NUMERICAL SOLUTION OF THE EVOLUTION EQUATIONS OF DAMAGE AND RATE-DEPENDENT PLASTICITY

JON M. BASS† and J. T. ODEN
Texas Institute for Computational Mechanics, The University of Texas at Austin, Austin, TX 78712, U.S.A.

Abstract—The evolution equations of damage theory and viscoelasticity are stiff ordinary differential equations and present numerous difficulties when one attempts to solve them numerically. This is particularly true when cyclic loading effects are to be simulated as few numerical schemes are stable when significant jumps in stresses occur. The present paper explores the performance of several numerical schemes for solving these evolution equations and presents methods which provide both accurate and stable simulations of large classes of rate-dependent problems.

1. INTRODUCTION

In recent years, the success of modeling progressive damage and rate-dependent plasticity have led to the application of such theories to an increasing list of engineering problems. Typically, such theories are characterized by constitutive equations which include evolution equations for some type of internal variable which could represent such features as a loss of stiffness due to an increase in microcrack density, hardness, plastic strain, dislocation density, etc. While such phenomenological theories can be very effective in modeling history effects, damage, viscoplastic deformation, and other phenomena, the numerical integration of the equations often presents serious difficulties, particularly when cyclic loading cases are considered. These difficulties are related to the mathematical stiffness inherent in damage theories and in internal-state-variable formulations, with the result that many of the standard numerical schemes, particularly the explicit schemes, encounter serious stability or convergence problems.

Several computational schemes have been proposed in the literature for solving initial-boundary-value problems in viscoplasticity. These schemes include explicit and implicit methods of time integration which are used in conjunction with both constant stiffness [1-7], and tangent stiffness [8-11] formulations of the equilibrium equations. Due to the stiffness of the evolution constitutive equations, many of these schemes are only conditionally stable and may produce results which diverge rapidly from the true solution when applied to an arbitrary history of loading.

For this reason, an investigation of several such schemes is taken up in the present paper and two schemes are identified which appear to yield acceptable results when applied to problems with arbitrary loading histories. These techniques include an Euler forward predictor with trapezoidal corrector and time step control, given in [12], and Gear’s stiffy stable methods with time steps selected numerically approximating the truncation error; see Refs [13, 14]. These integration techniques have been implemented in a new algorithm which appears to be more computationally efficient than those previously proposed in the literature. Our primary mission here is to review schemes which can be used successfully for these classes of problems and to present results of applications to representative problems.

This paper is divided into seven sections. Following this introduction, a brief discussion of typical models is presented. This overview includes a discussion of the general mathematical structure of these models, and a synopsis of several recently proposed theories of this type. Section 3 discusses the problem of mathematical stiffness of the governing equations and
J. M. BASS and J. T. ODEN

presents a weak formulation of the problem. Section 4 and 5 are devoted to the efficiency and reliability of several computational schemes for solving initial-boundary-value problems in internal-state-variable viscoplasticity. Here we are able to show that some methods are unsuitable for use in general purpose finite element codes while others appear to be robust, stable and efficient for certain problems. The final two sections present the results of some numerical test cases for two representative constitutive theories.

2. MODELS OF VISCOPLASTICITY AND DAMAGE

This section contains a brief discussion of internal-state-variable models for metals exhibiting time-dependent nonelastic deformation.

Mathematical structure

Many of the internal-state-variable theories and damage theories for infinitesimal nonelastic deformation have the same general structure, typified by the following properties:

1. A strain-rate decomposition of the form

\[ \dot{\epsilon} = \dot{\epsilon}' + \dot{\epsilon}'' \]

where \( \dot{\epsilon} \) is the total strain rate tensor, \( \dot{\epsilon}' \) is the elastic strain rate tensor, and \( \dot{\epsilon}'' \) is the nonelastic strain rate tensor which includes both a time independent inelastic component and a time dependent anelastic component.

