

452

## A POSTERIORI ERROR ESTIMATORS FOR THE STOKES AND OSEEN EQUATIONS\*

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**Abstract.** The problem of obtaining a posteriori estimates of the discretization error when one uses finite element methods to approximate problems with an incompressibility constraint is discussed. A general approach to the treatment of the constraint condition and to the (possible) non-self-adjointness of the associated momentum equations is presented. A posteriori error estimates are derived for adaptive  $h$ ,  $p$ , and  $h$ - $p$  type finite element schemes. Key features are that the local error residual problems are *not* subject to an incompressibility constraint thereby avoiding the need for special finite element schemes and that the analysis is valid for essentially any discretization scheme, including continuous and discontinuous pressure spaces. The estimator bounds the actual error measured in an energy-like norm.

**Key words.**  $h$ - $p$  version finite element method, a posteriori error estimate, incompressible fluid flow

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**1. Introduction.** The importance of a posteriori error estimates for controlling adaptive finite element algorithms is now well established. For self-adjoint, second-order elliptic systems many a posteriori error estimation techniques are known. However, for problems involving supplementary conditions or constraints the situation is less well understood. In particular, the issue of how to take proper account of the incompressibility condition in the Stokes approximation of the steady-state Navier–Stokes equations is unclear. Equally well, in the Oseen approximation of the Navier–Stokes equations one has a non-self-adjoint operator in the momentum equations. It is of key importance to obtain a posteriori error estimates for both Stokes and Oseen equations. For instance, solving the steady-state Navier–Stokes equations often involves starting with the Stokes approximation of the problem and then using an iterative scheme based on the Oseen approximation to converge with the solution of the original problem. To obtain a good initial approximation requires adaptive refinement using a posteriori error estimators for the Stokes problem. As the iteration proceeds, the mesh is adaptively refined at each step, requiring an a posteriori error estimator for Oseen equations.

The problem of deriving a posteriori error estimates for the Stokes problem has already received attention. The basic approaches may be divided roughly into two types. Baranger and El Amri [8] and Verfürth [14, 15] obtain a posteriori error indicators in the form of suitably scaled integrals of the square of the residuals in the equations. Bank and Welfert [6, 7] and Verfürth [14] follow an alternative approach based on the *element residual method* proposed in [5, 11]. This consists of formulating elementwise Stokes problems with data composed of the residuals in the momentum equations and the incompressibility constraint along with boundary conditions depending on the discontinuity in the approximation of the interelement flux. These

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local Stokes problems are then solved approximately on each element yielding a pair of functions whose norm is then used as an estimate of the true discretization error.

The estimators based on evaluating integrals of the residuals provide only refinement *indicators* as opposed to actual direct estimates of the error. The approach based on solving local residual problems avoids this difficulty but obliges one to solve local Stokes problems on each of the subdomains. This hinders the development of a general purpose error estimation procedure since one must design stable schemes with which to approximate the local problem.

In the present work the approach based on solving local residual problems is considered. To begin with, a general approach is given to deal with the fundamental question of the norm in which the error will be estimated. One result is that one is led naturally to the conclusion that the estimator should involve the integral of the square of the amount by which the incompressibility condition is violated. Furthermore, and perhaps rather surprisingly, one finds that it is unnecessary to solve a local Stokes-type problem in order to obtain an a posteriori error estimate. The significance of this conclusion in the design of a general purpose a posteriori error estimation procedure for incompressible fluid flow solvers is vital: the approximation of the local residual problems can, to a large extent, be developed independently of the type of element used to approximate the original fluid flow problem. In fact, the analysis given is valid for essentially any approximation scheme, continuous or discontinuous pressure spaces (or even a mixture of the two), stable or unstable element combinations, incompressible or standard velocity spaces, and (our main interest) general  $h$ - $p$  finite element approximation spaces involving nonuniform meshes and elements of widely varying nonuniform polynomial degree.

The conclusions mentioned hitherto are valid for residual based a posteriori error estimators generally. In the present work, the a posteriori error estimator proposed in [2] is adapted to the Stokes and Oseen equations. Particular features of the approach are that the local problem encompasses a systematic approach to the determination of the boundary data and leads to an error estimator which provides a *guaranteed upper bound* on the true error in an energy-like norm. Under similar assumptions to those of Bank and Welfert [6, 7], the estimator is shown to give two-sided bounds on the actual discretization error. A simple example is given verifying the conclusions.

**1.1. Model problem.** Let  $\Omega \subset \mathbb{R}^2$  denote an open bounded Lipschitzian domain with piecewise smooth boundary  $\partial\Omega$ . The boundary is supposed to consist of a finite number of smooth arcs meeting with internal angle  $\theta \in (0, 2\pi)$ . The Sobolev spaces  $H^m(\Omega)$ ,  $m \in \mathbb{R}$  are defined in the usual way [1]. Function spaces  $\mathcal{X}$  and  $\mathcal{M}$  are introduced as

$$(1) \quad \mathcal{X} = H_0^1(\Omega) \times H_0^1(\Omega) \text{ and } \mathcal{M} = L_2(\Omega).$$

Let  $B : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$  and  $b : \mathcal{X} \times \mathcal{M} \rightarrow \mathbb{R}$  be the bilinear forms

$$(2) \quad b(\mathbf{v}, q) = - \int_{\Omega} q \operatorname{div} \mathbf{v} \, dx$$

and

$$(3) \quad B(\mathbf{v}, \mathbf{w}) = \int_{\Omega} \{ \nu \nabla \mathbf{v} \cdot \nabla \mathbf{w} + \mathbf{w} \cdot (\mathbf{U} \cdot \nabla) \mathbf{v} \} \, dx,$$

where  $\nu > 0$  is the viscosity parameter and  $\mathbf{U}$  is a smooth solenoidal vector field on  $\Omega$  (i.e.,  $\operatorname{div} \mathbf{U} = 0$ ). For given data  $\mathbf{f} \in L_2(\Omega) \times L_2(\Omega)$  we seek the solution of the following problem.

Find  $(\mathbf{u}, p) \in \mathcal{X} \times \mathcal{M}$  such that for all  $(\mathbf{v}, q) \in \mathcal{X} \times \mathcal{M}$

$$(4) \quad B(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) + b(\mathbf{u}, q) = L(\mathbf{v}),$$

where  $L : \mathcal{X} \rightarrow \mathbb{R}$  is the linear functional

$$(5) \quad L(\mathbf{v}) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx.$$

For ease of exposition we consider only homogeneous Dirichlet boundary conditions. More general conditions are dealt with in an analogous fashion. In order to describe sufficient conditions for the existence of a solution to (5) we introduce inner products  $a(\cdot, \cdot)$  and  $c(\cdot, \cdot)$  on  $\mathcal{X}$  and  $\mathcal{M}$ , respectively:

$$(6) \quad a(\mathbf{v}, \mathbf{w}) = \int_{\Omega} \nu \nabla \mathbf{v} \cdot \nabla \mathbf{w} \, dx$$

and

$$(7) \quad c(p, q) = \int_{\Omega} p q \, dx.$$

These inner products induce norms on  $\mathcal{X}$  and  $\mathcal{M}$  denoted by  $\|\cdot\|_{\mathcal{X}}$  and  $\|\cdot\|_{\mathcal{M}}$ , respectively. The following facts concerning  $B$  and  $b$  will be useful (see Girault and Raviart [10]):

- There exists a positive constant  $C_B$  such that

$$(8) \quad |B(\mathbf{v}, \mathbf{w})| \leq C_B \|\mathbf{v}\|_{\mathcal{X}} \|\mathbf{w}\|_{\mathcal{X}} \text{ for all } \mathbf{v}, \mathbf{w} \in \mathcal{X}.$$

•

$$(9) \quad |b(\mathbf{v}, q)| \leq \|\mathbf{v}\|_{\mathcal{X}} \|q\|_{\mathcal{M}} \text{ for all } (\mathbf{v}, q) \in \mathcal{X} \times \mathcal{M}.$$

•  $b$  satisfies an inf-sup condition: i.e., there exists a positive constant  $\alpha_b$  such that

$$(10) \quad \sup_{\mathbf{v} \in \mathcal{X}} \frac{|b(\mathbf{v}, q)|}{\|\mathbf{v}\|_{\mathcal{X}}} \geq \alpha_b \|q\|_{\mathcal{M}} \text{ for all } q \in \mathcal{M}.$$

- $B$  is coercive: i.e., owing to the vector field  $\mathbf{U}$  being solenoidal there holds

$$(11) \quad B(\mathbf{v}, \mathbf{v}) = \|\mathbf{v}\|_{\mathcal{X}}^2 \text{ for all } \mathbf{v} \in \mathcal{X}.$$

Under these conditions it follows [10] that there is a unique solution to (5).

