**h–p adaptive finite element methods in computational fluid dynamics**

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Received 30 August 1990

The principal ideas of h–p adaptive finite element methods for fluid dynamics problems are discussed. Applications include acoustics, compressible Euler and both compressible and incompressible Navier–Stokes equations. Several numerical examples illustrate the presented concepts.

1. Introduction: The search for exponential convergence

By the h–p finite element method we understand a method of piecewise polynomial approximations of functions in which a variation in local mesh size h can be combined with a local variation in the spectral order p of the approximation to yield an optimal rate of convergence in standard norms in terms of the number of degrees of freedom. The word 'spectral' refers to the spectral character of convergence of the p-method. Although various discontinuous versions of h–p methods are possible (see, e.g. [1]), in the 'classical' version of the h–p method the approximation is required to be continuous (we emphasize the importance of this requirement below).

Although advocated piecemeal in engineering literature in various forms for many years, the general idea of varying both h and p is attributed to the pioneering and continuing work of Ivo Babuška and his collaborators since 1980 (see [2–8]). In [5], Guo and Babuška considered a one-dimensional case and showed that, with the help of geometrically graded meshes, one can obtain an exponential rate of convergence for the interpolation error of both regular and singular functions. The result was later generalized by Guo and Babuška in [3, 6] to a two-dimensional case for functions with singular behavior typical of solutions of elliptic equations in two dimensions. In all this work, non-uniform but regular meshing was combined with a uniform order of approximation.

These restricted types of h–p methods have been recently extended to cases in which non-uniform orders of approximation can be combined with h-refined, irregular meshes while maintaining continuity of the approximation by means of constrained approximation [9–11]. The exponential rate of convergence of the interpolation error was once again confirmed through a series of numerical experiments.

In all these developments, the h–p method was applied to model second-order linear elliptic boundary-value problems. The key to the success in these cases is the crucial stability property of well-posed elliptic problems and internal approximations stated in the classical
Cea's lemma (see, e.g. [12–14]) as

\[ \|u - u_h\| \leq C\|u - \Pi_h u\|. \]  \hspace{1cm} (1.1)

where \(\|\|\) is the \(H_1\)-norm, \(u\) and \(u_h\) are the exact and approximate solutions, \(\Pi_h\) is the interpolation operator (thus \(\Pi_h u\) is the interpolant of solution \(u\)), and \(C\) is a stability constant generally independent of \(h\). In other words, the approximation error, understood as the norm of the difference between the exact and approximate solutions, is always bounded by the interpolation error, i.e., the norm of the difference between the solution and its interpolant, and therefore must exhibit the same rate of convergence as the interpolation error.

We demonstrate the point with two examples. The first, illustrated by Fig. 1 and reproduced from [15], compares rates of convergence for a model elliptic problem and various types of mesh refinements techniques, showing clearly the superior, exponential rate of convergence of the \(h-p\) extensions.

In the second example, we compare in Fig. 2 experimental rates of convergence for two \(h\)-adaptive boundary element methods applied to an exterior Neumann problem for the Helmholtz equation using two different formulations: the classical Helmholtz integral equation and a hypersingular formulation. In both cases, the error is measured in the \(L^2\)-norm and the same continuous quadratic elements have been used. Without going into details (see [16]), we emphasize only that the difference between the observed rates of convergence is that they are close to 3 for the first scheme, but only around 1 for the second formulation. As the interpolation error converges in \(L^2\) at a rate equal to 3, the first approximation is clearly optimal while the second (for whatever reasons) is not. This is the crucial point: without a

![Fig. 1. Performance of various types of finite element methods applied to a Poisson problem on a square domain in \(\mathbb{R}^2\), with log of the \(H^1\)-norm of the error plotted versus the logarithm of \(N\), the total number of degrees of freedom. Comparison of experimental rates of convergence for various refinements techniques: (a) \(h\)-uniform, (b) \(p\)-uniform, (c) \(h\)-adaptive, (d) \(p\)-adaptive, (e) \(h-p\)-adaptive (after [15]).](image-url)
method exhibiting good stability properties, the use of $h-p$ interpolations may be useless and the overhead paid for the sophistication of these types of approximation may be lost.

Thus, the first necessary condition for the success of $h-p$ approximation are the good stability properties of the method of approximation. The problem of inventing such methods for different boundary value problems, in particular in CFD, is a challenging one. Due to the irregularity of $h-p$ meshes, techniques for unstructured grids must be used and this implies a strong preference for implicit methods.

The second condition necessary for the success of an adaptive $h-p$ method is the availability of a reliable error estimator and/or error indicator. Again, the problem is not as easy as it looks. When using locally varying $h-p$ meshes, common superconvergence properties of finite element solutions exploited frequently in a posteriori error estimation for the $h$-method, is often lost, and only methods designed to work on unstructured grids survive.

Finally, the last essential ingredient of any adaptive method is an efficient and effective adaptive strategy. Driven usually by local error estimators/indicators, the adaptive strategy should deliver superior (possibly exponential) rates of convergence at a moderate cost. Typical problems concerning, say, $h$-adaptive methods are now complicated by an additional dilemma: how should new local degrees of freedom be adjusted, i.e., should $h$- or $p$-refinements be used? An efficient adaptive strategy should use all available information about the problem being solved, and this may even lead to applications of expert systems methodologies to adaptivity, as suggested by Rank and Babuška in [7].

