Theory and approximation of quasistatic frictional contact problems

C.Y. Lee and J.T. Oden

Texas Institute for Computational Mechanics, The University of Texas at Austin,
Austin, TX 78712, USA

Received 15 September 1992

A variational formulation of a class of quasistatic frictional contact problems is derived for which a compliant interface model of the contact surface is assumed. Proof of existence and uniqueness of solutions is given. Finite element approximations of a regularized version of the fractional problem are developed and a priori error estimates are derived. The theory is implemented and used to solve representative test problems of frictional contact. The numerical simulations show that frictional stress distributions can be very sensitive to load histories and surface properties of contact interfaces.

1. Introduction

In many respects, the mathematical theory of frictional contact is a relatively new subject, emerging only in the last decade with the introduction of new frictional models that were not only more physically realistic than classical theories, but which were also mathematically tractable. Earlier works focused on cases in which Coulomb's law was imposed, but for which the normal stresses were prescribed (e.g. [1]). The realization that the classical Coulomb friction law is based on global behavior of basically rigid bodies under the action of frictional forces has led to modifications of that law which are capable of modeling many of the principal features of frictional behavior, e.g., stick–slip motion, transition from equilibrium to sliding, etc. Representative of these frictional laws is the normal compliance model of Oden and Martins [2, 3], which was shown to be capable of producing results consistent with experimental evidence and of simulating stick–slip phenomena as well as textbook frictional behavior such as the sliding of blocks down inclined planes. Summary accounts of this subject together with a detailed historical account of contact and friction models can be found in the treatise of Kikuchi and Oden [4].

In recent years, a number of results have been published that further develop the theory of frictional contact on compliant surfaces. We mention, in particular, the work of Rabier et al. [5], and the important series of papers by Klarbring and Shillor and their collaborators [6, 7].

In the present paper, we re-examine the questions of existence and uniqueness of solutions of an incremental quasistatic model of frictional contact studied by Klarbring et al. [6]. We
develop a regularized version of this incremental formulation and use it as a basis for constructing general finite element approximations for these classes of problems. We then derive a priori error estimates for the discretized problem. Finally, we consider several numerical experiments involving slow motions of an elastic punch gradually pressed into contact with a rigid frictional surface.

The mathematical foundation of the theory of frictional contact is still somewhat incomplete. Available proofs of existence and uniqueness of solutions for normal compliance models pertain to the dynamic case and assume a linear viscoelastic material [8]. The presence of inertia and viscoelastic damping terms are essential in these proofs, and results for the quasistatic case are incomplete. This situation is not surprising when one considers the physical mechanisms involved in sticking and sliding of frictional surfaces. If a body is at rest on a frictional surface and is subjected to increasing forces tangent to that surface, motion (sliding) begins when the equilibrium configuration of the body becomes dynamically unstable (see [3]). Thus, inertia effects are essential in characterizing the dynamical transition from adhesion to sliding. Damping effects are useful in proving that the resulting motions are bounded in the time domain, but may not prove to be essential in an existence theory.

In order to obtain quasistatic models of frictional contact without inertial effects, Klarbring, Mikelic and Shillor proposed incremental and rate models. These were reduced from difference approximations of variational inequalities characterizing static and formally quasistatic cases; their proofs of existence and uniqueness of solutions, however, are based on the assumption that the displacement field exists and satisfies certain regularity conditions.

The present study is a continuation of the analysis presented in [9] and an extension of that theory to quasistatic frictional phenomena. Two different formulations are developed and compared: the first is an incremental theory for which incomplete results on existence and uniqueness of solutions exist, and the second is the dynamic theory which includes inertia effects and for which an existence theory has been developed. For slow contact and sliding, results obtained from the two theories are in very good agreement. Finally, finite element models are constructed and a priori error estimates are derived.

2. Formulations of the problem

We consider the motion of a linear-elastic body in contact with a rigid foundation. The motion is assumed to be very slow in comparison with the speed of elastic waves in the body, and acceleration is assumed to be negligible.

The body occupies a domain \( \Omega \) in \( \mathbb{R}^N \) \( (N = 2, 3) \) which has a Lipschitz boundary \( \Gamma \), and is subject to body forces \( f \) throughout \( \Omega \) and surface tractions \( t \) applied to a portion \( \Gamma_F \) of \( \Gamma \). The body is fixed along a portion \( \Gamma_D \) of \( \Gamma \) and \( \Gamma_C \) denotes a candidate contact surface. The initial gap between the body and the foundation is defined by a function \( g, g \geq 0 \). Points in \( \Omega \) and on the boundary \( \Gamma \) are denoted by \( \mathbf{x} = (x_1, x_2, \ldots, x_N) \) and \( s_i, 1 \leq i \leq N \), respectively. Time is denoted by \( t \) and we consider a time interval \( t \in [0, T] \). The usual summation convention is used throughout this work.