**1.2. Norm on  $\mathcal{X} \times \mathcal{M}$ .** The usual norm on the space  $\mathcal{X} \times \mathcal{M}$  is obtained by summing the squares of the norms on  $\mathcal{X}$  and  $\mathcal{M}$ . It will be convenient to establish an alternative norm on  $\mathcal{X} \times \mathcal{M}$  which is nonetheless equivalent to the usual norm. Let  $(\mathbf{e}, E) \in \mathcal{X} \times \mathcal{M}$  be *arbitrary*. The pair  $(\phi, \psi) \in \mathcal{X} \times \mathcal{M}$  is defined to be the Ritz projection of the residuals

$$(12) \quad a(\phi, \mathbf{v}) + c(\psi, q) = B(\mathbf{e}, \mathbf{v}) + b(\mathbf{v}, E) + b(\mathbf{e}, q)$$

for all  $(\mathbf{v}, q) \in \mathcal{X} \times \mathcal{M}$ . The existence and uniqueness of the pair  $(\phi, \psi)$  follows from the continuity of the forms  $B$  and  $b$ . Therefore, we may define

$$(13) \quad \|(\mathbf{e}, E)\|_* = \left\{ \|\phi\|_{\mathcal{X}}^2 + \|q\|_{\mathcal{M}}^2 \right\}^{1/2}.$$

The following theorem confirms that this quantity is a norm on  $\mathcal{X} \times \mathcal{M}$  equivalent to the obvious norm.

**THEOREM 1.1.** *Under the foregoing assumptions and definitions, there exist positive constants  $k_1$  and  $k_2$  such that*

$$(14) \quad k_1 \|(\mathbf{e}, E)\|_*^2 \leq \|\mathbf{e}\|_{\mathcal{X}}^2 + \|E\|_{\mathcal{M}}^2 \leq k_2 \|(\mathbf{e}, E)\|_*^2,$$

where  $k_1$  depends only on  $C_B$  and  $k_2$  depends only on  $C_B$  and  $\alpha_b$ .

*Proof. Right-hand inequality.* Making use of (10), (12), (8), and the Cauchy-Schwarz inequality yields

$$(15) \quad \|E\|_{\mathcal{M}} \leq \frac{1}{\alpha_b} \{ \|\phi\|_{\mathcal{X}} + C_B \|\mathbf{e}\|_{\mathcal{X}} \}.$$

Using (11), (12) (with  $q = -E$  and  $\mathbf{v} = \mathbf{e}$ ), and the Cauchy-Schwarz inequality,

$$(16) \quad \|\mathbf{e}\|_{\mathcal{X}}^2 \leq \|\phi\|_{\mathcal{X}} \|\mathbf{e}\|_{\mathcal{X}} + \|\psi\|_{\mathcal{M}} \|E\|_{\mathcal{M}}.$$

From (15) and (16) one finds

$$(17) \quad \frac{1}{2} \|\mathbf{e}\|_{\mathcal{X}}^2 \leq \frac{1}{2} \left( \|\phi\|_{\mathcal{X}} + \frac{C_B}{\alpha_b} \|\psi\|_{\mathcal{M}} \right)^2 + \frac{1}{\alpha_b} \|\phi\|_{\mathcal{X}} \|\psi\|_{\mathcal{M}}.$$

Combining (17) with (15) and once again using (17) gives the result with  $k_2$  a constant depending on  $\alpha_b$  and  $C_B$ .

*Left-hand inequality.* Using (12), (8), (9), and the Cauchy-Schwarz inequality gives

$$(18) \quad \|\phi\|_{\mathcal{X}} \leq C_B \|\mathbf{e}\|_{\mathcal{X}} + \|E\|_{\mathcal{M}}.$$

Using (12) and (9) gives

$$(19) \quad \|\psi\|_{\mathcal{M}}^2 = b(\mathbf{e}, \psi) \leq \|\mathbf{e}\|_{\mathcal{X}} \|\psi\|_{\mathcal{M}}.$$

Therefore, combining (18) and (19) yields the estimate claimed, where  $k_1$  depends only on  $C_B$ .  $\square$

## 2. Discretization

**2.1. Partitioning.** Let  $\mathcal{P}$  be a partitioning of the domain  $\Omega$  into the union of  $N$  subdomains  $K$  such that

- $N < \infty$ ,
- $\bar{\Omega} = \cup_{K \in \mathcal{P}} \bar{K}$ ,
- $K \cap J$  is empty whenever  $K \neq J$ ,
- each  $K$  is a convex Lipschitzian domain with piecewise smooth boundary  $\partial K$ .

The common boundary between subdomains  $K$  and  $J$  is denoted by

$$(20) \quad \Gamma_{KJ} = \partial K \cap \partial J.$$

Associated with each subdomain is a space  $P(K)$  of functions mapping  $K \times K \times K$  to  $\mathbb{R}^3$ . Typically, if  $Q_d(K)$  denotes some appropriate space of polynomials on  $K$  with the parameter  $d$  representing the polynomial degree, then

$$(21) \quad P(K) = Q_{d_u}(K) \times Q_{d_v}(K) \times Q_{d_p}(K),$$

where  $d_u$ ,  $d_v$ , and  $d_p$  need not necessarily all take the same value and may actually vary from subdomain to subdomain. Finally,  $D(K)$  denotes a collection of continuous linear functionals on  $P(K)$  such that the pair  $D(K), P(K)$  is *unisolvent* [9]. A finite element is then defined to be the triple  $(K, P(K), D(K))$ . Following the usual construction [9] leads us to a global finite element space  $\hat{\mathcal{X}} \times \hat{\mathcal{M}}$  given by

$$(22) \quad \hat{\mathcal{X}} = \prod_{K \in \mathcal{P}} [Q_{d_u}(K) \times Q_{d_v}(K)]$$

and

$$(23) \quad \hat{\mathcal{M}} = \prod_{K \in \mathcal{P}} Q_{d_p}(K).$$

It is supposed that the functionals  $D(K)$ , and correspondingly  $P(K)$ , are chosen in such a way that the inclusion  $\hat{\mathcal{X}} \times \hat{\mathcal{M}} \subset \mathcal{X} \times \mathcal{M}$  holds. The finite element approximation to (4) is then as follows:

find  $(\hat{u}, \hat{p}) \in \hat{\mathcal{X}} \times \hat{\mathcal{M}}$  such that for all  $(\hat{v}, \hat{q}) \in \hat{\mathcal{X}} \times \hat{\mathcal{M}}$ ,

$$(24) \quad B(\hat{u}, \hat{v}) + b(\hat{v}, \hat{p}) + b(\hat{u}, \hat{q}) = L(\hat{v}).$$

A few remarks concerning the construction of  $\hat{\mathcal{X}} \times \hat{\mathcal{M}}$  are in order. It will be noted that there has been no requirement for a discrete *inf-sup* condition to hold. Unstable discretizations do not hinder the a posteriori error analysis since only stability of the underlying continuous problem is required. Of course, it is generally unwise to actually use unstable discretizations. One restriction we shall impose on  $\hat{\mathcal{X}}$ , but *not* on  $\hat{\mathcal{M}}$ , is that it contains continuous piecewise polynomial functions on the partition of degree at least one. Many standard schemes satisfy this restriction.