In this presentation, we summarize recent results in applying our $h-p$ methodology to various problems in CFD. Emphasis is placed on transient formulations. Thus, even when solving for steady state solutions, time-marching schemes are used. As a second practical assumption, we restrict ourselves only to methods of approximation resulting from an FE
approximation in space variables combined with an FD approximation in time. This approach considerably limits the means of analysis omitting, in particular, all space–time elements formulations.

The plan of presentation is as follows. In Section 2 we begin with simple but illustrative examples of linear equations of transient acoustics, turning to the compressible Euler equations in Section 3. In Sections 4 and 5, the idea of viscous operator splitting is reviewed as an approach to both incompressible and compressible Navier–Stokes equations. Some conclusions are finally drawn in Section 6.

2. Linear acoustics

By linearizing the equations of isentropic, compressible flow about a solution to the incompressible Euler equations, we arrive at a nonclassical system of first order hyperbolic equations of linear acoustics with coefficients depending upon the background, incompressible flow (comp. [17]). Symbolically, we can write

$$\text{compressible Euler} = \text{incompressible Euler} + \text{Ma acoustics}.$$ 

where Ma is a reference Mach number. In other words, in a low Mach number regime, the solution to the compressible Euler equations can be decomposed into two parts: a solution to the incompressible Euler equations and a solution to the linear acoustics equations. If the incompressible background flow reduces to an equilibrium state with zero velocity and constant pressure \( p_0 \), the generalized equations of acoustics reduce to the classical form:

$$p_{,t} + c_n \text{div} u = 0, \quad u_{,t} + \frac{1}{c_n} \text{grad} p = 0 \quad \text{in} \ \Omega. \quad (2.1)$$

where \( \Omega \) is a domain of interest, \( c_n \) is the sound speed corresponding to the equilibrium pressure \( p_0 \), and \( p \) and \( u \) are perturbations in pressure and velocity from the equilibrium position. Thus, the equations of acoustics, important enough in themselves, may also serve as a good model problem for compressible Euler equations in a subsonic flow regime with low Mach number.

Valuable insight on the algebraic structure of (2.1) is gained when the equations are cast into the operator form of an abstract Cauchy problem

$$U_{,t} + iA U = 0, \quad U(0) = U^0. \quad (2.2)$$

Here \( U = (p, u^i)' \) is the group variable and \( A \) is an unbounded operator in the \( L^2(\Omega) \) space with properly defined domain (including regularity assumption on \( U \) and built in, homogeneous boundary conditions: for details see [17]), and \( U^0 \) is a vector specifying initial conditions. The operator \( A \) is self-adjoint (in the complex sense), which permits the spectral decomposition of \( A \) in the form

$$A U = \int_{-\infty}^{\infty} \lambda \, dI_{\lambda}(U). \quad (2.3)$$
Here $dI_A$ denotes the spectral family of $A$. For bounded domains $\Omega$, the spectrum of $A$ reduces to eigenvalues $\lambda_0, \pm \lambda_1, \pm \lambda_2, \ldots$ where $\lambda_j$ is a monotone sequence of positive numbers converging to infinity. The spectral representation reduces then to the series representation

$$AU = \sum_{k=\pm \infty} \lambda_k \ dI_k(U) ,$$

where $dI_k$ can now be identified as orthogonal projections onto eigenspaces corresponding to eigenvalues $\lambda_k$. The spectral representation of $A$ is crucial in solving (2.2) in much the same way as one solves simple ordinary differential equations of first order with constant coefficients. The solution to (2.2) takes on the form

$$U(t) = e^{-iA(t)} U(0) = \int_{-\infty}^{\infty} e^{-iA(t)} \ dI_k(U) ,$$

which, in particular, implies that $U(t)$ is the norm preserving (energy conservation)

$$\|U(t)\| = \|e^{-iA(t)} U(0)\| = \|U(0)\| .$$

When approximating (2.2), two paths depicted symbolically in Fig. 3 are possible. In the first method, usually called the method of lines, system (2.2) is first approximated in space variables resulting in a system of ODEs in the form

$$\dot{U}_h + iA_h U_h = 0 , \quad U_h(0) = U^0_h ,$$

where $\dot{U}_h = dU_h/dt, A_h$ is a discrete operator from a finite dimensional finite element subspace $X_h$ of $L^\infty(\Omega)$ into itself that results from the (say $h-p$) FE approximation in space variables.

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**Fig. 3.** Method of lines and method of discretization in time (after [17]).
The system is next discretized using an ODE solver leading to a fully discrete transient operator $T_h$ taking the approximate solution $U_h^n$ at time level $t^n = n \Delta t$ into $U_h^{n+1}$.

In an alternative procedure known as the method of discretization in time (see [18]), the original problem is first approximated in time resulting in a transient operator $T$ defined at the infinite dimensional level: $U^{n+1} = T(U^n)$. An appropriate discretization in space next follows leading to a fully discrete approximation. As discussed in [17], the two formulations yield the same result if no relaxation is used or, simply speaking, if no integration by parts is performed when formulating the infinite-dimensional transient problem.