We confine our attention to infinitesimal deformations of the body and assume that the body has a linear elastic behavior characterized by the generalized Hooke's law,
\[ \sigma_i(u) = E_{ijkl} u_{kl}, \quad 1 \leq i, j, k, l \leq N. \]  

where \( u = u(x, t) \) is the displacement field in \( \Omega \). \( E_{ijkl} \) are the usual elasticity coefficients of the material, \( \sigma_{ij} \) are the components of the Cauchy stress tensor, and \((\quad)_{,i} \) and \((\quad)_{,t} \) denote the partial differentiations with respect to \( x_i \) and time, respectively. We denote by \( n \) the unit outward normal vector on the boundary \( \Gamma \) and by \( \sigma_n \) and \( \sigma_T \) the normal stress and the tangent stress vector on the boundary whose values at a displacement \( u \) are, respectively.

\[ \begin{align*} 
\sigma_n(u) &= \sigma_n(u)n_j n_j, \\
\sigma_T(u) &= \sigma_T(u)n_j - \sigma_n(u)n_j. 
\end{align*} \]

Similarly, the displacement \( u \) on the boundary \( \Gamma \) will be decomposed into normal and tangential components \( u_n \) and \( u_T \), respectively, according to

\[ \begin{align*} 
u_n &= u_i n_i, \quad u_T = u_i - u_n n_i. 
\end{align*} \]

The frictional contact behavior on the contact surface \( \Gamma_c \) is assumed to be governed by the compliant surface model studied by Oden and Martins [3]

\[ \sigma_n = -c_n(u_n - g)^{m_n}_+ \]

when \( u_n \leq g \), \( \sigma_T(u) = 0 \), and when \( u_n > g \).

\[ \begin{align*} 
|\sigma_T(u)| < c_T(u_n - g)^{m_T}_+ & \Rightarrow \dot{u}_T = 0, \\
|\sigma_T(u)| = c_T(u_n - g)^{m_T}_+ & \Rightarrow \exists \lambda \geq 0 \text{ such that } \dot{u}_T = -\lambda \sigma_T(u). 
\end{align*} \]

where \( c_n = c_n(s) \), \( c_T = c_T(s) \), \( m_n \) and \( m_T \) are material parameters characterizing the interface and are to be determined experimentally, and \((\quad)_+ \) denotes the positive part of the argument.

We now state the following systems of equations and inequalities governing the displacement field \( u \), for a time interval \([0, T]\):

- Linear momentum equations:

\[ -\rho \ddot{u}_i + \sigma_{ij}(u)_{,j} + f_i = 0 \quad \text{in } \Omega \times (0, T), \quad 1 \leq i, j \leq N. \]

- Boundary conditions:

\[ \begin{align*} 
u &= 0 \quad \text{on } \Gamma_D \times (0, T), \quad 1 \leq i \leq N, \\
\sigma_i(u)n_j &= t_i \quad \text{on } \Gamma_N \times (0, T), \quad 1 \leq i, j \leq N. 
\end{align*} \]

- Contact boundary conditions (4):

- Initial conditions:

\[ \begin{align*} 
u(x, 0) &= u_0(x) \quad \text{in } \Omega, \quad \dot{u}(x, 0) = u_1(x) \quad \text{in } \Omega; 
\end{align*} \]

where \( \rho = \rho(x) \) is the mass density of the material of which the body is composed.
NOTE. We shall assume homogeneous boundary conditions on $\Gamma_D$ for simplicity, but this is not essential in our theory.

We introduce the space $V$ of admissible displacements defined by

$$ V = \{ v \in [H^1(\Omega)]^N : v = 0 \text{ on } \Gamma_D \} , $$

and we also assume that

$$ f_i \in L^2(\Omega) , \quad t_i \in L^2(\Gamma_T) , \quad g \in H^{1/2}(\Gamma_C) . $$

The elasticity coefficients are assumed to satisfy the following conditions:

$$ E_{ijkl}(x) \in L^\infty(\Omega) , $$

$$ E_{ijkl} = E_{jikl} = E_{klij} , \quad 1 \leq i, j, k, l \leq N . $$

$$ E_{ijkl} \xi_{ij} \xi_{kl} \geq \alpha |\xi|^2 \quad \forall \xi \neq 0 , $$

where $\alpha$ is a positive constant. Also, in order to make boundary integrals on $\Gamma_C$ well defined, we put restrictions on the coefficients $c_n(s), c_T(s), m_n$ and $m_T$:

$$ c_n(s), c_T(s) \in L^\infty(\Gamma_C) , $$

$$ 1 \leq m_n, m_T < \infty \quad \text{ when } N = 2 , \quad 1 \leq m_n, m_T \leq 3 \quad \text{ when } N = 3 . $$

These restrictions on $m_n$ and $m_T$ basically come from the embedding theorem which states that for $v \in [H^1(\Omega)]^N$, $\gamma(v) \in [L^q(\Gamma)]^N$ with $2 \leq q \leq \infty$ for $N = 2$, and with $2 \leq q \leq 4$ for $N = 3$.