**2.2. Mesh-dependent forms and spaces.** It will be convenient to reduce the global spaces and forms into sums of contributions from each of the subdomains in the partition  $\mathcal{P}$ . With this in mind, define the broken Sobolev spaces for  $m \in \mathbb{Z}^+$ ,

$$(25) \quad H^m(\mathcal{P}) = \{v \in L_2(\Omega) : v|_K \in H^m(K) \text{ for all } K \in \mathcal{P}\}$$

Here, and in what follows,  $v_K$  denotes the restriction of  $v$  to a single subdomain  $K$ . The associated norm is

$$(26) \quad \|v\|_{m,\mathcal{P}} = \left\{ \sum_{K \in \mathcal{P}} \|v_K\|_{m,K}^2 \right\}^{1/2}.$$

For each subdomain  $K \in \mathcal{P}$  let

$$(27) \quad \mathcal{V}_K = \{v \in H^1(K) : \gamma v = 0 \text{ on } \partial\Omega \cap \partial K\},$$

$$(28) \quad \mathcal{X}_K = \mathcal{V}_K \times \mathcal{V}_K,$$

and

$$(29) \quad \mathcal{M}_K = L_2(K).$$

The bilinear forms  $B_K : \mathcal{X}_K \times \mathcal{X}_K \rightarrow \mathbb{R}$  and  $b_K : \mathcal{X}_K \times \mathcal{M}_K \rightarrow \mathbb{R}$  are defined as follows:

$$(30) \quad b_K(\mathbf{v}, q) = - \int_K q \operatorname{div} \mathbf{v} \, dx$$

and

$$(31) \quad B_K(\mathbf{v}, \mathbf{w}) = \int_K \{ \nu \nabla \mathbf{v} : \nabla \mathbf{w} + \mathbf{w} \cdot (\mathbf{U} \cdot \nabla) \mathbf{v} \} \, dx.$$

Similarly,  $L_K : \mathcal{X}_K \rightarrow \mathbb{R}$  is defined by

$$(32) \quad L_K(\mathbf{v}) = \int_K \mathbf{f} \cdot \mathbf{v} \, dx.$$

Hence for  $\mathbf{v}, \mathbf{w} \in \mathcal{X}$ , and  $q \in \mathcal{M}$ ,

$$(33) \quad b(\mathbf{v}, q) = \sum_{K \in \mathcal{P}} b_K(\mathbf{v}_K, q_K),$$

$$(34) \quad B(\mathbf{v}, \mathbf{w}) = \sum_{K \in \mathcal{P}} B_K(\mathbf{v}_K, \mathbf{w}_K),$$

and

$$(35) \quad L(\mathbf{v}) = \sum_{K \in \mathcal{P}} L_K(\mathbf{v}_K).$$

The inner products are decomposed as sums of contributions from each subdomain in the partition in an analogous fashion. The broken version of the space  $\mathcal{X} \times \mathcal{M}$  is defined by

$$(36) \quad \mathcal{X}(\mathcal{P}) \times \mathcal{M}(\mathcal{P}) = \{ (\mathbf{v}, q) \in L_2(\Omega) \times L_2(\Omega) \times L_2(\Omega) : (\mathbf{v}, q)|_K \in \mathcal{X}_K \times \mathcal{M}_K \text{ for all } K \in \mathcal{P} \}.$$

Examining the previous notations reveals that  $\mathcal{M}(\mathcal{P}) = \mathcal{M}$ . Later, we shall be led into considering the space of continuous linear functionals  $\tau$  on  $\mathcal{X}(\mathcal{P}) \times \mathcal{M}(\mathcal{P})$  which vanish on the subspace  $\mathcal{X} \times \mathcal{M}$ . Therefore, let  $\mathbb{H}(\mathbf{div}, \Omega)$  denote the space

$$(37) \quad \mathbb{H}(\mathbf{div}, \Omega) = \{ \mathbf{A} \in L_2(\Omega)^{2 \times 2} : \mathbf{div} \mathbf{A} \in L_2(\Omega)^2 \}$$

equipped with norm

$$(38) \quad \|A\|_{\mathbb{H}(\mathbf{div}, \Omega)} = \left\{ \|\mathbf{A}\|_{0, \Omega}^2 + \|\mathbf{div} \mathbf{A}\|_{0, \Omega}^2 \right\}^{1/2}.$$

The following theorem generalizes a result in [13].

**THEOREM 2.1.** *A continuous linear functional  $\tau$  on the space  $\mathcal{X}(\mathcal{P}) \times \mathcal{M}(\mathcal{P})$  vanishes on the subspace  $\mathcal{X} \times \mathcal{M}$  if and only if there exists  $\mathbf{A} \in \mathbb{H}(\mathbf{div}, \Omega)$  such that*

$$(39) \quad \tau[(\mathbf{v}, q)] = \sum_{K \in \mathcal{P}} \oint_{\partial K} \mathbf{n}_K \cdot \mathbf{A} \cdot \mathbf{v}_K \, ds,$$

where  $\mathbf{n}_K$  denotes the unit outward normal on the boundary of  $K$ .

*Proof* (cf. [13]). By the Riesz representation theorem, any continuous functional on  $H^1(K) \times H^1(K) \times L^2(K)$  is of the form

$$(40) \quad (\mathbf{v}, q) \rightarrow \sum_{j=1}^2 \int_K \left\{ \sum_{i=1}^2 A_{ij} \frac{\partial v_j}{\partial x_i} + a_j v_j \right\} \, dx + \int_K a_0 q \, dx,$$

where  $A_{ij}$ ,  $a_j$ , and  $a_0 \in L_2(K)$ . Therefore, for any  $(\mathbf{v}, q) \in \mathcal{X}(\mathcal{P}) \times \mathcal{M}(\mathcal{P})$ ,

$$(41) \quad \tau[(\mathbf{v}, q)] = \sum_{K \in \mathcal{P}} \left[ \sum_{j=1}^2 \int_K \left\{ \sum_{i=1}^2 A_{ij} \frac{\partial v_j}{\partial x_i} + a_j v_j \right\} dx + \int_K a_0 q dx \right],$$

where  $A_{ij}$ ,  $a_j$ , and  $a_0$  now denote elements of the *global* space  $L_2(\Omega)$ . Owing to the hypothesis on  $\tau$  it follows that for any  $(\mathbf{v}, q) \in \mathcal{X} \times \mathcal{M}$ ,

$$(42) \quad 0 = \sum_{j=1}^2 \int_{\Omega} \left\{ \sum_{i=1}^2 A_{ij} \frac{\partial v_j}{\partial x_i} + a_j v_j \right\} dx + \int_{\Omega} a_0 q dx.$$

Hence, in the sense of distributions

$$(43) \quad - \sum_{i=1}^2 \frac{\partial A_{ij}}{\partial x_i} + a_j = 0, \quad j = 1, 2,$$

and  $a_0 = 0$ . Rewriting (43) reveals  $\mathbf{A} \in \mathbb{H}(\text{div}, \Omega)$ :

$$(44) \quad \text{div } \mathbf{A} = - \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in L_2(\Omega)^2.$$

Therefore using the Green identity and (41),

$$(45) \quad \tau[(\mathbf{v}, q)] = \sum_{K \in \mathcal{P}} \sum_{i,j=1}^2 \int_K \left\{ A_{ij} \frac{\partial v_j}{\partial x_i} + \frac{\partial A_{ij}}{\partial x_i} v_j \right\} dx = \sum_{K \in \mathcal{P}} \oint_{\partial K} \mathbf{n}_K \cdot \mathbf{A} \cdot \mathbf{v}_K ds.$$

The converse is shown using similar arguments.  $\square$

The import of this result is that one may identify  $\tau$  with an element  $\mathbf{A}$  of the space  $\mathbb{H}(\text{div}, \Omega)$  and vice versa. In view of Theorem 2.1, we shall abuse the nomenclature slightly and refer to an element of  $\mathbb{H}(\text{div}, \Omega)$  as being a linear functional.

