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DIRECT NUMERICAL APPROXIMATIONS OF
THE TRANSIENT NAVIER-STOKES EQUATIONS

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ADAPTIVE CONTROL OF THE ERROR IN DIRECT NUMERICAL APPROXIMATIONS OF THE TRANSIENT NAVIER-STOKES EQUATIONS

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Abstract. Advances in Computational Fluid Dynamics and progress in computer performances have justified the use of direct numerical simulations to investigate transitional incompressible flows at the onset of turbulence. However, such time-dependent simulations produce large amounts of output information, and one faces the major issue of assessing the accuracy, and a fortiori the validity, of numerical approximations of fluid flow problems. Here, we develop an error estimator for time-dependent approximations of the Navier-Stokes equations, based on residual methods, and propose a strategy to automatically control the numerical error within some preset tolerances. The performance of the methodology is demonstrated for two-dimensional channel flows past a cylinder in the periodic regime.

1 Introduction

The error estimation method considered here belongs to the family of Error Residual Methods. These methods were originally developed for linear elliptic problems [1], and were later extended to the Stokes [2, 3, 4], and steady-state Navier-Stokes problem [2, 5, 6]. In the present approach, we introduce two residuals, one deriving from the momentum equation and the other from the continuity equation. These residuals represent the sources of errors in the numerical approximations, and, as such, are related to the actual errors in some appropriate measures in order to provide meaningful error estimates. The objective in error control is to contain the estimated numerical errors within some preset tolerances. This is possible by reducing the effects of the source terms, i.e., the residuals, as soon as the errors exceed the prescribed tolerances. In finite element methods this amounts to performing mesh adaptation. One advantage of our method is the possibility to prescribe two different tolerances, one for each of the two residuals, in order to adequately control their respective effects. Numerical experiments performed on channel flows past a cylinder at moderate Reynolds numbers show the efficiency of such a strategy.

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2 Preliminaries

Let \( \Omega \) denote an open bounded domain in \( \mathbb{R}^n \), \( n = 2 \) or \( 3 \), with boundary \( \partial \Omega \). The flow of a viscous incompressible fluid in \( \Omega \) is modeled by the Navier-Stokes equations, given here in nondimensionalized form,

\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u - \text{Re}^{-1} \Delta u + \nabla p &= f \quad \text{in } \Omega \times (0,T) \\
\nabla \cdot u &= 0 \quad \text{in } \Omega \times (0,T) 
\end{align*}
\]

(1)

with boundary condition \( u(x,t) = g(x,t) \), for all \( x \in \partial \Omega \) and \( t \in (0,T) \), and initial condition \( u(x,0) = u_0(x) \), for all \( x \in \Omega \). Here \( u = u(x,t) \) and \( p = p(x,t) \) are respectively the velocity vector and the pressure scalar at point \( x \in \Omega \) and at time \( t \in [0,T) \), \( \text{Re} \) is the Reynolds number, \( f = f(x,t) \) is a prescribed body force and \( u_0 = u_0(x) \) a prescribed initial velocity field that satisfies the continuity equation \( \nabla \cdot u_0 = 0 \).

For the sake of simplicity in the mathematical development, we consider only homogeneous boundary conditions \( g = 0 \) on \( \partial \Omega \). We then introduce the trial spaces of velocities \( V \) and pressures \( Q \) with associated norms and inner products:

\[
V = H_0^1(\Omega) = (H_0^1(\Omega))^n, \quad |v|^2 = (\nabla v, \nabla v) = \int_{\Omega} \nabla v \cdot \nabla v \, dx,
\]

\[
Q = \{ q \in L^2(\Omega) : \int_{\Omega} q \, dx = 0 \}, \quad \|q\|_0^2 = (q,q) = \int_{\Omega} q^2 \, dx.
\]

We also introduce the bilinear forms \( a \) and \( b \), as well as the trilinear form \( c \), such that for all \( u, v, w \in V \) and \( q \in Q \):

\[
a(u,v) = \text{Re}^{-1} \int_{\Omega} \nabla u : \nabla v \, dx,
\]

\[
b(v,q) = -\int_{\Omega} q \nabla \cdot v \, dx,
\]

\[
c(u,v,w) = \int_{\Omega} (u \cdot \nabla) v \cdot w \, dx.
\]