2. The nonelastic strain rate is a function of the stress, the damage, and a set of internal state variables

\[ \dot{\epsilon}'' = f(\sigma, d, z_k) \]

where \( \sigma \) is the stress tensor, \( d \) is the damage, and \( z_k \) is a set of state variables, and both \( d \) and \( z_k \) may be tensors and/or scalars.

3. The state variables vary along a loading path according to certain laws, and the history dependence of the rate of nonelastic strain, up to the current time, is completely characterized by the current values of the damage and the state variables. The constitutive relations for the evolution of the damage and the state variables are of the general form

\[ \dot{d} = D(\sigma, d, z_k) \]
\[ \dot{z}_i = g_i(\sigma, z_k). \]

4. Often the nonelastic deformation rate is deviatoric.

\[ \text{tr} \dot{\epsilon}'' = 0. \]

5. There need not exist yield criteria nor loading or unloading conditions. Hence, nonelastic deformation is assumed to occur at all stages of loading.

These five properties characterize the general structure of most of the damage and/or internal-state variable models, although some models may deviate slightly from the above descriptions. A list of some representative examples follows:

The Bodner and Partom/Bodner and Stouffer's Theories [15, 22]

In the period 1979–1983, an anisotropic hardening law was proposed by Bodner and Stouffer [18, 21] in which a tensor relationship for the nonelastic strain rate and a single scalar state variable equation appear. This model also uses a hardness tensor which is related to the single internal state variable and is responsible for the anisotropic material characterization. The nonelastic deformation rate in an anisotropic formulation is given by

\[ \dot{\epsilon}_{ij}'' = \frac{D_0 S_{ij}}{\sqrt{J_2}} \exp \left( -0.5 \left( \frac{z^2_{ij}}{3J_2} \right)^n \left( \frac{n + 1}{n} \right) \right). \]
Here $D_0$ is a scale factor, $z_q$ a hardness tensor, $S_q$ are the deviatoric stress components, $J_2$ the second invariant of the stress deviator, and $n$ is a material constant related to the rate sensitivity.

In this model, the single internal state variable is the plastic work, which has the constitutive form

$$\dot{\varepsilon} = S_q \varepsilon_{ij}.$$

**Hart's Theory** [23, 27]

The equations for the nonelastic deformation rate are given by

$$\dot{\varepsilon} = \frac{\hat{\sigma}^*}{||S - \mu a||} \left(\frac{||S - \mu a||}{||S - \mu a||}^M \right) (S - \mu a)$$

where $\hat{\sigma}^*$, $M$ and $\mu$ are material constants. $S$ is the stress deviator. $a$ is a tensor internal state variable, and $||a|| = \sqrt{a_{ij}a_{ij}}$. The evolution equations for the internal state variables, $\sigma^*$ and $a$, are

$$\dot{\sigma}^* = \sigma^* \Gamma \frac{\varepsilon^*/(\ln(\sigma^*/||\mu a||))^{1/\lambda}}{||\mu a||}$$

and

$$\dot{a} = \dot{\varepsilon} - \frac{\varepsilon^*/(\ln(\sigma^*/||\mu a||))^{1/\lambda}}{||\mu a||} \mu a.$$ 

Here $\Gamma$ is a material function of $\sigma^*$ and $||\mu a||$. $\lambda$ is a material constant, and $\dot{\varepsilon}^*$ is a function dependent on the temperature and $\sigma^*$.

**The Gillis and Jones Theory** [28]

This theory, proposed for polycrystalline metals, is limited to materials having a linear dependence of dislocation velocity on the stress and contains only one internal state variable, the nonelastic strain.

The nonelastic strain rate is

$$\dot{\varepsilon}^* = \phi f \rho^* \nu_0 (\beta + \alpha \varepsilon^*) (\sigma/\sigma_0 - (1 + h \varepsilon^*))$$

Here $\phi$ is an orientation factor, $b$ is the magnitude of the Burgers vector, $f$ is the fraction of mobile dislocations, $\rho^* \beta$ is the initial dislocation density, $\rho^* \alpha$ is a dislocation multiplication coefficient, $\nu_0$ is the dislocation speed produced by a stress of magnitude $\sigma_0$, and $h$ is a strain hardening coefficient. In this expression, $\langle \cdot \rangle$ denotes use of the step function $\langle \psi \rangle = 0$ if $\psi < 0$ and $\langle \psi \rangle = \psi$ if $\psi > 0$.