### 3. A posteriori error analysis.

**3.1. Preliminaries.** Suppose  $(\hat{\mathbf{u}}, \hat{p}) \in \hat{\mathcal{X}} \times \hat{\mathcal{M}}$  is the finite element approximation to  $(\mathbf{u}, p) \in \mathcal{X} \times \mathcal{M}$ . In view of the inclusion  $\hat{\mathcal{X}} \times \hat{\mathcal{M}} \subset \mathcal{X} \times \mathcal{M}$ , the discretization error  $(\mathbf{e}, E)$ , where

$$(46) \quad \mathbf{e} = \mathbf{u} - \hat{\mathbf{u}} \text{ and } E = p - \hat{p},$$

lies in the space  $\in \mathcal{X} \times \mathcal{M}$ . Pursuing the approach suggested in section 1.2, define a pair  $(\phi, \psi) \in \mathcal{X} \times \mathcal{M}$  such that

$$(47) \quad a(\phi, \mathbf{v}) + c(\psi, q) = B(\mathbf{e}, \mathbf{v}) + b(\mathbf{v}, E) + b(\mathbf{e}, q)$$

for all  $(\mathbf{v}, q) \in \mathcal{X} \times \mathcal{M}$ . In view of Theorem 1.1, the norm of the discretization error is given by

$$(48) \quad \|(\mathbf{e}, E)\|_*^2 = \|\phi\|_{\mathcal{X}}^2 + \|\psi\|_{\mathcal{M}}^2.$$

The problem is now that of estimating  $\|\phi\|_{\mathcal{X}}$  and  $\|\psi\|_{\mathcal{M}}$  numerically. In principle one could approximate the problem (47) directly and thereby obtain an error

estimator. However, the cost associated with solving the global problem would make this approach impractical. However, an alternative approach is to attempt to reduce the single global problem (47) into a sequence of independent problems posed locally over each element. The aim would be that each of these smaller problems might then be approximated comparatively inexpensively and even in parallel. With this goal in mind, we aim to recast the *global statement* (47) as a *sequence of local problems* posed on each subdomain  $K \in \mathcal{P}$ .

It will be useful to introduce the stresslike tensor  $\sigma(\mathbf{v}, q)$  formally defined to be

$$(49) \quad \sigma_{ij}(\mathbf{v}, q) = \nu \frac{\partial v_i}{\partial x_j} - q\delta_{ij},$$

where  $\delta_{ij}$  is the Kronecker symbol.

For  $(\hat{\mathbf{u}}, \hat{p}) \in \hat{\mathcal{X}} \times \hat{\mathcal{M}}$  it is often the case that  $\sigma(\hat{\mathbf{u}}, \hat{p})$  is smooth (i.e., continuous) on each of the subdomains  $K \in \mathcal{P}$  but suffers jump discontinuities across the interelement boundaries. In order to define the value of the normal component of the stress on the interelement boundaries it is convenient to introduce notations for the jump on  $\Gamma_{KJ}$ :

$$(50) \quad [\mathbf{n} \cdot \sigma(\hat{\mathbf{v}}, \hat{q})] = \mathbf{n}_K \cdot \sigma(\hat{\mathbf{v}}_K, \hat{q}_K) + \mathbf{n}_J \cdot \sigma(\hat{\mathbf{v}}_J, \hat{q}_J).$$

Furthermore, an averaged normal stress on  $\Gamma_{KJ}$  is defined as

$$(51) \quad \langle \mathbf{n}_K \cdot \sigma(\hat{\mathbf{v}}, \hat{q}) \rangle = \begin{bmatrix} \alpha_{KJ}^{(1)} & 0 \\ 0 & \alpha_{KJ}^{(2)} \end{bmatrix} \mathbf{n}_K \cdot \sigma(\hat{\mathbf{v}}_K, \hat{q}_K) + \begin{bmatrix} \alpha_{JK}^{(1)} & 0 \\ 0 & \alpha_{JK}^{(2)} \end{bmatrix} \mathbf{n}_K \cdot \sigma(\hat{\mathbf{v}}_J, \hat{q}_J),$$

where  $\alpha_{KJ}^{(i)} : \Gamma_{KJ} \rightarrow \mathbb{R}$  are smooth (polynomial) functions, the choice of which will be discussed later. Naturally, should the stress be continuous then it is required that the averaged stress coincide with this value. Therefore, on  $\Gamma_{KJ}$ ,

$$(52) \quad \begin{bmatrix} \alpha_{KJ}^{(1)} & 0 \\ 0 & \alpha_{KJ}^{(2)} \end{bmatrix} + \begin{bmatrix} \alpha_{JK}^{(1)} & 0 \\ 0 & \alpha_{JK}^{(2)} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

As a notational convenience, the averaged stress on  $\partial\Omega$  is understood throughout to be identically zero.

The notation  $[\cdot]$  is also used to denote jumps in the elements of  $\mathcal{X}(\mathcal{P})$  between subdomains: after introducing an ordering on the elements (for instance using the element numbers in the finite element code), define

$$(53) \quad [\mathbf{v}] = \begin{cases} \mathbf{v}_K - \mathbf{v}_J, & K > J, \\ \mathbf{v}_J - \mathbf{v}_K, & K < J \end{cases}$$

and

$$(54) \quad [\mathbf{n}] = \begin{cases} \mathbf{n}_K - \mathbf{n}_J, & K > J, \\ \mathbf{n}_J - \mathbf{n}_K, & K < J. \end{cases}$$

The following identity, valid for  $\mathbf{v} \in \mathcal{X}(\mathcal{P})$ , is readily verified:

$$(55) \quad \sum_{K \in \mathcal{P}} \oint_{\partial K} \langle \mathbf{n}_K \cdot \sigma(\hat{\mathbf{u}}, \hat{p}) \rangle \cdot \mathbf{v} \, ds = \sum_{\Gamma_{KJ}} \int_{\Gamma_{KJ}} \langle \mathbf{n}_K \cdot \sigma(\hat{\mathbf{u}}, \hat{p}) \rangle \cdot [\mathbf{v}] \, ds.$$



**3.2. Localization.** The process of decomposing the global problem (47) into smaller, local problems posed over the elements can now be discussed. Two basic steps are involved in breaking up the problem. Firstly, the unknowns  $(\mathbf{u}, p)$  in (47) are replaced by appealing to (4), which is valid for all  $(\mathbf{w}, q) \in \mathcal{X} \times \mathcal{M}$ :

$$(56) \quad \begin{aligned} a(\phi, \mathbf{w}) + c(\psi, q) &= B(\mathbf{e}, \mathbf{w}) + b(\mathbf{w}, E) + b(\mathbf{e}, q) \\ &= \sum_{K \in \mathcal{P}} \{L_K(\mathbf{w}) - B_K(\hat{\mathbf{u}}, \mathbf{w}) - b_K(\mathbf{w}, \hat{p}) - b_K(\hat{\mathbf{u}}, q)\}. \end{aligned}$$

The next step is to deal with the global space  $\mathcal{X} \times \mathcal{M}$ . The essence is to decompose this space of globally smooth functions into functions which are smooth on each of the elements but not necessarily continuous across the interelement boundaries. If this could be accomplished then one would be able to compute the solutions on each of the elements independently, thereby reducing the complexity of the problem substantially. To analyze this approach mathematically, we begin by extending the functional given by (56) to the broken space  $\mathcal{X}(\mathcal{P}) \times \mathcal{M}(\mathcal{P})$ . For any  $(\mathbf{w}, q) \in \mathcal{X}(\mathcal{P}) \times \mathcal{M}(\mathcal{P})$  define the linear functional  $\mathcal{R} : \mathcal{X}(\mathcal{P}) \times \mathcal{M}(\mathcal{P}) \rightarrow \mathbb{R}$  by

$$(57) \quad \begin{aligned} \mathcal{R}[(\mathbf{w}, q)] &= \sum_{K \in \mathcal{P}} \left\{ L_K(\mathbf{w}) - B_K(\hat{\mathbf{u}}, \mathbf{w}) - b_K(\mathbf{w}, \hat{p}) - b_K(\hat{\mathbf{u}}, q) \right. \\ &\quad \left. + \oint_{\partial K} \langle \mathbf{n}_K \cdot \boldsymbol{\sigma}(\hat{\mathbf{u}}, \hat{p}) \rangle \cdot \mathbf{w}_K \, ds \right\} \\ &\quad - \sum_{\Gamma_{KJ}} \int_{\Gamma_{KJ}} \langle \mathbf{n}_K \cdot \boldsymbol{\sigma}(\hat{\mathbf{u}}, \hat{p}) \rangle \cdot \llbracket \mathbf{w} \rrbracket \, ds. \end{aligned}$$