The forms are all continuous on their respective spaces of definition. Moreover, the bilinear form \( b \) is known to satisfy the standard \( \text{inf-sup condition} \) [7], and in particular, there exists a constant \( \beta > 0 \) such that:

\[
\sup_{v \in V \setminus \{0\}} \frac{|b(v,q)|}{|v|_1} \geq \beta \|q\|_0, \quad \forall q \in Q.
\]

(2)

Solutions of the Navier-Stokes problem are given by the pair of functions \( (u,p) \in V \times Q \),
for all \( t \in [0, T) \), which satisfy:

\[
(\partial_t u, v) + c(u, u, v) + a(u, v) + b(v, p) = \langle f, v \rangle, \quad \forall v \in V,
\]

\[
b(u, q) = 0, \quad \forall q \in Q,
\]

\[
u = u_0, \quad \text{at} \ t = 0.
\]

The above problem is then approximated using \( h-p \) finite element spaces \( V^h \subset V \) and \( Q^h \subset Q \) [8] and any numerical time marching scheme we wish. Next, we define and study numerical errors in finite element approximations \((u_h, p_h) \in V^h \times Q^h\), for \( t \in [0, T)\), of the Navier-Stokes equations.

### 3 Error Estimation

The numerical error is defined as the pair \((e, E) = (u - u_h, p - p_h)\). It belongs to the space \( V \times Q \) for each time \( t \in [0, T) \). Substituting \( u \) and \( p \) respectively with \((u_h + e)\) and \((p_h + E)\) in (3), the error \((e, E)\), due to the discretization in space, is shown to satisfy the following time evolution equation and constraint:

\[
(\partial_t e, v) + c(e, u_h, v) + c(u_h, e, v) + c(e, e, v) + a(e, v) + b(v, E) = R^m_e(v), \quad \forall v \in V,
\]

\[
b(e, q) = R^c_e(q), \quad \forall q \in Q,
\]

where the residual \( R^m_e \) in the momentum equation and the residual \( R^c_e \) in the continuity equation are the linear functionals

\[
R^m_e(v) = \langle f, v \rangle - (\partial_t u_h, v) - c(u_h, u_h, v) - a(u_h, v) - b(v, p_h),
\]

\[
R^c_e(q) = -b(u_h, q).
\]

The objective in error estimation is to relate, inexpensively but accurately, the residuals to the errors in some relevant measures. We establish here the relationship between the quantity \(|e|_1\) to the norms of the residuals \( R^m_e \) and \( R^c_e \), defined as

\[
\|R^m_e\|_* = \sup_{v \in V \setminus \{0\}} \frac{|R^m_e(v)|}{|v|_1}, \quad \|R^c_e\|_* = \sup_{q \in Q \setminus \{0\}} \frac{|R^c_e(q)|}{\|q\|_0}.
\]

The key point in our approach is to decompose the error \( e \) into two unique vectors \( e_d \in J \) and \( e_\perp \in J^\perp \) such that \( e = e_d + e_\perp \). The norm of \( e \) is then given by \( |e|_1 = |e_d|^2 + |e_\perp|^2 \) since
the space $J$ is the subspace of $V$ ($H^1_0(\Omega)$) which contains all the divergence-free functions of $V$, whereas $J^\perp$ is the orthogonal complement of $J$ with respect to the inner product $(\nabla \cdot, \nabla \cdot)$ (see [7]).