**Robinson's Theory** [29, 30]

In this model, the multiaxial representation for the nonelastic deformation rate is given by

$$\dot{\varepsilon}_{ij} = \frac{1}{2} \left[ \frac{(S_{kl} - \alpha_{kl})(S_{kl} - \alpha_{kl})(M - 1)^2}{2K^2} \right] \varepsilon_{ij} - \alpha_{ij}.$$

where $S_{ij}$ is the deviatoric stress tensor, $\alpha_{ij}$ is the tensor-valued internal state variable $(\alpha_{ij} = z_{ij})$, and $M$ and $K$ are material constants. The evolution equations for the internal state variable, given in a work hardening-recovery format, are

$$\dot{\alpha}_{ij} = \frac{2\mu H \varepsilon_{ij}}{(\alpha_{kl} \alpha_{kl})^{\mu^2}} - R \left( \frac{\alpha_{kl} \alpha_{kl}}{2K^2} \right)^{(n - \beta - 1)/2} \alpha_{ij}.$$

Here $\mu$, $H$, $n$, $\beta$ and $R$ are also material constants.

**Miller's Theory** [31, 32]

The development of this model stems from a desire to accurately model steady-state creep deformation. Using this as a basis, the following one-dimensional generalized form for the
nonelastic deformation rate is obtained:

\[ \dot{\varepsilon} = B \theta \left[ \sinh \left( \frac{|\sigma - R|^{1.5}}{D} \right) \right]^n \text{sgn} (\sigma - R), \]

where

\[ \theta' = \exp \left[ \left( \frac{-Q}{0.6KT_m} \right) - \left( \ln \left( \frac{0.6T_m}{T} \right) + 1 \right) \right]; \quad T \leq 0.6T_m \]

\[ = \exp \left[ \frac{-Q}{KT} \right]; \quad T > 0.6T_m \]

In these expressions, \( B \) and \( n \) are material constants, \( R \) and \( D \) are internal state variables representing the back stress and drag stress, respectively, \( \sigma \) is the applied stress, \( Q \) is an activation energy, \( K \) is Boltzmann's constant, \( T \) is the temperature, \( T_m \) is the melting temperature, and \( \text{sgn} \) represents the signum function. The evolution equations for the internal state variables are

\[ \dot{R} = H_1 \varepsilon^n - H_1 B \theta' \left( \sinh (A_1 |R|) \right)^n \text{sgn} (R) \]

\[ \dot{D} = H_2 \varepsilon^n \left( C_2 + |R| - A_2/A_1 \right) D^3 - H_2 C_2 B \theta' \left( \sinh (A_2 D^3) \right)^n. \]

Here \( H_1, H_2, C_2, A_1, \) and \( A_2 \) are also material constants.

**The Krieg, Swearengen and Rhodes Theory [33]**

In 1978, Krieg et al. proposed a unified model for creep and plasticity in metals, using two internal state variables to reflect the current microstructural state. The model is applicable to metals under isothermal conditions in the temperature range of 0.3 \( T_m \) to 0.7 \( T_m \), where \( T_m \) is the homologous temperature.

The constitutive equations for the non-elastic deformation rate are based on a general function form for dislocation glide given by

\[ \dot{\varepsilon}^n = \dot{\varepsilon}_0 \left[ \frac{||S - \alpha||}{R} \right]^n \left( \frac{S - \alpha}{||S - \alpha||} \right). \]

In this expression, \( S \) is the deviatoric applied stress, \( \alpha \) is the back stress tensor, \( R \) is the drag stress, \( \dot{\varepsilon}_0 \) and \( m \) are temperature-dependent material constants, and \( ||\sigma|| = \sqrt{\sigma_{ij}\sigma_{ij}} \). The evolution equations for the two internal state variables are postulated as

\[ \dot{a} = A_a \varepsilon^n - \frac{r_a \alpha}{||\alpha||} \quad \text{and} \quad \dot{R} = A_R ||\varepsilon^n|| - r_R \]

where \( A_a, A_R, r_a \) and \( r_R \) are hardening and recovery functions, respectively; Specific forms for these hardening and recovery functions are given in [33].