**Error estimation for the method of lines**

If the usual Galerkin approximation is used, the approximate operator $A_h$ takes the form

$$A_h = P_h \circ A \big|_{X_h} ,$$

where $P_h$ is the $L^2$-projection on the FE space $X_h$ and $A \big|_{X_h}$ denotes the restriction of operator $A$ to $X_h$. The approximate operator $A_h$ can easily be extended to the whole domain of $A$ by setting

$$A_h = P_h \circ A \circ P_h .$$

It is easy to check that $A_h$ is also self-adjoint. Subtracting (2.7) from (2.2) we get

$$(U - U_h) \cdot + i A_h (U - U_h) = -i(A - A_h)U ,$$

which leads to the following representation formula for the error $E_h = U - U_h$:

$$E_h(t) = e^{-iA_h t} E_h^0 - \int_0^t i e^{-iA_h(t-\tau)}(A - A_h)U(\tau) \, d\tau .$$

and an a priori error estimate of the form

$$\|E_h(t)\| \leq \|e^{-iA_h t} E_h^0\| + \int_0^t \|e^{-iA_h(t-\tau)}(A - A_h)U(\tau)\| \, d\tau$$

$$\leq \|E_h^0\| + t \max_{0 \leq \tau \leq t} \|(A - A_h)U(\tau)\|$$

$$\leq \|E_h^0\| + t \max_{0 \leq \tau \leq t} \|AU(\tau) - P_h AP_h U(\tau)\| .$$

Estimate (2.12) can be interpreted as a stability result for the method of lines: the finite element error is bounded by the sum of the error in the initial condition and the error made in approximating operator $A$ by $A_h$. For $AU(t)$ *sufficiently regular*, both terms can yield estimates with exponential rates of convergence.

A total discretization error estimate should, of course, involve an estimate due to the discretization in time.

By rearranging slightly the terms in (2.10), we get

$$(U - U_h) \cdot + iA(U - U_h) = -i(A - A_h)U_h .$$

(2.13)
which leads to a simple a posteriori error estimate of the form

$$\| E_h(t) \| \leq \| E_h^0 \| + \int_0^t \| \dot{U}_h(\tau) + iAU_h \| \, d\tau .$$  

(2.14)

Of course, such an error estimate has only a global character, but usually the element contributions to both terms on the right-hand side of (2.14) can be used as reasonable error indicators. Again, a full a posteriori error estimate should be completed by including an a posteriori error estimate for solving the system of ODEs (using, e.g., the Richardson technique [19]).

**Error estimation for the method of discretization in time**

Use of an implicit Runge-Kutta method for the discretization in time (see [19] for details), due to the spectral decomposition of operator $A$, results in the following formula for the transient operator $T(t)$ at the infinite-dimensional level

$$T(t)U^0 = \int_{-\infty}^{\infty} f(-i\lambda t) \, dI_{\lambda}(U^0) .$$  

(2.15)

where $f(\cdot)$ is a rational function approximating the exponential function. Various versions of Runge-Kutta schemes result in different functions $f$, including Padé or restricted Padé approximations. An explicit formula for the error $E(t) = U(t) - T(t)U^0$ is thus

$$E(t) = \int_{-\infty}^{\infty} (e^{i\lambda t} - f(-i\lambda t)) \, dI_{\lambda}(U^0) .$$  

(2.16)

For an approximation in time of order $m$

$$|e^t - f(x)| \leq c|x|^{m+1} .$$  

(2.17)

we easily get the estimate

$$\| E(t) \| \leq c \int_{-\infty}^{\infty} |\lambda|^{m+1} \, dI_{\lambda}(U^0) t^{m+1} .$$  

(2.18)

provided of course that the right-hand side is finite, i.e., that the function $U_0$ is regular enough. This proves, unfortunately, that the error behaves only algebraically in time, even if no approximation in space is present. An a posteriori version of (2.18) is not readily available.

As before, a complete error estimate must include yet an error estimate for (2.15) due to the approximation in space. An example of such a technique, based on the second order in time Taylor expansion and the element residual method is presented in [17].

As the method of lines yields more readily available a posteriori error estimates and guarantees an optimal (exponential) rate of convergence, it seems to provide a better basis for adaptive methods for transient problems. The situation is different if a steady state solution is to be obtained. The steady state equation
\[ AU = \int_{-\infty}^{\infty} \lambda \, dI_k(U) = 0 \]  

(2.19)

is then replaced with the approximation

\[ U - T(t)U = \int_{-\infty}^{\infty} (1 - f(-i\lambda t)) \, dI_k(U) = 0 \]  

(2.20)

As long as \( 1 - f(-i\lambda t) \neq 0 \) for \( \lambda \neq 0 \), both equations reduce to the condition that \( U \) must come from the null space of \( A \) (of infinite dimension). The point is, however, that both (2.19) and (2.20) are yet approximated in space and operator (2.20) with a proper choice of function \( f \) may have better approximate properties than operator \( A \) itself. Thus the method of discretization in time may provide natural, regularized problems to solve for the steady state solutions.