**Theorem 1** Let $e_\perp \in J^\perp$ be the error component in the numerical velocity $u_h$. Then, for all $t \in (0, T)$,

$$\beta |e_\perp(t)|_1 \leq \| R_h^m \|_* \leq \sqrt{n} |e_\perp(t)|_1 \cdot \tag{6}$$

\[ \square \]

Such a result shows that $\| R_h^m \|_*$ provides a reasonable estimate of $|e_\perp|_1$. In order to evaluate the quantity $|e_d|_1$, we assume that $e_\perp$ is maintained so as to be negligible with respect to $e_d$. Replacing $e$ and $v$ by $e_d$ in the time evolution (4), we obtain:

$$\frac{1}{2} \frac{d}{dt} \| e_d \|^2 = -c(e_d, u_h, e_d) - Re^{-1} |e_d|^2 + R_h^m(e_d), \tag{7}$$

where $\| \cdot \|$ denote the $L^2(\Omega)$ norm in $V$. Applying Kolmogorov's scaling theory [9], which conjectures the existence of a dissipation length scale below which viscosity dominates the dynamics, we are allowed to neglect the inertial term with respect to the viscous term, so that:

$$\frac{1}{2} \frac{d}{dt} \| e_d \|^2 \approx - Re^{-1} |e_d|^2 + R_h^m(e_d). \tag{8}$$

When the error "increases", i.e. $d \| e_d \|^2 / dt \geq 0$, we can further get the bound:

$$|e_d|_1 \leq Re \| R_h^m \|_*. \tag{9}$$

Therefore, the cost in evaluating the error estimates amounts to calculating the norm of $R_h^m$ and $R_h^c$. On one hand, the computation of $\| R_h^c \|_*$ is shown to be exact and cheap, as for all $t \in [0, T)$,

$$\| R_h^c \|_* = \| \nabla \cdot u_h(t) \|_0. \tag{10}$$

On the other hand, the calculation of $\| R_h^m \|_*$ is more complicated. By virtue of the Riesz Representation theorem, there exists a unique function $\varphi \in V$ such that:

$$|\varphi|_1 = \| R_h^m \|_* \tag{11}$$

Unfortunately, such a function $\varphi$ can not in general be computed exactly, but low-cost iterative methods have been developed to calculate accurate approximations $\varphi_h$ of $\varphi$ in some suitable finite element subspaces of $V$, such that $|\varphi_h|_1 \approx |\varphi|_1$. 
4 Adaptive Control of the Error

The objective in adaptive control is to contain the error within some preset tolerances by reducing the local effects of the residuals $R_h^m$ and $R_h^e$. In time-dependent simulations we expect the numerical errors to grow when degrees of freedom are lacking. Therefore, we need to define local quantities in terms of the residuals which would serve as indicators for element refinement as soon as they exceed some threshold values. The refinement procedure consists here in dividing a given element into smaller elements. An alternative would be to increase the polynomial degree of the shape functions in the same element. In what follows we devise elementwise indicators as well as a strategy to control the numerical error. Let $N_e$ be the number of elements in the finite element mesh. By defining the quantities $\eta_{e,K}$ and $\eta_{m,K}$ for each element $\Omega_K$, $K = 1, \ldots, N_e$:

$$\eta_{e,K}^2 = \int_{\Omega_K} |\nabla \cdot u_h|^2 \, dx, \quad \eta_{m,K}^2 = \int_{\Omega_K} \nabla \phi_h \cdot \nabla \phi_h \, dx,$$

the norms of the residuals are decomposed as:

$$\|R_h^e\|^2 = \sum_{K=1}^{N_e} \eta_{e,K}^2, \quad \|R_h^m\|^2 = \sum_{K=1}^{N_e} \eta_{m,K}^2.$$  \hfill (13)

It is then natural to define the elementwise indicators $\xi_{e,K}$ and $\xi_{m,K}$ for each element $\Omega_K$ by the following quantities:

$$\xi_{e,K} = \sqrt{N_e \frac{\eta_{e,K}}{|u_h|}}, \quad \xi_{m,K} = \text{Re} \sqrt{N_e \frac{\eta_{m,K}}{|u_h|}}$$  \hfill (14)

and, given some constants $C^e$ and $C^m$, we check that $\xi_{e,K} \leq C^e$, and $\xi_{m,K} \leq C^m$. When there are elements for which these criteria are not satisfied, they need to be refined. Otherwise the relative global errors:

$$E_e = \frac{|e_\perp|}{|u_h|}, \quad E_m = \frac{|e_d|}{|u_h|}$$  \hfill (15)