Then the weak form of the dynamic frictional contact problem is: Find $u(t) : [0, T] \rightarrow V$ such that

$$ \langle \rho \ddot{u}(t), v - \dot{u} \rangle + a(u(t), v - \dot{u}) + j_n(u(t), v - \dot{u}) + j_T(u(t), v) - j_T(u(t), \dot{u}) \geq F(v - \dot{u}) \quad \forall v \in V , $$

with the initial conditions $u(0) = u_0, \dot{u}(0) = \dot{u}_1$, where

$$ \langle u, v \rangle = \int_\Omega u_i v_i \, dx , \quad a(u, v) = \int_\Omega E_{ijkl} u_k i v_i j \, dx . $$

$$ j_n(u, v) = \int_{\Gamma_C} c_n(s)(u_n - g) v_n \, ds , $$

$$ j_T(u, v) = \int_{\Gamma_C} c_T(s)(u_n - g)^{n_T} v_T \, ds , \quad F(v) = \int_\Omega f_i v_i \, dx + \int_{\Gamma_C} t_i v_i \, ds . $$

The formal equivalence of (12) with (5)–(7) is established by standard methods as in [1, 4]. As described in the Introduction, the existence and uniqueness of this formulation is proved in
and for the case of Coulomb's friction law and prescribed normal stresses, the proofs can be found in [1].

We now consider the cases when the inertia term is small enough to be neglected; for example, when the density of the material is small or the acceleration of the whole body is small. Then we can omit the inertia term and we have the quasistatic formulation: Find \( u \in V \), \( \forall v \in V \) such that

\[
a(u(t), v - u) + j_n(u(t), v - u) + j_T(u(t), v) - j_T(u(t), u) \geq F(v - u),
\]

with the initial conditions \( u(0) = u_0 \).

The existence and uniqueness of the solutions to (14) are open issues: however, we can derive a time discretized approximation to the quasistatic problem, the so-called incremental problem, for which conditions for the existence and uniqueness of solutions have been established by Klarbring et al. [6].

### 3. The incremental model of the static problem

We derive a time discretized approximation of problem (14) as in [6]. Let the time interval \([0, T]\) be divided into \( n \) successive intervals \((t_i, t_{i+1})\) for \( i = 0, \ldots, n - 1 \) and \( 0 = t_0 < \cdots < t_n = T \). We introduce the standard finite difference approximation of displacements and velocities such as

\[
\begin{align*}
    u(t_\theta) &= u(t_i) + \theta(u(t_{i+1}) - u(t_i)), \\
    u(t_\theta) &= u(t_i) + \theta(u(t_{i+1}) - u(t_i)).
\end{align*}
\]

where \( t_\theta = t_i + \theta(t_{i+1} - t_i), \quad \Delta t = t_{i+1} - t_i. \)

Considering the equilibrium at \( t = t_\theta \) and denoting

\[
\begin{align*}
    u^l &= u(t_i), \\
    w &= \theta(u^{l+1} - u^l), \\
    v &= \theta \Delta t v,
\end{align*}
\]

we have

\[
\begin{align*}
    a(w + u^l, v - w) + j_n(w + u^l, v - w) + j_T(w + u^l, v) - j_T(w + u^l, u) &\geq F_o(v - w), \\
\end{align*}
\]

where

\[
F_o(v) = \int_{\Omega} f_i(x, t_\theta) v_i \, dx + \int_{\Gamma} t_i(s, t_\theta) v_i \, ds.
\]

Here we assume that \( u^l \) is known and treat it as a given data at each time. At each time step, we have a time independent problem which is similar to static contact problems discussed in [9].

**Incremental problem**

Find \( w \in V \) such that

\[
\begin{align*}
    a(w, v - w) + j_n(w + u^l, v - w) + j_T(w + u^l, v) \\
    - j_T(w + u^l, w) &\geq F_o(v - w) - a(u^l, v - w) \quad \forall v \in V.
\end{align*}
\]
From the similarity of (19) to the static problems, we may follow the same procedure described in [6, 9, 10] to prove the uniqueness and existence of the solutions to (19). In order to simplify the above formulation we denote

\[ g' = g - u'_n. \]

\[ j_n(u, v) = \int_{I_C} c_n(u_n - g')^m v_n \, ds, \quad j_T(u, v) = \int_{I_C} c_T(u_n - g')^m T |v_T| \, ds, \]

\[ F(v) = F_o(v) - a(u', v) \quad \forall u, v \in V. \]

Then we simply have: Find \( w \in V \) such that

\[ a(w, v - w) + j_n(w, v - w) + j_T(w, v) - j_T(w, w) \geq F(v - w) \quad \forall v \in V. \]

This is exactly the same form as the static problem which was proved to have a unique solution [9] under certain constraints: conditions (9)-(11) and we assume that

\[ \sigma_n(u') \in H^{-1/2}(\Gamma_C), \quad \sigma_T(u') \in [H^{-1/2}(\Gamma_C)]^N, \quad u'_n \in H^{1/2}(\Gamma_C). \]

We note that since \( u' \) is treated as a given data, we effectively make regularity assumptions on data in proving the existence of unique solutions to (21).