**Numerical example - The vibrating cylinder problem**

The problem is defined as follows:

1. Governing equations (2.1) are to be solved in the domain

\[ \Omega = \mathbb{R}^2 \setminus \{(r, \theta) | r \leq a\} \]  

(2.21)

2. Boundary condition

\[ u_n = A \sin \omega t [H(t) - H(t - T/2)] \]  

(2.22)

where \( A, \omega, T \) are constants and \( H(\cdot) \) is the Heaviside function.

3. Sommerfeld radiation condition at \( r = \infty \).

4. Initial condition

\[ U_0 = 0 \]  

(2.23)

For computations we accept the computational domain

\[ \Omega = \{(r, \theta) | a < r < r_z, -\alpha < \theta < \alpha\} \]  

(2.24)

The problem was solved using the Taylor-Galerkin method with \( \alpha = 0.5 \) and a constant time step \( \Delta t = 0.015625 \) (see [17]). The following boundary conditions were used in one time-step calculations: boundary condition (2.22) at \( r = a \), \( -\alpha < \theta < \alpha \) and

\[ u_n = 0 \quad \text{at} \quad a < r < r_z, \quad \theta = \pm \alpha \]  

\[ \partial U / \partial n = 0 \quad \text{at} \quad r = r_z, \quad -\alpha < \theta < \alpha \]  

(2.25)

The adaptive strategy was based on \( h \)-refinements. Figure 4 shows a typical evolving mesh with the corresponding solution (pressure) presented in Fig. 5.
3. Compressible Euler equations

The compressible inviscid flow of a perfect gas is governed by the system of conservation laws (Euler equations) of the form

$$ U_t + F(U)_x + G(U)_y = 0 \quad (x, y) \in \Omega \ , \ t > 0 \ . $$

(3.1)
accompanied by the initial condition

$$U(x, y; 0) = U_0(x, y), \quad (x, y) \in \Omega$$  \hspace{1cm} (3.2)

and appropriate boundary conditions.

As a starting point in approximating (3.1), we accept the method of discretization in time and the following finite difference formula resulting from a Taylor series expansion:

$$U(t + \Delta t) - \alpha \frac{\Delta t^2}{2} U_{tt}(t + \Delta t) = U(t) + \Delta t U_t(t) + \alpha \frac{\Delta t^2}{2} U_{tt}(t) + o(\Delta t^3).$$  \hspace{1cm} (3.3)

Let us emphasize that at this point one has complete freedom in selecting the coefficient of implicitness $\alpha \in [0, 1]$. In particular for $\alpha = 0$ the formula will lead to the classical Lax-Wendroff scheme, whereas for $\alpha = \frac{1}{2}$ one obtains the well-known Taylor-Galerkin method of Donea [20].

Next, using the original equation (3.1), we calculate the time derivatives in terms of spatial derivatives as follows:

$$U_t = -F(U)_x - G(U)_y$$
$$U_{tt} = -F(U)_{xx} - G(U)_{yy} = -(AU_t)_x - (BU_t)_y$$

$$= (A^2 U_x + ABU_y)_x + (BAU_x + B^2 U_y)_y.$$  \hspace{1cm} (3.4)

where $A = F_U$, $B = G_V$ are the Jacobian matrices corresponding to fluxes $F$ and $G$.

Replacing the time derivatives in (3.3) by formulas (3.4), we arrive at the following system of second order equations with purely spatial derivatives:

$$U^{n+1} - \alpha \frac{\Delta t^2}{2} [(A^2 U_x^{n+1} + ABU_y^{n+1})_x + (BAU_x^{n+1} + B^2 U_y^{n+1})_y]$$

$$= U^n - \Delta t [F(U^n)_x + G(U^n)_y] + (1 - \alpha) \frac{\Delta t^2}{2} [(A^2 U_x^n + ABU_y^n)_x$$

$$+ (BAU_x^n + B^2 U_y^n)_y].$$  \hspace{1cm} (3.5)

Here $U^{n+1}$ and $U^n$ denote $U(t + \Delta t)$ and $U(t)$, respectively. Formally, the Jacobian matrices $A$ and $B$ on the left-hand side depend on $U^{n+1}$, which makes the problem highly nonlinear. In order to avoid this complication, we linearize the left-hand side by evaluating $A$ and $B$ as functions of $U^n$. Since

$$A(U(t + \Delta t)) = A(U(t) + o(\Delta t)) = A(U(t)) + o(\Delta t).$$  \hspace{1cm} (3.6)

the resulting error, due to the factor of $\Delta t^2$ in front of the second order derivatives, is of third order and the formal second order accuracy-in-time is preserved. Note that for $\alpha = \frac{1}{2}$, the corresponding third order accuracy is lost unless the problem is linear or, alternatively, if a higher order approximation for $A(U^{n+1})$ is used.
Variational formulation

Multiplying (3.5) by a test function \( W = W(x, y) \), integrating over domain \( \Omega \), and integrating by parts the first and second order terms, we arrive at the following weak formulation:

Find \( U^{n+1} \) such that

\[
\int_{\Omega} W^t U^{n+1} \, dx \, dy \\
+ \alpha \frac{\Delta t^2}{2} \int_{\Omega} \left( W^t_{,x} A^2 U_{,x} + W^t_{,x} A B U_{,y} + B A U_{,x} + W^t_{,y} B^2 U_{,y} \right) \, dx \, dy \\
+ \text{boundary terms}
\]

\[
= \int_{\Omega} W^t U^n \, dx \, dy + \Delta t \int_{\Omega} \left( W^t_{,x} F(U^n) + W^t_{,y} G(U^n) \right) \, dx \, dy \\
- \Delta t \int_{\partial \Omega} W^t (F(U^n)n_x + G(U^n)n_y) \, ds \\
- (1 - \alpha) \frac{\Delta t^2}{2} \int_{\Omega} \left( W^t_{,x} A^2 U^n_{,x} + W^t_{,y} A B U^n_{,y} + W^t_{,x} B A U^n_{,x} + W^t_{,y} B^2 U^n_{,y} \right) \, dx \, dy \\
+ \text{boundary terms}
\]

for every admissible test function \( W \). For a detailed discussion of possible different boundary conditions and the resulting boundary terms in (3.7) we refer to [21].

Equation (3.7) is next approximated in space variables replacing both solutions \( U^{n+1}, U^n \), and test function \( W \) with their corresponding \( h-p \) approximations. The resulting fully discrete method is implicit as for every next step solution \( U^{n+1} \) one has to solve the linear system of equations obtained after the Galerkin approximation of (3.7). As the first order, nonlinear fluxes are fully left on the explicit side, the method is frequently referred to as being only \textit{weakly implicit}. It has been shown in [21] that for \( \alpha \geq 0.5 \) the method is \textit{unconditionally linearly stable}, thus allowing (at least when marching toward a steady state solution) for large time steps.

\[ F(U)_{,x} + G(U)_{,y} - \frac{\Delta t}{2} \left[ (A^2 U_{,x} + AB U_{,y})_{,x} + (BA U_{,x} + B^2 U_{,y})_{,y} \right] = 0. \]  

(3.8)

Two immediate observations follow:
- Different values of parameter \( \alpha \), affecting the stability properties of the time marching scheme, play no rule for the steady state limits which are independent of \( \alpha \).
- Every solution to the steady state Euler equations, if sufficiently regular, is automatically a solution to (3.8). Thus if both problems admit unique solutions, the two formulations are equivalent to each other. The problem is, of course, very complicated as no mathematical
theory exists that guarantees that such an existence and uniqueness result holds in multi-
dimensional case for systems of conservation laws.

In practice, (3.5), (3.7) or (3.8) must be augmented by artificial viscosity terms to avoid
spurious oscillations at shocks or contact discontinuities. If those terms depend proportionally
upon the time step \( \Delta t \), the corresponding solutions may depend very strongly upon its choice
and large time steps may be prohibited.

**A posteriori error estimation**

Formally, (3.8) can be rewritten in the operator form

\[
L(U) = 0 .
\]  
(3.9)

where \( L \) is a nonlinear, differential operator defined from a vector space, say \( V \), into its dual
\( V' \). Replacing \( U \) with an approximate \( h-p \) approximation \( U_{h,p} \), we have formally the
nonvanishing residual

\[
L(U_{h,p}) \neq 0 .
\]  
(3.10)

identified as an element of the dual space \( V' \). A natural candidate for an error measure is thus
a norm of this residual defined as

\[
\| L(U_{h,p}) \|_{V'} = \sup_{W \in V} \langle L(U_{h,p}), W \rangle / \| W \|_{V'}.
\]  
(3.11)

where \( \langle , \rangle \) denotes the duality pairing between \( V \) and its dual and \( \| \|_{V'} \) is a proper norm
defined on \( V \). Thus the residual norm depends upon the choice of the norm in the original
space. In [29] we generalized the element residual method (comp. [10, 23]) to both Euler and
Navier–Stokes equations proposing for the choice of the norm in \( V \) the linearized entropy
function

\[
\| W \|_{V}^2 = \int_{\Omega} W^t A_0(U_{h,p}) W \, dx \, dy .
\]  
(3.12)

where

\[
A_0 = H_{tt} .
\]  
(3.13)

is the second order derivative of the entropy function defined as

\[
H = -\rho \dot{s} ,
\]  
(3.14)

with \( \dot{s} \) the physical, specific entropy and \( \rho \) the density functions.

If the nonlinear equation (3.9) is linearized about some reference flow \( V \), the resulting
residual (3.11) is equal to the norm of the FE approximation error measured in norm (3.12).
For a mesh \( \Omega_h = \bigcup \{ K \mid K \in \text{partition } Q_{h,p} \} \), the residual error estimate takes the form

\[
\| L(U_{h,p}) \|_{V'} \leq \left( \sum_K \eta_k^2 \right)^{1/2} .
\]  
(3.15)
where the element error indicators
\[ \eta_K^2 = \int_K \varphi_K^1 A_0 \varphi_K \, dx \, dy \] (3.16)
are evaluated using indicator functions \( \varphi_K \) determined for some local problems including local residuals corresponding to (3.10) (see [10, 29] for details). Typically, these local element contributions to the global error estimate (3.15) are used as a basis for refinements using the equidistribution principle, i.e., simply elements with higher error indicators are refined first. The process of refining and converging to a steady state solution is continued until the estimate of the error is small enough, typically not exceeding five percent of the total entropy (the energy norm squared) corresponding to the solution.