Notice that whenever  $(\mathbf{w}, q) \in \mathcal{X} \times \mathcal{M}$ ,

$$(58) \quad \mathcal{R}[(\mathbf{w}, q)] = a(\phi, \mathbf{w}) + c(\psi, q).$$

**LEMMA 3.1.** *Under the above notations and conventions, there exists  $\hat{\mu} \in \mathbb{H}(\text{div}, \Omega)$  such that for all  $(\mathbf{w}, q) \in \mathcal{X}(\mathcal{P}) \times \mathcal{M}(\mathcal{P})$ ,*

$$(59) \quad \hat{\mu}[(\mathbf{w}, q)] = \sum_{\Gamma_{KJ}} \int_{\Gamma_{KJ}} \langle \mathbf{n}_K \cdot \boldsymbol{\sigma}(\hat{\mathbf{u}}, \hat{p}) \rangle \cdot \llbracket \mathbf{w} \rrbracket \, ds.$$

*Proof.* The right-hand side of equation (59) vanishes on  $\mathcal{X} \times \mathcal{M}$ . Applying Theorem 2.1, the result follows immediately.  $\square$

Applying Lemma 3.1 yields

$$(60) \quad \begin{aligned} \mathcal{R}[(\mathbf{w}, q)] &= \sum_{K \in \mathcal{P}} \left\{ L_K(\mathbf{w}) - B_K(\hat{\mathbf{u}}, \mathbf{w}) - b_K(\mathbf{w}, \hat{p}) - b_K(\hat{\mathbf{u}}, q) \right. \\ &\quad \left. + \oint_{\partial K} \langle \mathbf{n}_K \cdot \boldsymbol{\sigma}(\hat{\mathbf{u}}, \hat{p}) \rangle \cdot \mathbf{w}_K \, ds \right\} \\ &\quad - \hat{\mu}[(\mathbf{w}, q)] \quad \text{for all } (\mathbf{w}, q) \in \mathcal{X}(\mathcal{P}) \times \mathcal{M}(\mathcal{P}). \end{aligned}$$

**3.3. Variational analysis.** Introduce the Lagrangian functional  $\mathcal{L} : \mathcal{X}(\mathcal{P}) \times \mathcal{M}(\mathcal{P}) \times \mathbb{H}(\text{div}, \Omega) \rightarrow \mathbb{R}$  to be

$$(61) \quad \mathcal{L}[(\mathbf{w}, q), \mu] = \frac{1}{2} \{a(\mathbf{w}, \mathbf{w}) + c(q, q)\} - \mathcal{R}[(\mathbf{w}, q)] - \mu[(\mathbf{w}, q)]$$

and note that

$$(62) \quad \sup_{\mu \in \mathbf{H}(\operatorname{div}, \Omega)} \mathcal{L}[(\mathbf{w}, q), \mu] = \begin{cases} \frac{1}{2} \{a(\mathbf{w}, \mathbf{w}) + c(q, q)\} - \mathcal{R}[(\mathbf{w}, q)] & \text{if } (\mathbf{w}, q) \in \mathcal{X} \times \mathcal{M}, \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover, for  $(\mathbf{w}, q) \in \mathcal{X} \times \mathcal{M}$ ,

$$(63) \quad \begin{aligned} & \frac{1}{2} \{a(\mathbf{w}, \mathbf{w}) + c(q, q)\} - \mathcal{R}[(\mathbf{w}, q)] \\ &= \frac{1}{2} \{a(\mathbf{w} - \phi, \mathbf{w} - \phi) + c(q - \psi, q - \psi) - a(\phi, \phi) - c(\psi, \psi)\} \\ &\geq -\frac{1}{2} \{a(\phi, \phi) + c(\psi, \psi)\} = -\frac{1}{2} \|(\mathbf{e}, E)\|_*^2. \end{aligned}$$

Therefore,

$$(64) \quad \begin{aligned} & -\frac{1}{2} \|(\mathbf{e}, E)\|_*^2 \\ &= \inf_{(\mathbf{w}, q) \in \mathcal{X}(\mathcal{P}) \times \mathcal{M}(\mathcal{P})} \sup_{\mu \in \mathbf{H}(\operatorname{div}, \Omega)} \mathcal{L}[(\mathbf{w}, q), \mu] \\ &= \sup_{\mu \in \mathbf{H}(\operatorname{div}, \Omega)} \inf_{(\mathbf{w}, q) \in \mathcal{X}(\mathcal{P}) \times \mathcal{M}(\mathcal{P})} \mathcal{L}[(\mathbf{w}, q), \mu] \\ &\geq \inf_{(\mathbf{w}, q) \in \mathcal{X}(\mathcal{P}) \times \mathcal{M}(\mathcal{P})} \mathcal{L}[(\mathbf{w}, q), \hat{\mu}] \\ &= \sum_{K \in \mathcal{P}} \inf_{\mathbf{w}_K \in \mathcal{X}_K} \left\{ \frac{1}{2} a(\mathbf{w}_K, \mathbf{w}_K) - L_K(\mathbf{w}) + B_K(\hat{\mathbf{u}}, \mathbf{w}) + b_K(\mathbf{w}, \hat{p}) \right. \\ &\quad \left. - \oint_{\partial K} \langle \mathbf{n}_K \cdot \boldsymbol{\sigma}(\hat{\mathbf{u}}, \hat{p}) \rangle \cdot \mathbf{w}_K \, ds - \frac{1}{2} c_K(\operatorname{div} \hat{\mathbf{u}}, \operatorname{div} \hat{\mathbf{u}}) \right\}, \end{aligned}$$

where the infimum over the space  $\mathcal{M}(\mathcal{P})$  has been computed explicitly. The order of the inf-sup may be changed here since a saddle point is obtained when the multiplier  $\mu$  is the true interelement flux. This choice is a valid multiplier, as can be seen by applying Theorem 2.1. Summarizing, we have shown the following.

**THEOREM 3.2.** *Let  $\mathcal{J}_K : \mathcal{X}_K \rightarrow \mathbb{R}$  be the quadratic functional*

$$(65) \quad \begin{aligned} \mathcal{J}_K(\mathbf{w}_K) &= \frac{1}{2} a(\mathbf{w}_K, \mathbf{w}_K) - L_K(\mathbf{w}) + B_K(\hat{\mathbf{u}}, \mathbf{w}) \\ &\quad + b_K(\mathbf{w}, \hat{p}) - \oint_{\partial K} \langle \mathbf{n}_K \cdot \boldsymbol{\sigma}(\hat{\mathbf{u}}, \hat{p}) \rangle \cdot \mathbf{w}_K \, ds. \end{aligned}$$

Then

$$(66) \quad \|(\mathbf{e}, E)\|_*^2 \leq \sum_{K \in \mathcal{P}} \left\{ -2 \inf_{\mathbf{w}_K \in \mathcal{X}_K} \mathcal{J}_K(\mathbf{w}_K) + c_K(\operatorname{div} \hat{\mathbf{u}}_K, \operatorname{div} \hat{\mathbf{u}}_K) \right\}.$$

**3.4. Analysis of local error residual problems.** The analysis contained in the previous section leads to problems on each subdomain of the form

$$(67) \quad \inf_{\mathbf{w}_K \in \mathcal{X}_K} \mathcal{J}_K(\mathbf{w}_K).$$

Suppose for a moment that a minimum exists; then the minimizing element is characterized by finding  $\phi_K \in X_K$  such that

$$(68) \quad a(\phi_K, \mathbf{v}) = L_K(\mathbf{v}) - B_K(\hat{\mathbf{u}}, \mathbf{v}) - b_K(\mathbf{v}, \hat{p}) + \oint_{\partial K} \langle \mathbf{n}_K \cdot \boldsymbol{\sigma}(\hat{\mathbf{u}}, \hat{p}) \rangle \cdot \mathbf{v} \, ds$$

for all  $\mathbf{v} \in X_K$ . This problem is simply a Poisson-type problem with Neumann boundary conditions. The result of the foregoing analysis is that one can obtain a local a posteriori error estimator for the Stokes problem by solving auxiliary Neumann-type problems for the velocity, since the influence of the pressure can be explicitly calculated. This has a considerable impact in the computation of the error estimator since one need not solve a local Stokes-type problem, as for example is the case with [6, 7, 14, 15]. The approach suggested above simplifies that of [6, 7, 14, 15] and could be used in the context of those papers.