**Remark.** It is not difficult to show that formulation (19) is equivalent to the following boundary value problem:

\[ \sigma_{ij}(w)_{ij} + F_n + \sigma_{ij}(u')_{ij} = 0 \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \Gamma_D, \]

\[ \sigma_{ij}(w)n_i = t_n - \sigma_{ij}(u')n_i \quad \text{on } \Gamma_N. \quad \sigma_n(w + u') = -c_n(w_n - g')^m \quad \text{on } \Gamma_C, \]

when \( w_n \leq g' \). \( \sigma_T(w + u') = 0 \) and when \( w_n > g' \).

\[ |\sigma_T(w + u')| < c_T(w_n - g')^m \quad \Rightarrow \quad w_T = 0. \]

\[ |\sigma_T(w + u')| = c_T(w_n - g')^m \quad \Rightarrow \quad \exists \lambda > 0 \text{ such that } w_T = -\lambda \sigma_T(w + u'). \]

4. Finite element approximation and a priori error estimations

4.1. \( h-p \) finite elements

We now consider approximations of frictional contact boundary-value problems using \( h-p \) finite elements. Full details of these types of methods can be found in [11–13]. We describe these methods for the two-dimensional case for simplicity.

First, we consider a partition \( \mathcal{T}_h \) of domain \( \Omega \) which divides the domain into a mesh of quadrilateral elements \( K \) each of which is the image of a master element \( K \) under a smooth invertible map \( F_K \):
where \( P_j(\xi) \) is the Legendre polynomial of degree \( j \). With these notations, we define the master element shape functions in three groups:

- Node functions:

\[
x^\Delta(\xi, \eta) = \frac{1}{2} (1 + \xi)(1 + \eta), \quad \Delta = 1, 2, 3, 4, \quad \xi = \xi_1, \quad \eta = \xi_2:
\]

- Edge functions:

\[
x^{1, k}_H(\xi, \eta) = \frac{1}{2} (1 + \eta) \phi_k(\xi), \quad x^{2, k}_H(\xi, \eta) = \frac{1}{2} (1 - \xi) \phi_k(\eta),
\]

\[
x^{3, k}_H(\xi, \eta) = \frac{1}{2} (1 - \eta) \phi_k(\xi), \quad x^{4, k}_H(\xi, \eta) = \frac{1}{2} (1 + \xi) \phi_k(\eta), \quad k = 2, 3, \ldots, p;
\]

- Bubble functions:

\[
x^{B, k}_H(\xi, \eta) = \phi_j(\xi) \phi_k(\eta), \quad 2 \leq j, k \leq p.
\]

The space \( Q^p(\hat{K}) \) of master element shape functions is now defined by

\[
Q^p(\hat{K}) = \text{span}\{x^\Delta, x^{I, k}_H, x^{B, k}_H: 1 \leq I, \Delta \leq 4, 2 \leq j, k \leq p\}.
\]

Then each \( \hat{u} \in Q^p(\hat{K}) \) is representable in the form

\[
\hat{u} = \sum_i a_i \hat{\phi}_i(\xi),
\]

with \( \hat{\phi}_i \) the shape functions defining the basis of \( Q^p(\hat{K}) \).

The element shape functions described above are hierarchical. We also treat the mesh size \( h \) as a parameter by employing standard \( h \)-adaptive techniques for bisection of elements. Here, we introduce typical interpolation properties of \( hp \)-spaces.

**THEOREM 1.** Consider a mesh of elements \( K \) which are affine equivalent to a master element \( \hat{K} \), with

\[
\hat{K} = [-1, 1]^2, \quad \hat{\Omega} = \bigcup_{K \in \mathcal{K}_h} K, \quad x = T_K \hat{x} + a_K, \quad x \in K, \quad \hat{x} \in \hat{K},
\]

where \( T_K \) is an invertible (constant) matrix and \( a_K \) is a translation vector. Let
\[ \rho_K = \sup \{ \text{dia}(S) \mid S = \text{sphere contained in } K \} . \]  
\[ h_K = \text{dia}(K) . \]  

and suppose
\[ \frac{h_K}{\rho_K} \leq \sigma = \text{const.} \quad \forall K \in \mathcal{T}_h . \]  

Then, given any function \( u \in H^s(K) \), there exists a sequence of interpolants \( w^{hp} \in \mathcal{P}_{p_K}(K) \), the space of polynomials of degree \( \leq p_K \) defined on \( K \), \( p_K = 1, 2, \ldots \), and a constant \( C \) independent of \( u \), \( p_K \) or \( h \), such that for any \( r \), \( 0 \leq r \leq s \) and \( \forall K \)
\[ \| u - w^{hp} \|_{r,K} \leq C \frac{h_K^{\mu - r}}{p_K^{s-r}} \| u \|_{s,K} . \quad \mu = \min(p_K + 1, s). \]  