**Numerical examples**

We conclude this section with two numerical solutions to the classical blunt body problem. The following data were chosen:

- ratio of specific heats \( \gamma = 1.4 \),
- free stream Mach number \( Ma = 6.0 \),
- angle of attack = 0.

Figures 6 and 7 present two adaptive meshes obtained using linear and cubic elements with the corresponding density contours. In both cases, the generalized Lapidus artificial viscosity due to Löhner et al. [24] was used but with different constants. \( c = 1 \) for linear elements and \( c = 0.075 \) for the mesh of third order elements. For more examples and details of implementation we refer to [21].

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Fig. 6. Blunt body problem on an \( h \)-adaptive mesh of linear elements. An optimal mesh after two levels of refinements and the corresponding density contours.
4. Compressible Navier–Stokes equations

Strang's operator splitting

One idea of operator splitting resulting in an approximation which is second order accurate in time is due to Strang [25] and is often referred to as viscous splitting or Strang's splitting. The main idea is explained as follows. Consider an arbitrary system of partial differential equations of the form

\[ u_t = c(D^n u, x, t) = \alpha(D^n u, x, t) + b(D^n u, x, t) \quad (4.1) \]

where \( u = u(x, t) \) is a vector of state variables and \( c \) represents an arbitrary function of derivatives \( D^n u \) of \( u \) (up to a certain order), including the function \( u \) itself and the independent space variables \( x \) and time \( t \). As a first step, we split the function \( c \) into two arbitrary contributions \( a \) and \( b \), as indicated in (4.1), and identify two fractional evolution problems of the form

\[ u_\tau = a(D^n u, x, t + \tau), \quad u(x, 0) = u_0(x) \quad (4.2) \]

and

\[ u_\tau = b(D^n u, x, t + \tau), \quad u(x, 0) = u_0(x) \quad (4.3) \]

Introducing the corresponding evolution operators

\[ (M_{\tau}(\tau)u_0)(x) \overset{\text{def}}{=} u(x, \tau) \quad (4.4) \]

where \( u(x, \tau) \) is the solution to the Cauchy problem (4.2) and
with \( u(x, \tau) \) – the solution of (4.3) – we have the fundamental result by Strang:

\[
|(E_1(\tau) - M_{t+\tau/2}(\frac{1}{2}\tau)N_1(\tau)M_{t}(\frac{1}{2}\tau))u_0(x)| \leq c(x)\tau^{3}.
\]

where \( E_1(\tau) \) is the evolution operator corresponding to the original problem (i.e., \( u(x, \tau) = (E_1(\tau)u_0(x)) \)). In other words, to within terms of order \( \tau^3 \),

\[
u(x, t) = (M_{t+\tau/2}(\frac{1}{2}\tau)N_1(\tau)M_{t}(\frac{1}{2}\tau)u_0)(x).
\]
from physical problems with an increasing complexity (Euler equations → Navier–Stokes equations → chemically reactive flows → radiative chemically reactive flows, etc.). In such a situation, there is a good chance that an existing method (code) can be modified by adding the additional two fractional time steps, to solve the more complex phenomena.

Another important observation is the possibility of separation of operators of conflicting nature (transport from diffusion, etc.). Once the operators are separated, highly specialized methods can (at least formally) be used to solve each of the fractional steps.

The challenge in applying Strang's splitting is connected with the study of boundary conditions. As (4.7) is purely a local estimate, incorrect boundary conditions may cause \( c(x) \) in (4.6) to grow toward infinity, which would lead to completely erroneous results. In general, a study of boundary conditions combined with (4.7) should lead to some global estimates.

**Viscous splitting for compressible Navier–Stokes equations**

Given a domain \( \Omega \subset \mathbb{R}^N \) (\( N = 2 \) or \( 3 \)), we consider a system of compressible gas dynamics equations of the form

\[
U_t + \sum_{i=1}^{N} F_i(U)_i = \sum_{i=1}^{N} \left( \sum_{j=1}^{N} K_{ij}(U)U_j \right)_i .
\]  

(4.9)

where

\[
U = \{ \rho, m_1, \ldots, m_N, e \}^T
\]

(4.10)

is the vector of conservation variables with density \( \rho \), momentum components \( m_i = \rho u_i \) (\( u = (u_1) \) being the velocity vector), \( e \) the total energy per unit volume, and commas denoting differentiation with respect to time \( t \) and the spatial coordinates \( (\ )_i = \partial / \partial x_i \). Both Eulerian fluxes \( F_i(U) \) and viscous matrices \( K_{ij}(U) \), \( i, j = 1, \ldots, N \) are (vector-valued) algebraic functions of vector \( U \).