The necessary and sufficient conditions for the existence of a minimum are that the data satisfy the following compatibility or *equilibration* condition:

$$(69) \quad 0 = L_K(\boldsymbol{\theta}) - B_K(\hat{\mathbf{u}}, \boldsymbol{\theta}) - b_K(\boldsymbol{\theta}, \hat{p}) + \oint_{\partial K} \langle \mathbf{n}_K \cdot \boldsymbol{\sigma}(\hat{\mathbf{u}}, \hat{p}) \rangle \cdot \boldsymbol{\theta} \, ds$$

for all  $\boldsymbol{\theta} \in \text{Ker}[a, \mathcal{X}_K]$ , where

$$(70) \quad \text{Ker}[a, \mathcal{X}_K] = \{\boldsymbol{\theta} \in \mathcal{X}_K : a_K(\mathbf{w}, \boldsymbol{\theta}) = 0 \text{ for all } \mathbf{w} \in \mathcal{X}_K\}.$$

When the subdomain  $K$  lies on the boundary  $\partial\Omega$  then the local problem (68) will be subject to a homogeneous Dirichlet condition on a portion of their boundaries and thus will be automatically well posed. However, elements away from the boundary are subject to pure Neumann conditions and the null space of the operator  $a(\cdot, \cdot)$  will contain the rigid body motions

$$(71) \quad \text{Ker}[a, \mathcal{X}_K] = \text{Span}\{\boldsymbol{\theta}_1, \boldsymbol{\theta}_2\},$$

where

$$(72) \quad \boldsymbol{\theta}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{\theta}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Fortunately, we shall be able to construct data which satisfy the equilibration condition (69) thanks to the freedom to choose the averaged interelement flux. First, define

$$(73) \quad \begin{bmatrix} \lambda_{KJ}^{(1)} & 0 \\ 0 & \lambda_{KJ}^{(2)} \end{bmatrix} = \begin{bmatrix} \alpha_{JK}^{(1)} & 0 \\ 0 & \alpha_{JK}^{(2)} \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so that the consistency condition (52) becomes

$$(74) \quad \begin{bmatrix} \lambda_{KJ}^{(1)} & 0 \\ 0 & \lambda_{KJ}^{(2)} \end{bmatrix} + \begin{bmatrix} \lambda_{JK}^{(1)} & 0 \\ 0 & \lambda_{JK}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The averaged interelement stress may then be rewritten

$$(75) \quad \langle \mathbf{n}_K \cdot \boldsymbol{\sigma}(\hat{\mathbf{v}}, \hat{q}) \rangle = \langle \mathbf{n}_K \cdot \boldsymbol{\sigma}(\hat{\mathbf{v}}, \hat{q}) \rangle_{1/2} + [\mathbf{n} \cdot \boldsymbol{\sigma}(\hat{\mathbf{v}}, \hat{q})] \begin{bmatrix} \lambda_{KJ}^{(1)} & 0 \\ 0 & \lambda_{KJ}^{(2)} \end{bmatrix},$$

where  $\langle \mathbf{n}_K \cdot \boldsymbol{\sigma}(\hat{\mathbf{v}}, \hat{q}) \rangle_{1/2}$  denotes the interelement averaged stress obtained using the symmetrical weighting corresponding to  $\alpha = 1/2$ . The equilibration condition then becomes

$$(76) \quad \begin{aligned} & L_K(\boldsymbol{\theta}) - B_K(\hat{\mathbf{u}}, \boldsymbol{\theta}) - b_K(\boldsymbol{\theta}, \hat{p}) + \oint_{\partial K} \langle \mathbf{n}_K \cdot \boldsymbol{\sigma}(\hat{\mathbf{u}}, \hat{p}) \rangle \cdot \boldsymbol{\theta} \, ds \\ &= - \sum_{J \in \mathcal{P}} \int_{\Gamma_{KJ}} [\mathbf{n} \cdot \boldsymbol{\sigma}(\hat{\mathbf{v}}, \hat{q})] \begin{bmatrix} \lambda_{KJ}^{(1)} & 0 \\ 0 & \lambda_{KJ}^{(2)} \end{bmatrix} \cdot \boldsymbol{\theta} \, ds \end{aligned}$$

for all  $\boldsymbol{\theta} \in \text{Ker}[a, \mathcal{X}_K]$ .

Let  $\{\chi_A\}$  be chosen so that

$$\text{Span} \{\chi_A\} \times \text{Span} \{\chi_A\} \subset \hat{\mathcal{X}}$$

and scaled so that

$$(77) \quad \sum_A \chi_A(\mathbf{x}) \equiv 1.$$

For example, one might choose the piecewise bilinear pyramid functions associated with the interior nodes in the partition. The relation (77) must hold at all points  $\mathbf{x}$  contained in elements which do not intersect the boundary of the domain.

The functions  $\lambda_{KJ}^{(k)} : \Gamma_{KJ} \rightarrow \mathbb{R}$  are chosen to be of the form

$$(78) \quad \lambda_{KJ}^{(k)}(s) = \sum_A \lambda_{KJ,A}^{(k)} \chi_A(s),$$

where  $\lambda_{KJ,A}^{(k)}$  are constants to be determined. Owing the constraint (74), it is required that

$$(79) \quad \lambda_{KJ,A}^{(k)} + \lambda_{JK,A}^{(k)} = 0$$

for each  $A$ .

LEMMA 3.3. *Suppose for each  $\chi_A$  the constants  $\{\lambda_{KJ,A}^{(k)}\}$  can be chosen to satisfy*

$$(80) \quad - \sum_{J \in \mathcal{P}} \lambda_{KJ,A}^{(k)} \rho_{KJ,A}^{(k)} = b_{K,A}^{(k)}$$

for  $k = 1, 2$ , where

$$(81) \quad b_{K,A}^{(k)} = L_K(\chi_A \boldsymbol{\theta}_k) - B_K(\hat{\mathbf{u}}, \chi_A \boldsymbol{\theta}_k) - b_K(\chi_A \boldsymbol{\theta}_k, \hat{p}) + \oint_{\partial K} \chi_A(s) \langle \mathbf{n}_K \cdot \boldsymbol{\sigma}(\hat{\mathbf{u}}, \hat{p}) \rangle \cdot \boldsymbol{\theta}_k \, ds$$

and

$$(82) \quad \rho_{KJ,A}^{(k)} = \int_{\Gamma_{KJ}} [\mathbf{n} \cdot \boldsymbol{\sigma}(\hat{\mathbf{v}}, \hat{q})] \cdot \boldsymbol{\theta}_k \, ds.$$

Then

$$(83) \quad 0 = L_K(\boldsymbol{\theta}) - B_K(\hat{\mathbf{u}}, \boldsymbol{\theta}) - b_K(\boldsymbol{\theta}, \hat{p}) + \oint_{\partial K} \langle \mathbf{n}_K \cdot \boldsymbol{\sigma}(\hat{\mathbf{u}}, \hat{p}) \rangle \cdot \boldsymbol{\theta} \, ds$$

for all  $\boldsymbol{\theta} \in \text{Ker}[a, \mathcal{X}_K]$ .