**Cernecky and Krempl's Theory [34, 38]**

In 1980, Krempl and associates proposed a coupled infinitesimal theory of thermo-visco-plasticity. This constitutive model is based on a nonlinear, multiaxial generalization of the standard linear solid, which consists of a spring in parallel with a Maxwell element. This generalization leads to a proposed constitutive form of

\[ \sigma_{pq} + K_{pqmn} (\sigma, \varepsilon, T) \varepsilon_{mn} = G_{pq} (\varepsilon, T) + M_{pqmn} (\sigma, \varepsilon, T) \dot{\varepsilon}_{mn} \]

where \( K_{pqmn}, G_{pq}, \) and \( M_{pqmn} \) are material functions determined from experimental data, and

\[ K_{lmnp} M_{mklt} = D_{ijkl}. \]

\( D_{ijkl} \) being the tensor of linear elastic moduli. Solving eqn (2.1) for \( \sigma_{mn} \) and using equation (2.2) gives the following relation for the nonelastic strain rate

\[ \dot{\varepsilon}_y^m = M_{ijmn} (\sigma, \varepsilon, T) [\sigma_{mn} - G_{mn} (\varepsilon, T)]. \]
Also by using the relation

\[ \dot{\varepsilon}_{ma} = D_{iin} \dot{\alpha}_{kl} + \dot{\varepsilon}_{ma}^* \]

for the total strain rate in (2.3), a final form for the nonelastic strain rate is obtained which is a function only of the stress and internal variable.

The Cescotto and Leckie Theory [39]

The constitutive forms for the nonelastic strain rate and internal variables are

\[ \dot{\varepsilon}^n = f\left( \frac{\sigma - \alpha_2}{\alpha_3} \right) \text{sgn}(\sigma - \alpha_2) \]

\[ \dot{\alpha}_2 = \frac{3}{2} h_a \varepsilon^n - r_a \alpha_2 \]

and

\[ \dot{\alpha}_3 = h_k - r_k \]

where \( f \) is the nonelastic strain rate function, \( h_a \) and \( h_k \) are hardening functions, and \( h_a \) and \( r_k \) are recovery functions. Here \( \text{sgn} \) represents the signum function which takes on the values of 1, 0, and -1, depending on whether the argument is positive, zero or negative. No particular form of the functions \( f, h_a, h_k, r_a \) and \( r_k \) is assumed \textit{a priori}, and a set of experiments is required to define those functions.

3. STIFFNESS AND APPROXIMATION

A wide range of techniques has been proposed in the literature to numerically integrate the evolution equations described above. These methods include constant stiffness and tangent stiffness formulations of the equilibrium equations used in conjunction with either explicit or implicit time integration techniques. Testing of these techniques is often done in the context of a monotonic loading situation and for restricted forms of the constitutive equations, even though arbitrary loading histories, especially cyclic loading situations, often present very severe numerical difficulties.

In this section, we examine several computational methods and integration techniques. We focus on computational issues, stability and overall performance of representative algorithms. For definiteness, we confine our attention to internal state variable theories appropriate for quasi-static, infinitesimal, isothermal deformations.