**PROOF.** See [14]. \( \Box \)

Now, we define the space of piecewise continuous polynomials:
\[ V^{hp}(\Omega) \subset H^1(\Omega) , \quad \tilde{\Omega} = \bigcup \{ K \in \mathcal{T}_h \} , \]  
\[ V^{hp}(\Omega) = \{ v \in \mathcal{C}^0(\Omega) : v|_K \in \mathcal{P}_{p_K}(K) \} . \]  

**COROLLARY 2.** Let the conditions (27)-(30) hold. Then, for uniform \( h \) and \( p \), there exists a sequence of interpolants \( w^{hp} \in V^{hp}(\Omega) \), and a constant \( C \) independent of \( u \), \( p \) or \( h \), such that for any \( r \), \( 0 \leq r \leq s \),
\[ \| u - w^{hp} \|_{r,\Omega} \leq C \frac{h^{\mu - r}}{p^{s-r}} \| u \|_{s,\Omega} . \quad \mu = \min(p + 1, s). \]  

**PROOF.** See [14]. \( \Box \)

### 4.2. Finite element approximation

Suppose that the domain \( \Omega \) is exactly covered by finite elements \( K \), i.e.,
\[ \tilde{\Omega} = \bigcup_{K \in \mathcal{G}_h} K . \]  

Let \( V_h \subset V \) be the \( hp \)-finite element space defined by
\[ V_h = [V^{hp}]^N \cap V . \]  

The discrete problems corresponding to (19) are characterized as follows: Find \( w_h \in V_h \) such that \( \forall v_h \in V_h \)
Here we state the a priori estimation theorem.

**Theorem 3.** Let \( w \in [H^s(\Omega)]^N \cap V \) be a solution of problem (19) and \( w_h \) be a solution of the finite element approximation (37). Assume that for \( s > 3/2 \),

\[
(w_n - g^l)^{\mu_t} \in H^{s-3/2}(\Gamma_c), \quad \sigma_T(w + u^l) \in [H^{s-3/2}(\Gamma_c)]^N.
\]

\[ C_\kappa > C_1 \mu_n + C_4 \mu_T. \]

where \( C_\kappa, \mu_n, \mu_T \) are defined as

\[
a(v, v) = C_\kappa \|v\|_1, \quad C_\kappa > 0 \quad \forall v \in V.
\]

\[
\mu_n = \|c_n(s)\|_{L^\infty(\Gamma_c)}, \quad \mu_T = \|c_T(s)\|_{L^\infty(\Gamma_c)}.
\]

Then, there exists positive constants \( K_1 \) and \( K_2 \) such that

\[
\|w - w_h\|_{1,\Omega} \leq K_1 \|w - v_h\|_{1,\Omega} + K_2 \left( \|w - v_h\|_{-s+2,\Omega} \right)^{1/2} \quad \forall v_h \in V_h.
\]

**Proof.** See [9]. \( \square \)

5. A regularized problem

With the same argument stated in [9], we first consider a smooth approximation of the function \( \cdot : [L^q(\Gamma_c)]^N \rightarrow L^q(\Gamma_c) \) and its directional derivative such as

\[
\psi_\epsilon(\xi) = \begin{cases} 
\frac{1}{\epsilon} \left( \binom{\frac{1}{\epsilon}}{\xi} - \frac{1}{3} \binom{\frac{1}{\epsilon}}{\xi} \right) & \text{if } |\xi(s)| \leq \epsilon \ , \\
\frac{1}{\epsilon} \left( \binom{\frac{1}{\epsilon}}{\xi} \right) & \text{if } |\xi(s)| > \epsilon \ . 
\end{cases}
\]

\[
\phi_\epsilon(\xi) = \begin{cases} 
\frac{1}{\epsilon} \left( \binom{\frac{1}{\epsilon}}{\xi} - \frac{1}{3} \binom{\frac{1}{\epsilon}}{\xi} \right) \xi & \text{if } |\xi(s)| \leq \epsilon \ , \\
\frac{1}{|\xi|} \xi & \text{if } |\xi(s)| > \epsilon \ . 
\end{cases}
\]

**Note.** Here, due to numerical difficulties to be explained later, we use a cubic smoothing approximation instead of a quadratic approximation used in [9].