*Proof.* The result follows immediately by forming appropriate linear combinations of (80) and using (78) and (74).  $\square$

In view of Lemma 3.3 in order to satisfy the compatibility condition one need only choose the space  $\text{Span}\{\chi_A\}$  so that  $1 \in \text{Span}\{\chi_A\}$ . It remains to construct  $\{\lambda_{KJA}^{(k)}\}$  so that conditions (79) and (80) hold. In [3] a numerical procedure for solving the system efficiently is presented, the cost of which is shown to be modest.

In summary, the numerical procedure is to first calculate the parameters  $\alpha$  which ensure that the local problems (67) are well posed. These problems are then solved numerically, using for example a  $p$  version finite element scheme on each subdomain to give an approximate solution  $\phi_K$ . It will be assumed that this solution is computed sufficiently accurately that the error between this and the true solution to the local problems may be neglected. The process then yields an a posteriori error estimate  $\epsilon_K$  on the subdomain  $K$ ,

$$(84) \quad \epsilon_K = \{a_K(\phi_K, \phi_K) + c_K(\text{div } \hat{\mathbf{u}}, \text{div } \hat{\mathbf{u}})\}^{1/2}.$$

A global error estimate may be obtained by summing the local estimates. Theorem 3.2 guarantees that the estimate bounds the true error  $\|(\mathbf{e}, E)\|_*$  from above.

**4. Reliability of estimator.** In this section the question of the reliability of the estimator is addressed. The main result will be that the estimator is bounded above by a constant multiple of the actual error under certain assumptions.

**4.1. Assumptions.** In addition to the previous conditions on the partition  $\mathcal{P}$  we require the following:

- Each subdomain  $K \in \mathcal{P}$  is convex and satisfies a minimum angle condition.
- Let  $\mathcal{N}_K$  be the set consisting of subdomains neighboring  $K$ :

$$(85) \quad \mathcal{N}_K = \{J \in \mathcal{P} : \partial K \cap \partial J \text{ is nonempty}\}.$$

It is assumed that there exists a constant  $C$ , which does not depend on  $\mathcal{P}$  such that

$$(86) \quad \text{card}(\mathcal{N}_K) \leq C \text{ for all } K \in \mathcal{P}.$$

- Let  $h_K = \text{diam}(K)$ . It is assumed that there exists a constant  $\kappa$  which does not depend on  $\mathcal{P}$  such that

$$(87) \quad \frac{1}{\kappa} \leq \frac{h_K}{h_J} \leq \kappa \text{ for all } J \in \mathcal{N}_K.$$

Together, these requirements force the partition to be locally quasi uniform. However, the partition may still be irregular (albeit with the degree of irregularity bounded) and strongly refined in portions of the domain  $\Omega$ .

These restrictions are placed on the partition  $\mathcal{P}$ . In terms of the spaces  $P(K)$  it is assumed that

- the spaces  $P(K)$  consist of piecewise polynomials on each subdomain  $K \in \mathcal{P}$ . The polynomial degree is allowed to vary from subdomain to subdomain but overall must be bounded above independently of  $\mathcal{P}$  by  $\bar{p}$  (say).

While this assumption precludes the  $p$  and  $h$ - $p$  version finite element methods proper, it is a reasonable assumption in a practical context. The final restriction represents a regularity assumption on the true solution  $(\mathbf{u}, p)$ .

• There exists a piecewise polynomial  $(\mathbf{u}^*, p^*)$  on  $\mathcal{P}$  with degree not exceeding  $\bar{p} + 1$  such that

$$(88) \quad \|\mathbf{u} - \mathbf{u}^*\|_X^2 + \|p - p^*\|_M^2 + \sum_{K \in \mathcal{P}} \oint_{\partial K} h_K \langle \mathbf{n}_K \cdot \boldsymbol{\sigma}(\mathbf{u} - \mathbf{u}^*, p - p^*) \rangle^2 ds \leq \beta^2 \left\{ \|\mathbf{e}\|_X^2 + \|E\|_M^2 \right\},$$

where  $\beta$  is a positive constant.

This assumption is similar to the saturation assumption made by Bank and Weiser [5]. As remarked in Bank and Welfert [6], it is a relatively weak assumption but does require that the solution is more regular than the smoothness implied by the inclusion  $(\mathbf{u}, p) \in \mathcal{X} \times \mathcal{M}$ . In fact, our assumption is slightly weaker than that of Bank and Welfert, since it is not assumed that  $\beta \rightarrow 0$  as the finite element space is refined. Alternatively, one can avoid the saturation assumption altogether by following the approach of Nochetto [12].

**4.2. Analysis of reliability.**

LEMMA 4.1. *Under the above assumptions there exists a constant  $C > 0$ , independent of the partition  $\mathcal{P}$ , such that*

$$(89) \quad \sum_{K \in \mathcal{P}} \oint_{\partial K} h_K \langle \mathbf{n}_K \cdot \boldsymbol{\sigma}(\mathbf{e}, E) \rangle^2 ds \leq C^2 \left\{ \|\mathbf{e}\|_X^2 + \|E\|_M^2 \right\}.$$

*Proof.* Applying the triangle inequality,

$$(90) \quad \left[ \sum_{K \in \mathcal{P}} \oint_{\partial K} h_K \langle \mathbf{n}_K \cdot \boldsymbol{\sigma}(\mathbf{e}, E) \rangle^2 ds \right]^{1/2} \leq \left[ \sum_{K \in \mathcal{P}} \oint_{\partial K} h_K \langle \mathbf{n}_K \cdot \boldsymbol{\sigma}(\mathbf{u} - \mathbf{u}^*, p - p^*) \rangle^2 ds \right]^{1/2} + \left[ \sum_{K \in \mathcal{P}} \oint_{\partial K} h_K \langle \mathbf{n}_K \cdot \boldsymbol{\sigma}(\hat{\mathbf{u}} - \mathbf{u}^*, \hat{p} - p^*) \rangle^2 ds \right]^{1/2}.$$

Now, using standard inverse estimates [9] applied *locally* to the polynomials  $\hat{\mathbf{u}} - \mathbf{u}^*$  and  $\hat{p} - p^*$  on each subdomain yields the estimate

$$(91) \quad \sum_{K \in \mathcal{P}} \oint_{\partial K} h_K \langle \mathbf{n}_K \cdot \boldsymbol{\sigma}(\hat{\mathbf{u}} - \mathbf{u}^*, \hat{p} - p^*) \rangle^2 ds \leq C \left\{ \|\hat{\mathbf{u}} - \mathbf{u}^*\|_X^2 + \|\hat{p} - p^*\|_M^2 \right\}.$$

Using the triangle inequality and applying (88) gives the result claimed.  $\square$

Consider the norm of the solution of the error residual problem on the subdomain  $K$ :

$$(92) \quad \begin{aligned} a(\boldsymbol{\phi}_K, \boldsymbol{\phi}_K) &= L_K(\boldsymbol{\phi}_K) - B_K(\hat{\mathbf{u}}, \boldsymbol{\phi}_K) - b_K(\boldsymbol{\phi}_K, \hat{p}) + \oint_{\partial K} \langle \mathbf{n}_K \cdot \boldsymbol{\sigma}(\hat{\mathbf{u}}, \hat{p}) \rangle \cdot \boldsymbol{\phi}_K ds \\ &= B_K(\mathbf{e}, \boldsymbol{\phi}_K) + b_K(\boldsymbol{\phi}_K, E) - \oint_{\partial K} \langle \mathbf{n}_K \cdot \boldsymbol{\sigma}(\mathbf{e}, E) \rangle \cdot \boldsymbol{\phi}_K ds. \end{aligned}$$

Let  $\bar{\boldsymbol{\phi}}_K = \int_K \boldsymbol{\phi}_K \, d\mathbf{x} / |K|$ ; then owing to the satisfaction of the compatibility conditions, we obtain

$$\begin{aligned}
 & a(\phi_K, \phi_K) \\
 &= B_K(\mathbf{e}, \phi_K - \bar{\phi}_K) + b_K(\phi_K - \bar{\phi}_K, E) - \oint_{\partial K} \langle \mathbf{n}_K \cdot \boldsymbol{\sigma}(\mathbf{e}, E) \rangle \cdot (\phi_K - \bar{\phi}_K) \, ds \\
 (93) \leq & \|\phi_K\|_X \left\{ C_B \|\mathbf{e}\|_{X,K} + C_b \|E\|_{M,K} + C \left[ \oint_{\partial K} h_K \langle \mathbf{n}_K \cdot \boldsymbol{\sigma}(\mathbf{e}, E) \rangle^2 \, ds \right]^{1/2} \right\},
 \end{aligned}$$

where the following standard approximation result has been used:

$$(94) \quad \oint_{\partial K} (\phi_K - \bar{\phi}_K)^2 \, ds \leq Ch_K \|\phi_K\|_{X,K}^2.$$

Applying Lemma 4.1 and summing over each subdomain in the partition then gives

$$(95) \quad \sum_{K \in \mathcal{P}} a(\phi_K, \phi_K) \leq C \left\{ \|\mathbf{e}\|_X^2 + \|E\|_M^2 \right\} \leq C \|(\mathbf{e}, E)\|_*^2.$$

Summarizing and incorporating the results of section 4 we have Theorem 4.2.

