13605E - 02$</td>
<td>$0.18162E - 02$</td>
<td>$0.21041E - 02$</td>
</tr>
<tr>
<td>4</td>
<td>$0.15191E - 02$</td>
<td>$0.20041E - 02$</td>
<td>$0.21268E - 02$</td>
</tr>
<tr>
<td>5</td>
<td>$0.16385E - 02$</td>
<td>$0.20709E - 02$</td>
<td>$0.21353E - 02$</td>
</tr>
<tr>
<td>6</td>
<td>$0.17357E - 02$</td>
<td>$0.20918E - 02$</td>
<td>$0.21381E - 02$</td>
</tr>
</tbody>
</table>

$N_r$

<table>
<thead>
<tr>
<th>$p$</th>
<th>Mesh 1</th>
<th>Mesh 2</th>
<th>Mesh 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$-0.21400E + 01$</td>
<td>$-0.36111E + 00$</td>
<td>$-0.57472E + 00$</td>
</tr>
<tr>
<td>3</td>
<td>$-0.36676E + 01$</td>
<td>$-0.50823E + 00$</td>
<td>$-0.51078E + 00$</td>
</tr>
<tr>
<td>4</td>
<td>$-0.35140E + 01$</td>
<td>$-0.58930E + 00$</td>
<td>$-0.50118E + 00$</td>
</tr>
<tr>
<td>5</td>
<td>$-0.70640E + 01$</td>
<td>$-0.57953E + 00$</td>
<td>$-0.50010E + 00$</td>
</tr>
<tr>
<td>6</td>
<td>$-0.16847E + 00$</td>
<td>$-0.54540E + 00$</td>
<td>$-0.50010E + 00$</td>
</tr>
</tbody>
</table>

$Q_r$

<table>
<thead>
<tr>
<th>$p$</th>
<th>Mesh 1</th>
<th>Mesh 2</th>
<th>Mesh 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$-0.74804E + 01$</td>
<td>$-0.92650E - 01$</td>
<td>$-0.85549E - 01$</td>
</tr>
<tr>
<td>3</td>
<td>$-0.94225E + 01$</td>
<td>$-0.88316E - 01$</td>
<td>$-0.83842E - 01$</td>
</tr>
<tr>
<td>4</td>
<td>$-0.92026E + 01$</td>
<td>$-0.86455E - 01$</td>
<td>$-0.83447E - 01$</td>
</tr>
<tr>
<td>5</td>
<td>$-0.90037E + 01$</td>
<td>$-0.85212E - 01$</td>
<td>$-0.83365E - 01$</td>
</tr>
<tr>
<td>6</td>
<td>$-0.88637E + 01$</td>
<td>$-0.84481E - 01$</td>
<td>$-0.83365E - 01$</td>
</tr>
</tbody>
</table>

$M_r$

<table>
<thead>
<tr>
<th>$p$</th>
<th>Mesh 1</th>
<th>Mesh 2</th>
<th>Mesh 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$0.0025$</td>
<td>$0.01$</td>
<td>$0.01$</td>
</tr>
<tr>
<td>3</td>
<td>$0.001$</td>
<td>$0.01$</td>
<td>$0.01$</td>
</tr>
<tr>
<td>4</td>
<td>$0.0005$</td>
<td>$0.001$</td>
<td>$0.001$</td>
</tr>
<tr>
<td>5</td>
<td>$0.0001$</td>
<td>$0.001$</td>
<td>$0.001$</td>
</tr>
<tr>
<td>6</td>
<td>$0.00$</td>
<td>$0.001$</td>
<td>$0.001$</td>
</tr>
</tbody>
</table>

Fig. 10. Stress resultant $N_r$ at $B$.

Fig. 11. The $p$-convergence variations in stress resultant $N_r$ for three meshes.

Fig. 12. Stress resultant $Q_r$ at $B$.

Fig. 13. The $p$-convergence variations in stress resultant $Q_r$ for three meshes.
and its intensity decreases as the first element size decreases. In particular, uniform mesh pattern shows big fluctuations even at the inside region distant from the boundary.

Numerical results obtained using a uniform mesh pattern considerably deviate from the exact values even at
relatively high approximation orders. On the contrary, for non-uniform mesh patterns, numerical accuracy increases as the approximation order increases. From the numerical results, we confirm that non-uniform mesh patterns with narrow thin boundary element provide accurate numerical analysis. Even though we can obtain better results by decreasing mesh size of uniform mesh patterns, but this approach is insufficient in cost and time aspects.

Even though numerical simulation obtained using uniform mesh patterns satisfies the predefined tolerance measured in energy norm $\|\mathbf{e}\|_{E(\Omega)}$, it may have very big error in localized $L^\infty$ error norm in the narrow thin region neighborhood of boundaries. Then, it is obvious that neglecting this big error with very small measure leads to crucial disaster on structural analysis because most failures of structures take place near boundary where maximum stress resultants occur.

6. Conclusions

In this paper we analyzed locking and boundary layer effects, and introduced an a posteriori locking detection method and effective meshes for capturing boundary layer.

Based on the two major theoretical results of locking phenomenon: (i) it occurs only for bending-dominated structures and (ii) it depends only on lower-order hierarchical models ($q \approx 1$), a simple but reliable a posteriori detection of locking is developed by estimating bending dominance from numerical data. Verification made on three model problems shows that estimated bending dominance is accurately correlated to the intensity of locking in the approximated results.

From the review of recent research results using Reissner–Mindlin plate theory, boundary layer has been found as a singular behavior in rotational components of displacement fields, which is exponentially decaying in the normal direction to the boundary and restricted within very thin narrow region neighborhood of the boundary. This singularity, from asymptotic analysis, strongly depends on the type of boundary conditions.

Numerical test for the effect of mesh pattern and approximation order on stress resultants provides that non-uniform mesh pattern with the narrow thin boundary element of size $0.1(d/a)$ is efficient to capture boundary singularity so that reasonable $p$-convergence in stress resultants is possible.

For a robust $hp$-finite element of thin structures, development of local a posteriori $L^\infty$ error estimator for stress resultants and technique for automatic mesh refinement for boundary layer are essential and deserve future study.

Acknowledgments

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References