Here, we present a basic property of the regularization function \( \phi_\epsilon \).
LEMMA 4.

\[ |\phi_r(\xi_1) - \phi_r(\xi_2)| \leq \frac{4}{\varepsilon} |\xi_1 - \xi_2| \quad \forall \xi_1, \xi_2 \in [L^q(T_C)]^N. \]

PROOF

We consider four possible cases:

(a) When \(|\xi_1| > \varepsilon\) and \(|\xi_2| > \varepsilon\):

\[ |\phi_r(\xi_1) - \phi_r(\xi_2)| = \left| \frac{1}{|\xi_1|} \xi_1 - \frac{1}{|\xi_2|} \xi_2 \right| = \left| \frac{1}{|\xi_1|} \xi_1 - \frac{1}{|\xi_2|} \xi_2 + \frac{1}{|\xi_1|} \xi_2 - \frac{1}{|\xi_2|} \xi_2 \right| \]

\[ \leq \frac{2}{|\xi_1|} |\xi_1 - \xi_2| \leq \frac{2}{\varepsilon} |\xi_1 - \xi_2|. \]

(b) When \(|\xi_1| \leq \varepsilon\) and \(|\xi_2| > \varepsilon\):

\[ |\phi_r(\xi_1) - \phi_r(\xi_2)| = \left| \frac{1}{\varepsilon} \left( 2 - \frac{|\xi_1|}{\varepsilon} \right) \xi_1 - \frac{1}{|\xi_2|} \xi_2 \right| \]

\[ = \left| \frac{1}{\varepsilon} \left( 2 - \frac{|\xi_1|}{\varepsilon} \right) \xi_1 - \frac{1}{|\xi_2|} \xi_1 + \frac{1}{|\xi_2|} \xi_1 - \frac{1}{|\xi_2|} \xi_2 \right| \]

\[ \leq \frac{2}{\varepsilon} |\xi_1 - \xi_2| \leq \frac{2}{\varepsilon} |\xi_1 - \xi_2|. \]

(c) When \(|\xi_1| > \varepsilon\) and \(|\xi_2| \leq \varepsilon\): The result coincides with case (b).

(d) When \(|\xi_1| \leq \varepsilon\) and \(|\xi_2| \leq \varepsilon\):

\[ |\phi_r(\xi_1) - \phi_r(\xi_2)| = \left| \frac{1}{\varepsilon} \left( 2 - \frac{|\xi_1|}{\varepsilon} \right) \xi_1 - \frac{1}{\varepsilon} \left( 2 - \frac{|\xi_2|}{\varepsilon} \right) \xi_2 \right| \]

\[ = \frac{1}{\varepsilon} \left| 2 \xi_1 - 2 \xi_2 - \frac{|\xi_1|}{\varepsilon} \xi_1 + \frac{|\xi_2|}{\varepsilon} \xi_2 \right| \]

\[ \leq \frac{2}{\varepsilon} |\xi_1 - \xi_2| + \frac{1}{\varepsilon} \left| |\xi_1| \xi_1 - |\xi_1| \xi_2 + |\xi_1| \xi_2 - |\xi_2| \xi_2 \right| \]

\[ \leq \frac{2}{\varepsilon} |\xi_1 - \xi_2| + \frac{1}{\varepsilon} \left( |\xi_1| |\xi_1 - \xi_2| + |\xi_1| - |\xi_2| \right| |\xi_2| \}

\[ \leq \frac{4}{\varepsilon} |\xi_1 - \xi_2|. \]

\(\square\)
Following the same procedure as in [9] and using this regularization function, we now formulate the regularized incremental contact problem: Find \( \omega_\varepsilon \in V \) such that

\[
a(\omega_\varepsilon, v) + j_\varepsilon(\omega_\varepsilon, v) + \langle j_\varepsilon(\omega_\varepsilon, \omega_\varepsilon), v \rangle = F(v) \quad \forall v \in V.
\]

(41)

where

\[
\langle j_\varepsilon(u, w), v \rangle \equiv \int_{\Gamma} c_T(s)(u_n - g^I)^{m_T} \phi_\varepsilon(w_T) \cdot v_T \, ds \quad \forall u, v, w \in V.
\]

(42)

The problem is formally equivalent to the problem defined by a system of equations analogous to (23) but with \( \omega_\varepsilon \) replacing \( \omega \). However, the friction conditions should be replaced by

\[
\sigma_T(w + \bar{u}) = -c_T(s)(u_n - g^I)^{m_T} \phi_\varepsilon(w_T) \quad \text{for} \quad \omega_\varepsilon.
\]

(43)

REMARK. When we think of reducing this quasistatic formulation to the static case, we will have \( u(t) = \text{const.} \) with respect to time. Hence \( u(t) = 0 \) and \( w = u^{I+1} - u^I = 0 \) and \( \sigma_T(w + \bar{u}) = 0 \). i.e., the regularized problem is reduced to a frictionless static problem. But this phenomenon originates from the quasistatic formulation (14). For example, suppose that there is a solution \( u(t) \) for the quasistatic problem (14) such that \( u(t) = 0 \), then

\[
a(u, v) + j_n(u, v) + j_T(u, v) \geq F(v) \quad \forall v \in V.
\]

(44)

Since \( j_T(u, v) \geq 0 \), \( \forall u, v \in V \), any solution of

\[
a(u, v) + j_n(u, v) = F(v) \quad \forall v \in V.
\]

which is the static frictionless formulation, will satisfy the reduced quasistatic problem. While there is no proof for the uniqueness of the quasistatic problem (14), the regularized problem gives a unique frictionless solution which might be undesirable. If so, one can terminate the evolution process whenever the external forces become constant with respect to time.