We define

1. The convection operator

\[
(E(t)U_0)(x) \overset{\text{def}}{=} U(x, t) .
\]

(4.11)

where \( U(x, t) \) is the solution to the system of Euler equations (transport step)

\[
U_t + \sum_{i=1}^{N} F_i(U)_i = 0 .
\]

(4.12)

with the initial condition

\[
U(x, 0) = U_0(x)
\]

(4.13)

and appropriate boundary conditions.
2. The diffusion operator

\[ (H(t)U_n)(x) = \frac{\partial}{\partial x}U(x, t) \quad (4.14) \]

where \( U(x, t) \) is the solution to the system of equations (viscous step)

\[ U_t = \sum_{i=1}^{N} \left( \sum_{j=1}^{N} K_{ij}(U)U_{,i} \right)_{,i} \quad (4.15) \]

with initial condition \( (4.13) \) and appropriate boundary conditions.

The three-step Strang's procedure is defined as follows:

\[ U^{n+1} = S(\Delta t)U^n \quad (4.16) \]

where

\[ S(\Delta t) = H(\frac{1}{2}\Delta t)E(\Delta t)H(\frac{1}{2}\Delta t) \quad (4.17) \]

In other words, in order to advance the solution from \( U^n \) to a next step solution \( U^{n+1} \), we solve first with a half-time time step equations \( (4.15) \), then a full time step solution of the Euler equations follows and finally another half-time step solution of \( (4.15) \) concludes the step.

Boundary conditions for both steps are formulated in such a way that the Euler step is solved with boundary conditions suitable for the Euler equations. Thus, at least theoretically, any solver for the Euler equations can be used to solve the transport step. Boundary conditions for the viscous step are formulated in such a way that the overall boundary conditions would converge to well posed boundary conditions for the Navier–Stokes equations. For details we refer to [22].

**Numerical examples**

As an illustration of the method, we present a solution to the classical flat plate problem with Carter's data:

- free stream Mach number \( M_\infty = 3 \)
- Reynolds number \( Re_L = 1000 \)
- Prandtl number \( Pr = 0.72 \)
- ratio of specific heats \( \gamma = 1.4 \) \( (4.18) \)
- free stream temperature \( \theta_\infty = 390 \, {^\circ}R \)
- solid wall temperature \( \theta_{\text{wall}} = 1092 \, {^\circ}R \)
The geometry of the problem is shown in Fig. 9. The problem is solved using the Taylor–Galerkin method to solve the Euler step and the Crank–Nicholson scheme to solve the viscous step (in this way second order time accuracy overall is preserved). Both discretizations in time were combined with an $h$–$p$ approximation in space. Figure 10 presents an initial $h$–$p$ mesh and three consecutive $h$-adaptive meshes obtained using the residual error estimate and adaptive strategy discussed in the previous section. Second and third order elements are generated only along the plate. Except for the immediate vicinity of the stagnation point the higher order approximation was restricted to the vertical direction only resulting in ‘$p$-anisotropic’ elements and essential savings in terms of the number of degrees of freedom.

Density and pressure contours are shown in Figs. 11 and 12 and corresponding heat flux coefficient profiles along the plate are shown in Fig. 13. Notice the improving convergent character of the solution.

**Compressible flow over a ramp**

An optimal $h$–$p$ mesh, with anisotropic $p$ in the boundary layer, is shown in Fig. 14 for the problem of compressible flow over a $20^\circ$-inclined ramp. Inflow Mach number is 5.0 with Re $= 30,000$. Density contours are shown in Fig. 15 and the recirculation velocity is depicted in Fig. 16.

**Three-dimensional results**

Figure 17 shows preliminary three-dimensional results obtained with the adaptive $h$–$p$ methods. Shown are density contours over a 3D-mesh of linear and cubic elements. Further discussion of the three-dimensional calculations is scheduled for a later paper. Figure 18 shows a typical three-dimensional mesh structure with some elements removed to expose spectral orders chosen on element sides, faces and interiors. The idea is that the user may interrogate the mesh to view, via a color circle, the particular distribution of element spectral orders that the code has selected to resolve the problem to a specified accuracy level.
Fig. 10. Carter’s problem. Various stages of adaptive meshes.
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Fig. 10. (continued).
Fig. 11. Carter's problem. Mesh 3. Density contours.

Fig. 12. Carter's problem. Pressure contours.
Fig. 13. Carter's problem. Heat flux coefficient along the plate.

Fig. 14. An optimal $h-p$ mesh for compressible viscous flow over a ramp: $Ma = 5.0$, $Re = 30,000$. 
Fig. 15. Density contours.

Fig. 16. Recirculation velocity $u_1$. 
Fig. 17. Three-dimensional $h-p$ adapted mesh for compressible flow over a wedge: mesh and density contours.