The general form of the constitutive relationships given above is:

\[ \dot{\varepsilon}^n = f(\sigma, Z_k) \]

\[ \dot{Z}_i = g_i(\sigma, Z_k) \]

\[ \dot{\sigma} = E(\dot{\varepsilon} - \dot{\varepsilon}^n) = E(\dot{\varepsilon} - f(\sigma, Z_k)) \]

where here \( Z_k \) can also represent a damage measure. These differential equations are to be integrated over a time interval \((0, T)\) subject to the constraint imposed by the equilibrium equations on the total strain rate. Assuming that the total strain rates are prescribed over time, it is obvious that \( \Delta \varepsilon^n \) is available upon inverting

\[ \Delta \sigma = E(\Delta \varepsilon - \Delta \varepsilon^n). \]

Thus it is desirable to integrate \( \dot{\sigma} \) and \( \dot{Z}_i \) subject to the equilibrium constraint.

If \( \dot{\varepsilon} \) is prescribed, then the above problem reduces to a system of the form

\[ \dot{\sigma} = k(\sigma, Z_k) \]

\[ \dot{Z}_i = g_i(\sigma, Z_k) \]

which can be rewritten as the dynamical system,

\[ \dot{y} = F(y). \]
Such a system of differential equations given by $\dot{y} = F(y)$ is said to be stiff (see Lambert [40]) if

1. $\lambda_i < 0$ for $i = 1, 2, \ldots, m$
2. $\max_i |\text{Re} \lambda_i| > \min_i |\text{Re} \lambda_i|

where $\lambda_i$ are the eigenvalues of the Jacobian $\partial F/\partial y$ and $m$ is the number of equations in the system.

Thus, a system of stiff differential equations is one in which the components of the solution may be changing or decaying at greatly different rates over a time interval. Then the evolution of the rapidly decaying component of the solution may require very small time steps to be used in a numerical scheme while the component associated with the largest eigenvalue may need to be integrated over a relatively large period of time.

In the classical numerical solutions to ordinary differential equations, the problem of numerical stability concerns the growth of truncation and round-off errors from one time step to another in a given numerical integration scheme. A numerical integration scheme is said to be absolutely stable for a given time step $\Delta t$ and differential equation, if the change in the solution due to a perturbation $\delta$ in one mesh value $y_n$ is no larger than $\delta$ in all subsequent values $y_m$ for $n < m$ (cf. Gear [41]).

For the standard test problem, $\dot{x} = \lambda x$, $\lambda$ being constant, one can define the region of absolute stability of a given algorithm as that set of values $\Delta t$ and $\lambda$ for which a perturbation in a single value $x_n$ will produce subsequent values which do not increase from time step to time step. Thus, for the forward Euler integration of the test equation $\dot{x} = \lambda x$ given by

$$x_{n+1} = x_n + \Delta t \dot{x}_n \times (1 + \lambda \Delta t)x_n.$$

there is no region of absolute stability for $\lambda \geq 0$. For $\lambda < 0$ it is necessary that $|1 + \lambda \Delta t| < 1$ so that $\lambda \Delta t$ lies in the unit circle centered at $-1$ in the $\lambda \Delta t$ complex plane. For these values of $\lambda \Delta t$ inside the unit circle, the integration may be performed without errors growing from one time step to another. Similarly, regions of absolute stability may be determined for any numerical method.

**Sample time step calculations**

For the one-dimensional form of the constitutive equations of Bodner et al., eigenvalue calculations have been performed for various materials and total strain rates. These eigenvalues were then used to estimate stable time steps for the forward Euler type integration. The results are presented in Figs 1 and 2 with the stable time steps set to 0.65 s in the initial "elastic region" where, otherwise, very large values would have been obtained. From this data large variations in eigenvalues can be seen for different regions of strain, for various materials and strain rates. It is notable that for the strain rate of $2.0 \times 10^{-4}$/s to a strain of 2%, say, may require between 12,000 and 20,000 time steps for the copper or aluminum specimens.

The shapes of these time step curves may also be used to explain the oscillations in some numerically obtained stress-strain curves such as that in Fig. 3. In this figure, time steps were initially selected slightly larger than the acceptable stable time step. Therefore, on exiting the elastic region, errors were introduced which were oscillatory in nature but not catastrophic. As the integration continued, the stable time step increased and the oscillations decayed as the time step entered the stable region.