are shown to be controlled within the tolerances $C^e$ and $C^m$, since:

$$|e_\perp| \approx \|R_h^e\| \leq C^e |u_h|,$$

$$|e_d| \leq \text{Re} \|R_h^m\| \approx \text{Re} |\phi_h| \leq C^m |u_h|.$$

The tolerances $C^e$ and $C^m$ are user prescribed. Such a flexibility in selecting different values for $C^e$ and $C^m$ allows us to control the errors $e_\perp$ and $e_d$ accordingly. Indeed, one wants to carefully control the component $e_\perp$ as it is responsible for generating undesirable numerical instabilities in the simulated flows. On the other hand, the perturbation $e_d$, comparable to perturbations in experimental flows, requires less precision in its control.
5 Numerical Experiments

The above error control strategy is tested on the simulation of channel flow past a cylinder. It is well-known that such a flow develops into a periodic vortex shedding as it becomes unstable to unsymmetric perturbations past a critical value of the Reynolds number. The flow domain $\Omega$ is initially discretized into the 160-element mesh shown in Figure 1. The cylinder is slightly moved off-center in order to generate unsymmetric perturbations. The Reynolds number is defined as $\text{Re} = U_c d / \nu$ where $d$ is the diameter of the cylinder and $U_c$ the maximal velocity of the parabolic profile at the inflow. We select $\text{Re} = 100$ in the present experiment. The basis functions for the velocity and pressure consist in piecewise

![Figure 1: Geometry and initial mesh.](image1)

![Figure 2: Evolution of the adapted mesh (close-up views around the cylinder) at (a) $t = 5$ (b) $t = 25$ (c) $t = 50$ (d) $t = 200$.](image2)
biquadratic and bilinear functions respectively. The numerical solution is advanced in time using the Adams-Bashforth Crank-Nicolson scheme (ABCN). The dimensionless timestep is fixed to the value $\Delta t = 0.01$ so that the errors due to the time discretization are kept small. Finally, the initial state $u_0$ is chosen as the Navier-Stokes solution computed at $Re = 1$.

We show in Figure 2 the adapted mesh at various times of the simulation. At $t = 50$, for example, the mesh is made of 532 elements while the final mesh contains 967 elements. We remark, however, that no adaptation was performed in the time period [100, 200] as the flow had reached the permanent periodic regime. The tolerances $C_e$ and $C_m$ are chosen as functions of time in order to allow for larger numerical errors during the transients. The estimated relative global errors $E_e$ and $E_m$, given in (15), are eventually reduced to about 5% and 10.5% respectively. The evolution of the tolerances and relative errors is shown in Figure 3. In comparison, the estimated global errors grow very fast during the transients and stabilize around 15% and 55%, when the numerical solution is calculated on the initial mesh without performing any error control, i.e. without mesh adaptation, as shown in Figure 3.

![Figure 3: Time evolution of the tolerances $C_e$ and $C_m$ and estimated relative errors $E_e$ and $E_m$ with and without error control.](image)

![Figure 4: Phase portraits (left) with error control (right) without error control.](image)

In addition, we use the time delay reconstruction technique to create phase portraits based on the global kinetic energy signal $K_e(t)$. In Figure 4, we show the phase portraits for
the numerical solutions calculated with error control (left) and without error control (right). They reveal that the solution obtained with error control is periodic with time, as expected, whereas the solution obtained on the initial mesh with no adaptation clearly converges to a steady-state regime. This confirms that it is essential to control the quality of the spatial discretization in order to capture the right dynamical flow properties.

6 Conclusion

We have proposed a fully automatic adaptive strategy for the control of the numerical error for the time-dependent Navier-Stokes equations. It has been successfully tested on the simulation of a channel flow past a cylinder in the periodic regime, for which a numerical solution has been controlled within some preset error tolerances. The proposed error estimation and adaptive strategy methods are nevertheless applicable to flows at any given Reynolds number and work is currently in progress to utilize them for direct numerical simulations of flow transition and turbulence.

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References