Fig. 18. Mesh slicing to reveal computer-selected polynomial degrees in an adapted 3D $h-p$ mesh.
5. Incompressible Navier–Stokes equations

The flow of a viscous incompressible fluid through a domain $\Omega \subset \mathbb{R}^N \ (N = 2, 3)$ is characterized by the incompressible Navier–Stokes equations

$$u_t + u \cdot \nabla u - \nu \Delta u + \nabla p = 0, \quad \text{div} \, u = 0,$$  \hfill (5.1)

accompanied by the initial condition

$$u(0) = u^0$$ \hfill (5.2)

and appropriate boundary conditions. Here $u = u(x, t)$ is the nondimensionalized velocity of the fluid at point $x \in \Omega$ and time $t > 0$. $p = p(x, t)$ is the pressure at $(x, t)$. $\Delta u = \nabla \cdot \text{div} \, u - \nabla \times \text{curl} \, u$ is the vector Laplacian of $u$, and $\nu > 0$ is the reciprocal of the Reynolds number.

Following Strang's idea, Beale and Majda [26] developed an operator splitting method of the form:

1. convection

$$u_t + u \cdot \nabla u + \nabla p = 0, \quad \text{div} \, u = 0.$$ \hfill (5.3)

2. diffusion

$$u_t - \nu \Delta u = 0.$$ \hfill (5.4)

and proved that the corresponding two step procedure (convection plus diffusion) is of first order in time while the Strang's splitting (half diffusion plus convection plus half diffusion) is of second order (this is not a trivial extension of Strang's result since (5.1) contains the incompressibility constraint and therefore is not in a form suitable for a direct application of Strang's method).

Another operator splitting method was proposed by Pironneau in [27]:

1. convection

$$u_t + u \cdot \nabla u = 0.$$ \hfill (5.5)

2. diffusion

$$u_t - \nu \Delta u + \nabla p = 0, \quad \text{div} \, u = 0.$$ \hfill (5.6)

The corresponding two step procedure (convection plus diffusion) is only first order in time and it does not seem that the three step procedure would improve the time accuracy. Even though it is a first order splitting only, it does have a few important advantages: the convection step is solved locally using the method of characteristics and, after applying the implicit Euler approximation to (5.6), the diffusion step reduces to the modified linear Stokes problem; the whole procedure is unconditionally stable (see [27] for details) and it allows for the $h-p$ approximations in the space variables.
Reduced integration penalty (RIP) method

The modified Stokes problem

\[ u^{n+1} - \nu \Delta t \Delta u^{n+1} + \nabla p = u^{n+1/2}, \quad \text{div } u^{n+1} = 0 \]  

(5.7)

where \( u^{n+1/2} \) is the solution of the transport step (5.5) (the solution \( u^n \) transported along the characteristics) can be solved using the RIP method, which leads to the following variational formulation:

Find \( u^{n+1} \) such that

\[ \int_{\Omega} u^{n+1} v \, dx + \nu \Delta t \int_{\Omega} \nabla u^{n+1} \cdot \nabla v \, dx + \frac{1}{\varepsilon} \text{RI}(\text{div } u^{n+1}, \text{div } v) + \text{BT} \]

(5.8)

\[ = \int_{\Omega} u^{n+1/2} v \, dx \text{ for every admissible } v, \]

where \( \text{RI} \) denotes the reduced integration rule for the penalty terms, \( \varepsilon \) is the penalty parameter and \( \text{BT} \) stand for boundary integrals resulting from boundary conditions.

Formally (5.8) can be approximated using an arbitrary \( h-p \) approximation. In practice, determining the reduced integration rules for locally variable order of approximation turns out to be very difficult and we have restricted ourselves to various but uniform order elements only. This, of course, still leaves place for using \( h \)-refinements. For meshes of elements of uniform order \( p \), the usual Gauss–Legendre \((p+1)^2\)-integration rule is used with the same rule but only \((p+1-r)^2\) points applied to integrate the penalty term. Parameter \( r \) is called the level of underintegration. For a numerical study on both \( h \) and \( p \) convergence as well as the effect of various choices of \( r \) we refer to [28].

Numerical example

The classical driven cavity problem for Reynolds number \( \text{Re} = 5000 \) in the square domain

![Fig. 19. \( h-p \) driven cavity problem with \( \text{Re} = 5000 \). An \( h \)-adaptive mesh for quartic elements (\( p = 4 \)) with second level of underintegration.](image-url)
Fig. 20. Driven cavity problem with Re = 5000. Steady state velocity pattern.

Fig. 21. Driven cavity problem with Re = 5000. Steady state solution – pressure contours.
Fig. 22. Driven cavity problem with $\text{Re} = 5000$. Steady state solution - velocity contours.

$\Omega = (0, 1)^2$ was solved using an $h$-refined mesh of fourth order elements ($p = 4$) shown in Fig. 19 (3521 single degrees of freedom). The second level of underintegration ($r = 2$) was used. Figure 20 presents the resulting steady state velocity field and the corresponding pressures are shown in Fig. 21. Contours of the $x$- and $y$-velocity components are given in Fig. 22. Results compare favorably with those reported in [22].

Acknowledgment

We wish to thank Dr. Waldek Rachowicz, Dr. Tad Liszka and Mr. A. Safjan for assistance in performing the numerical experiments reported in this work. Our work on error estimation and adaptive $h$--$p$ methods was supported under Office of Naval Research, Grants Nos. N00014-89-J-3109 and N00014-89-J-1451, NASA Contract NAS1-18746, and ARO Contract DAAL03-89-K-0120.

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