Using Lemma 4, we prove the strong convergence of \( \omega_\varepsilon \) to \( \omega \) as \( \varepsilon \to 0 \).

THEOREM 5. Let \( w \) and \( \omega_\varepsilon \) be solutions of (19) and (41), respectively. Moreover, let

\[
C_K > C_1 \mu_n + C_2 \mu_T.
\]

Then there exists a constant \( C \) independent of \( \varepsilon \) such that

\[
\|w - \omega_\varepsilon\|_{1,\Omega} \leq C \sqrt{\varepsilon}.
\]

Thus, \( \omega_\varepsilon \to w \) strongly in \( V \) as \( \varepsilon \to 0 \).

PROOF. See [9]. \( \square \)

Using the interpolation properties of \( hp \)-finite elements and following the same procedures as in Theorem 11 [9], we establish the a priori error estimation of the regularized solutions.

THEOREM 6. Let \( \omega_\varepsilon \in [H^1(\Omega)]^N \cap V \) be a solution of the problem (41) and \( \omega_{eh} \in V_h \) be a solution of the finite element approximation. Let the interpolation properties of Corollary 2 for \( hp \)-element spaces hold for \( V_h \) and suppose that
Then, there exists a positive constant $C$ such that

$$
\| w - w_{eh} \|_{1, \Omega} \leq C \left( \frac{h^l}{\rho^{\tau-1}} \| w \|_{s, \Omega} + t = \min\{ p, s - 1 \} \right).
$$

**PROOF.** See [9].

The results of Theorems 5 and 6 now combine to give the following.

**THEOREM 7.** Let the conditions of Theorems 5 and 6 hold. Then the error satisfies

$$
\| w - w_{eh} \|_{1, \Omega} \leq C \left( \sqrt{\epsilon} + \frac{h^l}{\rho^{\tau-1}} \| w \|_{1, \Omega} \right).
$$

**PROOF.** See [8].

6. Numerical examples

We introduced the quasistatic (incremental) problem and the regularization of the non-differentiability of the variation inequality formulation. By substituting variables and scaling, we recast the regularized quasistatic problem into the same form as the static case in [9]. Hence, we could use the proof of the existence and uniqueness of solutions and a priori error estimation of the static problem provided additional assumptions are made on the regularity of $u^l$. For the same reason, we refer the results of numerical experiments of a priori error estimations to the static case in [9].

Here, we focus on the behavior of the solutions of quasistatic problem, which lacks the 'full' proof of uniqueness or existence in time. Therefore we compare the resultant quasistatic solutions with those of the dynamic problem which was proven to have a unique solution. Also, we explore the role of the regularization parameter $\epsilon$ on the convergence behavior in both quasistatic and dynamic cases.

We consider a two-dimensional plane strain problem involving an infinitely long linearly elastic cylinder in contact with a rigid flat foundation. The configuration of the problem is illustrated in Fig. 1. Material properties are taken to be the same as those of [3], i.e., Young’s modulus $E = 1.4 \times 10^3 \left( 10^5 \text{ kg cm}^{-1} \text{ s}^{-2} \right)$, Poisson’s ratio $\nu = 0.25$, and the mass density $\rho = 7 \times 10^{-9} \left( 10^5 \text{ kg cm}^{-3} \right)$, $c_T = 0.3 \times 10^6 \left( 10^5 \text{ kg cm}^{-3} \text{ s}^{-2} \right)$, $m_a = m_T = 2$. No body forces are applied, and external pressure is exerted on the upper boundary of the body. From the symmetry of the problem, only half of the body is considered and the finite element mesh is shown in Fig. 2.

Two different types of loads are considered: ramp and sinusoidal load histories, as shown in Figs. 3 and 4. The time step is taken as 1 and the range of time is from 0 to 40 s. The initial conditions used were the following: the cylinder rests on the rigid flat foundation without any external force and it is in static equilibrium position as $u(x, 0) = 0$ and $\dot{u}(x, 0) = 0$, $x \in \Omega$.

First, we solve the problem using the quasistatic theory. We can consider this problem to be quasistatic, since the load is applied very slowly compared with the speed of elastic waves ($\approx 10^6 \text{ cm s}^{-1}$). The Newton–Raphson method was used to solve the resultant nonlinear
Fig. 1. Configuration of the problem: an elastic cylinder on a rigid foundation.