**Weak formulation**

Subsequent calculations are performed on systems resulting from a finite element approximation of weak forms of the momentum equations.

$$(E_{ijkl} \epsilon_{kl})_i = -\dot{b}_i \quad \text{in } \Omega \quad (3.1)$$
Fig. 1. Stable time steps for forward Euler integration of Bodner's constitutive equations for various materials.

Fig. 2. Stable time steps for forward Euler integration of Bodner's constitutive equations for titanium.

Fig. 3. Decay of oscillatory behavior for a conditionally stable time integration method.
where

\[ \dot{e}_k = \dot{u}_{(k,t)} - \dot{e}_k^{*} \]
\[ \ddot{e}_k = f_k(\sigma, Z_k) \]
\[ Z_i = g_i(\sigma, Z_k). \]

Here

\[ \sigma = \int_0^t \dot{\sigma} \, dt + \dot{\sigma}. \quad \dot{\sigma} = E \dot{e} \]
\[ Z_i = \int_0^t \dot{Z}_i \, dt + \dot{Z}_i. \]

A weak form of (3.1) is obtained in the usual way: multiply (3.1) by a suitably smooth test function \( v_j = v_j(x) \), integrate over \( \Omega \), and use the Green-Gauss divergence theorem to integrate by parts the stress-power terms. Let \( V \) denote the space of test functions

\[ V = \{ v_i \in W^{m,p}(\Omega) | v_i = 0 \text{ a.e. on } \partial\Omega, \ 1 \leq i \leq N \} \]

where \( W^{m,p}(\Omega) \) is the Sobolev space of order \((m, p)\), with \( m \geq 0, \ m \in \mathbb{R}, \ 1 \leq p \leq \infty \), and where specific values of \( m \) and \( p \) depend upon the particular forms of the constitutive equations governing the material under consideration, (for the cases considered here \( m = 1, \ p = 2 \)). The weak form of the boundary-initial-value problem (3.1)-(3.4) is then:

Find a displacement rate field \( t \rightarrow \dot{u}(x, t) \in V \) and \( \{ \ddot{u} \} \) such that for every \( t \in [0, T] \),

\[ \int_\Omega E_{ijkl} \ddot{u}_{k,l} v_{ij} \, d\Omega = \int_\Omega E_{ijkl} \dot{e}_k^{*}(u) v_{ij} \, d\Omega + \int_\Omega \dot{b}_i v_i \, d\Omega + \int_{\partial\Omega_2} \bar{T}_i v_i \, ds \quad \forall v_i \in V \]

with

\[ \dot{e}_k = f_k(\sigma, Z_k) \]

and \( \sigma \) and \( Z_i \) are given by (3.3) and (3.4). Here \( \partial\Omega \) and \( ds \) are volume and surface measures. \( \ddot{u} \) is any function defined on \( \Omega \) such that its trace on the boundary segment \( \partial\Omega_1 \) is \( \ddot{u} \) (where \( u_{i|\partial\Omega_1} = \ddot{u}_i \)), and \( \sigma \) and \( Z_i \) are understood to depend upon \( \dot{u}_{i,t} \) through (3.3) and (3.4), and to satisfy initial conditions \( \sigma_i(x, 0) = \dot{\sigma}_i(x) \), \( Z_i(x, 0) = \dot{Z}_i(x) \). It is easily verified that any sufficiently smooth solution of (3.5) will also satisfy the governing equations. Conversely, any solution of the governing equations and boundary conditions will also satisfy (3.5).

