Highly Accurate Adaptive hp-Methods for Linear Elastostatics

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Abstract. A new class of highly accurate, self-adaptive computational methods for two- and three-dimensional elastostatics problems is presented which can deliver solutions to general boundary-value problems in elasticity with any prescribed level of accuracy. The techniques feature the use of optimal adaptive control strategies to manage mesh size and spectral order as free discretization parameters. These are automatically selected to yield solutions with any preassigned accuracy with a near-minimum number of degrees of freedom.

Introduction

The goal of the mathematical theory of elasticity is to characterize and predict the behavior of elastic bodies under the action of prescribed forces and displacements. For over three decades, finite element methods have provided a powerful and general tool for generating approximate solutions of the equations governing elastostatics; but numerical solutions are, after all, only approximations of the "exact" solutions which are known in closed form for a few particular problems. These classical solutions can, in general, furnish quantitative descriptions of various features of elastic deformation to any desired accuracy only by expanding and evaluating series or integral forms of the solution generated by classical mathematical methods.

In the present study, adaptive $hp$-finite element methods are described which can automatically produce solutions with error (in appropriate norms) below any tolerance, with the provision that these target errors are given by a posteriori error estimators. These error estimates, on the other hand, can be shown to be asymptotically exact in many cases; i.e., as the mesh is appropriately refined, the approximate error in, say, strain energy can converge to the true error inherent in a given discrete model of the problem.

The problem class under consideration is characterized by the following weak or variational boundary-value problem:

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Find the displacement field \( u = u(x) \in V + \{ \hat{u} \} \), satisfying boundary conditions \( u = \hat{u} \) on \( \Gamma_D \) such that
\[
a(u, v) = F(v) \text{ for all } v \in V
\]
(1)

Here,
\[
V = \{ v = v(x) \text{ = test functions (virtual displacements) such that} \}
\]
\[
v \equiv 0, \text{ on } \Gamma_D \text{ and } v \text{ produces finite energy,}
\]
\[
\| v \|_E = a(v, v) < \infty
\]
\[
a(u, v) = \text{ the virtual work of the internal forces}
\]
\[
= \int_\Omega \text{tr} \nabla v^T E \nabla u \, dx
\]
\[
F(v) = \text{ the virtual work of the external forces (the body forces } f
\]
\[
\text{and the surface tractions } g \text{ on } \Gamma_{\sigma})
\]
\[
= \int_\Omega f \cdot v \, dx + \int_{\Gamma_{\sigma}} g \cdot v \, ds
\]

Here \( \Omega \) is a bounded domain in \( \mathbb{R}^n (n = 2 \text{ or } 3) \) with boundary \( \partial \Omega = \Gamma_D \cup \Gamma_{\sigma} \) and \( E \) is Hooke's tensor. Under standard assumptions on \( E \) and the regularity of the data, problem (1) has a unique solution. Under further assumptions on the smoothness of \( u \), the solution is also a solution of the classical elastostatics problem,
\[
- \nabla \cdot E \nabla u = f \text{ in } \Omega
\]
(2)

with boundary conditions \( n \cdot E \nabla u = g \) on \( \Gamma_{\sigma} \) and \( u = \hat{u} \) on \( \Gamma_D \).

\pAdaptive FEM

We construct a fully adaptive \( hp \)-finite element approximation of (1) which has the following features:

\hpmeshes. The domain \( \Omega \) is partitioned into elements \( \Omega_K (\bar{\Omega} = \bigcup_K \bar{\Omega}_K) \) which are images of a master cube \( \bar{\Omega} = [-1, 1]^3 \) under smooth invertible (e.g. isoparametric) maps, over which components of the displacement field are approximated using tensor products of Legendre polynomials of degree \( p \).

The finite elements are thus, hierarchic \( p \)-version elements (see, e.g. (Szabo and Babuska, 1991)), but these are used within the general \( hp \)-data structure described by (Demkowicz et al, 1989). This data structure accommodates automatic refinement and derefinement of the mesh size, \( h_K = \text{dia}(\Omega_K) \), and automatic enrichment or "de-enrichment" of the spectral order \( p_K \). Full continuity of the finite element approximation \( u^{hp} \) across interelement boundaries is maintained by enriching edge shape functions, as described by (Demkowicz et al. 1989). The resulting meshes have adjustable local mesh size \( h_{|\Omega_K} = h_K \) and adjustable spectral order (polynomial degree) \( p_{|\Omega_K} = p_K \). A typical two dimensional \( hp \)-mesh is shown in Figure 1 where various shades indicate various orders of \( p \).
Figure 1: hp-mesh

A Posteriori Error Estimation. Let $u^{hp} \in V^k \subset V$ be the hp-finite element approximation of $u$ corresponding to an arbitrary $hp$-mesh and let $u_K^{hp}$ be its restriction to an element $\Omega_K$. The interior residual is

$$ r_K = f|_K + \nabla \cdot E \nabla u_K^{hp} \quad \text{in} \quad \Omega_k $$

and the averaged local error flux on the interelement boundary $\Gamma_{KL} = \partial \Omega_K \cap \partial \Omega_L$ is characterized by the functional,

$$ < n_{KL} \cdot E \nabla u_K^{hp} > = n_K \cdot [(1 - \alpha_{KL}) E \nabla u_K^{hp} + \alpha_{KL} E \nabla u_L^{hp}] $$

where $\alpha_{KL}$ is a function with values in $[0,1]$ ($\alpha_{KL} + \alpha_{LK} = 1$). Using test functions $\omega \in V^{h_{K(p+1)}}$, we compute a local error indicator, $\varphi_K$, by

$$ a_K(\varphi_K, \omega) = \int_{\Omega_K} r_K \cdot \omega dx + \int_{\partial \Omega_K} \omega \cdot < n_K \cdot E u_K^{hp} > ds \quad (3) $$

Then the estimated global error in energy is bounded by (see Ainsworth and Oden, 1991)

$$ \| e \|_E^2 \approx \eta^2 = \sum_K a_K(\varphi_K, \varphi_K) \quad (4) $$

Under appropriate conditions, this estimator is asymptotically exact and, with proper choices of $\alpha_{KL}$, usually produces local effectivity indices $\gamma_K = \eta_K / \| e \|_{E,K}$ of near unity for arbitrary $p$. Further details are given in (Ainsworth and Oden, 1991).

Adaptive Strategy. An adaptive algorithm is used to adjust $h_K$ and $p_K$ until the global error estimate $\eta = \sqrt{\sum_K a_K(\varphi_K, \varphi_K)}$ is less than or equal to a target error $\eta_T$ (say $\eta_T = 5$ percent of $\| u \|_E$).
Figure 2: Performances of the adaptive scheme

Results
Numerous results have been obtained using the above strategy. In general, this adaptive process leads to exponential convergence rates and very good resolution of the solution with a near minimum number of degrees of freedom. Some typical results are summarized in Figure 2 for an analysis of torsion of an L-shaped bar with a $r^{3/2}$ singularity in the reentrant corner. The $hp$-adaptive scheme is seen to converge exponentially to a solution with a target error of 3 percent and to produce this level of accuracy with only around 100 degrees of freedom. To obtain this accuracy using conventional $h$-version FEM’s with $p = 1$ and uniform refinement requires around 10,000 degrees of freedom. A representative $p$-version required around 1000 degrees of freedom.

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References