Fig. 2. The finite element mesh used (84 quadratic elements, 365 dof).
system of equations at each time and backward differences \((\theta = 1)\) was used to discretize variables in time. We also tried various regularization parameters and the resultant stresses are shown in Figs. 5–10 for load type 1 and Figs. 17–22 for load type 2. From Figs. 5–7 and 17–19, we can easily notice that the normal contact pressure does not depend on the regularization parameter while the frictional forces on the boundary vary considerably according to the parameter in both load types.

On the other hand, numerical difficulties were encountered with the quadratic regularization function as its derivative is not continuous, and to overcome these, we employed the cubic regularization function instead of the quadratic smoother. We also observed the same behavior of the solutions described in [3, 8] when the regularization parameter \(\varepsilon\) is too small. We adopt the same technique to make the vibrating node ‘adherent’ by setting the displacement of the node to zero. Most of the time this accelerates convergence of the Newton–Raphson algorithm. But if the penalty parameter \(\varepsilon\) less than \(10^{-4}\) is used, then the scheme fails to converge. This can be explained by comparing Figs. 8–10 and 20–22. Even though the regularization function was designed to make the non-differential functional differentiable and our friction law reduces to Coulomb's law in the limit as \(\varepsilon \to 0\), these solutions are dependent
Fig. 6. Normal contact pressure with load type 1 ($e = 0.02$).

Fig. 7. Normal contact pressure with load type 1 ($e = 0.01$).

Fig. 8. Tangential friction force with load type 1 ($e = 0.1$).
C.Y. Lee, J.T. Oden, Quasistatic frictional contact problems

Fig. 9. Tangential friction force with load type 1 ($\varepsilon = 0.02$).

Fig. 10. Tangential friction force with load type 1 ($\varepsilon = 0.01$).

Fig. 11. Normal contact pressure with load type 1: dynamic case ($\varepsilon = 0.1$).
Fig. 12. Normal contact pressure with load type 1: dynamic case ($\varepsilon = 0.02$).

Fig. 13. Normal contact pressure with load type 1: dynamic case ($\varepsilon = 0.01$).

Fig. 14. Tangential friction force with load type 1: dynamic case ($\varepsilon = 0.1$).
Fig. 15. Tangential friction force with load type 1: dynamic case ($\varepsilon = 0.02$).

Fig. 16. Tangential friction force with load type 1: dynamic case ($\varepsilon = 0.01$).

Fig. 17. Normal contact pressure with load type 2 ($\varepsilon = 0.1$).
Fig. 18. Normal contact pressure with load type 2 ($\epsilon = 0.02$).

Fig. 19. Normal contact pressure with load type 2 ($\epsilon = 0.01$).

Fig. 20. Tangential friction force with load type 2 ($\epsilon = 0.1$).
Fig. 21. Tangential friction force with load type 2 ($\varepsilon = 0.02$).

Fig. 22. Tangential friction force with load type 2 ($\varepsilon = 0.01$).

Fig. 23. Normal contact pressure with load type 2: dynamic case ($\varepsilon = 0.1$).
Fig. 24. Normal contact pressure with load type 2: dynamic case ($\epsilon = 0.02$).

Fig. 25. Normal contact pressure with load type 2: dynamic case ($\epsilon = 0.01$).

Fig. 26. Tangential friction force with load type 2: dynamic case ($\epsilon = 0.1$).
on $\varepsilon$ and as $\varepsilon \to 0$, the solutions become rougher with respect to both time and space. Once again, we note that existence of solutions to the limiting quasistatic formulation has not been proved, and this corresponds to the extreme case for which $\varepsilon \to 0$.

Secondly, we solved the same problem with the full dynamic formulation. We used the Newmark method for time integration with $\beta = 0.25$, $\gamma = 0.5$, and the Newton–Raphson method is used for nonlinearity at each time.

As we expected, the influence of the inertia term is negligible, as shown in Figs. 11–16 and 23–28, and the resultant contact stresses are in good agreement with those of the quasistatic problem in both load types. The normal pressures seem almost independent of the regularization parameter as in the quasistatic case, and the behavior of the frictional forces as $\varepsilon \to 0$ is identical with that obtained by quasistatic formulation. We encounter similar numerical difficulties with the quadratic regularization function and again employ the cubic function instead. We also used the same technique to make vibrating nodes adherent. In this dynamic
problem, the uniqueness and existence of the solutions are proved in [8], but the numerical method is not convergent when $\varepsilon$ is less than $10^{-2}$.

For the examples considered, solutions of the quasistatic problem are essentially the same as those of the full dynamic problem, which is characterized by a system of second order hyperbolic equations. The solutions depend strongly on the regularization parameter and solutions of the quasistatic formulation with Coulomb's friction could not be obtained through our normal compliance law. This may represent another example illustrating the deficiencies of the classical version of Coulomb's friction law and further evidence that it is not appropriate for physically reasonable models of frictional contact.

Acknowledgment

The support of the AFOSR under Contract F49620-91-C is gratefully acknowledged. The partial support of C.Y. Lee under a grant from the Alcoa Foundation is also acknowledged.

References