Using standard notations, a finite element approximation of (3.5) leads to the discrete system of evolution equations. If the discrete velocity components are of the form

\[ \ddot{u}_k^{(i)} = \sum_{j=1}^{N} \ddot{u}_j^{(i)}(t) \phi_j(x) \]

where \( N \) is the number of nodes, \( i \) indicates the vector component, \( \phi_j \) is a basis function, \( v_j^{(i)} \) is the value of the test function \( v_j^{(i)} \) at node \( x_j \), and \( \ddot{u}_j^{(i)}(t) \) is the value of \( \ddot{u}_k^{(i)} \) at node \( x_j \) at time \( t \), then we wish to find the vector of nodal displacements \( \ddot{u} \) such that

\[ \int_\Omega B^T E B \ddot{u} \, d\Omega = \int_\Omega B^T E \dot{e} \, d\Omega + \int_\Omega \phi^T b \, d\Omega + \int_{\partial\Omega_2} \phi^T \bar{T} \, ds \]
\[ \dot{e} = f(\sigma(u), Z_k), \quad Z_k = g(\sigma, Z_k). \]

4. ALGORITHMS FOR INTEGRATING STIFF EVOLUTION EQUATIONS OF VISCOPLASTICITY

We now outline three popular algorithms for rate-dependent plasticity found in the literature and propose an alternative scheme that performs very well for cyclic loading cases.

The general strategy in these algorithms is as follows: with the initial distribution of the stress
and internal variables specified, use the equilibrium equations to supply the spatial variation of the constraint (the momentum equations). Then integrate the constitutive equations forward in time, evaluating principal variables at the integration points. With the updated values of the stress and internal variables at the new time, the constraint condition is again imposed. This sequence of determining the constraint, then advancing the constitutive equations in time is continued until the desired history of the initial-boundary-value problem has been traced.

The initial strain rate method

This method was originally proposed in 1972 in [7]. Starting with governing differential equations in the rate form, a finite element approximation of the equilibrium equations is constructed [as in (3.7)], giving

\[ \int_{\Omega} B^T E \dot{B}\mathbf{u} \, d\Omega = \int_{\Omega} B^T E \dot{\epsilon}^n \, d\Omega + \mathbf{F} \]  

(4.1)

where \( \mathbf{F} \) is the vector of force rates derived from surface traction rates and body force rates.

The algorithm is then:

1. Initialize \( \sigma, Z_i \), set \( t_n = 0 \).
2. Calculate \( \dot{\epsilon}^n = \mathbf{f}(\sigma, Z_i) \) at \( t = t_n \) [thus determining the right-hand-side of (4.1)].
3. Solve the equilibrium condition for \( \dot{\mathbf{u}}_h(t_n) \).
4. Calculate \( \dot{\mathbf{u}}(t_n) = B\dot{\mathbf{u}}_h(t_n) \).
5. Calculate \( \dot{Z}_i(t_n) = g_i(\sigma(t_n), Z_i(t_n)) \).
6. Calculate \( \dot{Z}_i(t_n) = g_i(\sigma(t_n), Z_i(t_n)) \).
7. Integrate \( \sigma, Z_i \) forward over some appropriate \( \Delta t \) to \( \sigma(t_n+1), Z_i(t_n+1) \).
8. If \( t_n + \Delta t < t \) to (2).
9. Stop.

Step (7) characterizes an explicit scheme, and this step can be displaced by a subroutine for implicit method, if needed. If predictor-corrector type integration schemes are used, a slight modification is necessary beginning with (7):

7. Estimate \( \sigma(t_n+1), Z_i(t_n+1) \) using a predictor.
8. Solve (4.1) for \( \dot{\mathbf{u}}_h(t_n+1) \) using latest entries for \( \sigma(t_n+1), Z_i(t_n+1) \).
9. Calculate \( \dot{\mathbf{u}}(t_n+1) \) and \( \dot{Z}_i(t_n+1) \).
10. Use a corrector to update \( \sigma(t_n+1), Z_i(t_n+1) \).
11. Check an appropriate tolerance index for convergence. If not achieved, go to (8) with a new estimate of \( \sigma(t_n+1) \) and \( Z_i(t_n+1) \).
12. Set \( \sigma(t_n+1) = \sigma(t_n+1) \) and \( Z_i(t_n+1) = Z_i(t_n+1) \).
13. If \( t + \Delta t < T \) to 2.