**THEOREM 4.2.** *Under the above assumptions, there exists a constant  $C > 0$  such that*

$$(96) \quad \|(\mathbf{e}, E)\|_*^2 \leq \sum_{K \in \mathcal{P}} \epsilon_K^2 \leq C \|(\mathbf{e}, E)\|_*^2.$$

**4.3. Numerical example.** In order to illustrate and verify the upper bound property, we present the result obtained in a simple test case. The boundary conditions are chosen so that the Stokes problem

$$\begin{aligned}
 & -\nu \Delta \mathbf{u} + \nabla p = \mathbf{0}, \\
 & \operatorname{div} \mathbf{u} = 0, \\
 (97) \quad & \mathbf{u} = \mathbf{g} \text{ on } \partial\Omega
 \end{aligned}$$

has (smooth) solution

$$\begin{aligned}
 & u = \sinh x \sin y + y^2, \\
 & v = \cosh x \cos y, \\
 (98) \quad & p = 2\nu y.
 \end{aligned}$$

The domain  $\Omega$  is the unit square and the viscosity parameter is taken to be  $\nu = 1/100$ . Knowing the true solution allows one to compute  $\|(\mathbf{e}, E)\|_*$  and to verify the conclusions of Theorem 3.2 in this case. The discretization scheme consists of subdividing the domain into uniform square elements. The spaces  $\mathcal{X}$  and  $\hat{\mathcal{P}}$  consist of continuous polynomials of total degree  $p$  and  $p - 1$ , respectively, ( $p > 1$ ). In Table 1 the results obtained using the estimator described above along the results obtained using an estimator in which all of the splitting parameters  $\alpha$  have been chosen to be  $1/2$ .

It is seen that the upper bound property for the estimator with equilibration is confirmed. The theory provides no guarantees regarding the performance of the latter scheme, which also yields upper bounds in this particular example.

TABLE 1

Performance of estimator obtained using equilibration  $\epsilon_\alpha$  and estimator obtained using symmetric averaging  $\epsilon_{1/2}$  for Stokes problem with known smooth solution.

Degree $p$	Mesh Size $h$	$\ (\mathbf{e}, E)\ _*$	$\epsilon_\alpha$	$\epsilon_{1/2}$
2	1/4	1.722(-3)	1.778(-3)	1.901(-3)
2	1/8	4.15(-4)	4.49(-4)	4.64(-4)
3	1/4	5.226(-5)	5.229(-5)	5.232(-5)
3	1/8	6.524(-6)	6.528(-6)	6.528(-6)

**5. Summary and conclusions.** An important point of the present work is that *the existing estimators based on solving error residual problems can be simplified*. The reason is that the analysis shows that one does not have to solve a local Stokes problem; it is sufficient to solve a pair of independent local Poisson problems. This means that one is solving a system of two equations (since the residual corresponding to the incompressibility condition can be treated directly) rather than the system of three coupled equations needed for existing techniques.

More importantly, when one comes to construct the basis functions used in approximating the local problems, there is no issue of stability (inf-sup) conditions. These conditions can be quite problematic if one is trying to solve a local Stokes problem using an appropriate space, requiring quite delicate stability analysis; see, for example, Bank and Welfert [7]. By adopting the approach suggested by the current paper this issue simply does not arise. *These features makes the computation of the estimators less expensive and more easily applicable to general finite element schemes for Stokes-type problems.*

The analysis suggests that the boundary data for the local residual type problems should be chosen to satisfy an equilibrium condition. However, the above comments are equally valid whether one is equilibrating the boundary fluxes or not. Of course, one loses the upper bound property if the equilibration condition is not satisfied, but this may not be of primary importance in some applications. The concept of equilibrating fluxes has been shown to be a key step in the development of reliable and robust error estimators. For example, in the work of Babuska and Strouboulis [4], one finds extensive numerical tests comparing estimators based on solving local problems with and without equilibration of fluxes. Their main recommendation is that one must use an equilibration procedure if one is to obtain a robust error estimator. Although their conclusions were based on Poisson problems, one can expect similar effects for more complicated problems. It is clearly desirable to derive an appropriate equilibration principle for a Stokes problem. This is carried out in the present paper and generalizes previous work that led to the equilibration principle for Poisson-type problems. Although it is not proved here, one might reasonably expect that the estimators with equilibration will be more robust. On a practical note, one cannot compute the solution of the local problems exactly. However, one usually does obtain an upper estimator [4] even if the problem is solved approximately.

One can question the usefulness of an upper bound in the unorthodox  $*$ -norm, albeit equivalent with the  $H^1$ -type norms. The analysis can be used to obtain an estimator in the  $H^1$ -norm. Instead, we derive an estimator for the  $*$ -norm. The energy of the actual solution can also be estimated in the same  $*$ -norm. This can be computed at the same time as the error estimator by modifying the right-hand sides used in the error estimation process by omitting the terms  $B_K(\hat{\mathbf{u}}, \mathbf{v})$  and  $b(\mathbf{v}, \hat{p})$  in equation (68). The process yields a sufficiently good estimate for practical purposes and may be used to obtain a numerical estimate of the relative error in the  $*$ -norm.



Although the  $*$ -norm is not necessarily ideal, it does allow one to perform rigorous and quantitative error control for Stokes and Oseen problems.

**5.1. Comparison with alternative error estimators.** Alternative estimators based on evaluating norms of the residuals are less expensive than solving residual problems. However, one obtains something quite different. Unfortunately, thanks to unknown constants in the estimator obtained by evaluating norms of residuals, one only obtains an *error indicator*, making it awkward to obtain quantitative estimates for the error. It is also the case that one may obtain pessimistic estimators because there is no possibility of cancellation between the interior and the boundary residual. Another difficulty is that of correctly balancing the contributions from the different terms. One can think of the process of solving the residual problem as correctly balancing the contributions between the boundary and interior residuals.

Estimators based on equilibration are more expensive to compute compared with classical element residual estimators in which the boundary fluxes are obtained using a simple averaging of the fluxes between neighboring elements. However, the cost of the equilibration procedure has been shown to grow only in direct proportion to the number of degrees of freedom in the problem [3]. Our opinion is very much that the effort of obtaining equilibrating fluxes is money well spent. A poor error estimator increases the likelihood of having to perform extra, expensive iterations of the refinement process. Our own experience is that this is especially true when one is dealing with highly irregular meshes, widely varying polynomial degree as in the case of  $h$ - $p$  finite element Navier–Stokes simulations.

In summary, it has been found that

- one does not have to solve a local Stokes problem to obtain an error estimator: this has a significant impact on the expense, complexity and applicability of existing error estimation techniques;
- the equilibration principle carries over from the scalar case. An appropriate equilibration principle for Stokes problem was formulated and analyzed;
- one can obtain rigorous error estimates and achieve quantitative control of the error in Stokes and Oseen problems. Moreover, the cost is not large in comparison with other stages of the finite element solution process.

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