The initial strain method

This method differs from that above in that the equilibrium equations are written in incremental form rather than rate form. This results in a finite element approximation of the equilibrium equations of the form

\[ \int_{\Omega} B^T E \Delta \mathbf{u} \, d\Omega = \int_{\Omega} B^T E \Delta \epsilon^n \, d\Omega + \Delta \mathbf{F} \]  

(4.2)

where \( \Delta \) represents an increment of the proposed quantity.

Thus the computational scheme becomes [for the case of explicit integration in steps (2) and (7)]:

1. Initialize \( \sigma, Z_i \), set \( t_n = 0 \).
2. Calculate \( \Delta \epsilon^n \) over \( \Delta t \).
3. Solve (4.2) for \( \Delta \mathbf{u}_h \).
4. Calculate \( \Delta \mathbf{u} \) over \( \Delta t \) from \( \Delta \mathbf{u} = B\Delta \mathbf{u}_h \).
(5) Calculate \( \Delta \sigma = E(\Delta \varepsilon - \Delta \varepsilon'') \).
(6) Calculate \( Z_i = g_i(\sigma(t_n), Z_k(t_n)) \).
(7) Integrate \( \dot{Z}_i \) forward over \( \Delta t \) to \( Z_i(t_{n+1}) \).
(8) Set \( \sigma(t_{n+1}) = \sigma(t_n) + \Delta \sigma \).
(9) If \( t_n + \Delta t < T \) to (2).
(10) Stop.

A similar revision to that discussed earlier for implementation of predictor-corrector methods can be made.

A forward gradient scheme

This method is considerably different from the methods introduced above. It involves a tangent stiffness matrix calculation updated from step to step, and a particular type of numerical integration specified for the time integration.

Beginning with the incremental form of the equilibrium equations, we have

\[
\int_{\Omega} B^T E B \Delta \varepsilon \, d\Omega = \int_{\Omega} B^T E \Delta \varepsilon'' \, d\Omega + \Delta F
\]

or

\[
\int_{\Omega} B^T \Delta \sigma \, d\Omega = \Delta F.
\]

It is assumed that \( \Delta \varepsilon'' \) and \( \Delta Z_i \) are given by

\[
\Delta \varepsilon'' = \Delta t \{(1 - \theta)\dot{\varepsilon}''(t_n) + \theta \dot{\varepsilon}''(t_{n+1})\}
\]

\[
\Delta Z_i = \Delta t \{(1 - \theta)\dot{Z}_i(t_n) + \theta \dot{Z}_i(t_{n+1})\}.
\]

Expanding \( \dot{\varepsilon}''(t_{n+1}) \) and \( \dot{Z}_i(t_{n+1}) \) in a Taylor series about \( t_n \) results in

\[
\Delta \varepsilon'' = \Delta t \{\dot{\varepsilon}''(t_n) + \theta A_n \Delta \sigma + \theta B_n \Delta Z_i\} \tag{4.3}
\]

\[
\Delta Z_i = \Delta t \{\dot{Z}_i(t_n) + \theta C_n \Delta \sigma + \theta D_n \Delta Z_i\} \tag{4.4}
\]

where

\[
A_n = \left. \frac{\partial \dot{\varepsilon}''}{\partial \sigma} \right|_{t_n} ; \quad B_n = \left. \frac{\partial \dot{\varepsilon}''}{\partial Z_i} \right|_{t_n} ; \quad C_n = \left. \frac{\partial \dot{Z}_i}{\partial \sigma} \right|_{t_n} ; \quad D_n = \left. \frac{\partial \dot{Z}_i}{\partial Z_k} \right|_{t_n}.
\]

Solving for \( \Delta Z_i \) gives

\[
\Delta Z_i = [I - \theta \Delta t D_n]^{-1} \Delta t \{\dot{Z}_i(t_n) + \theta C_n \Delta \sigma\}. \tag{4.5}
\]

Substituting this into (4.3) for \( \Delta Z_i \) and neglecting terms of order \( \Delta t^2 \) gives

\[
\Delta \varepsilon'' = \dot{\varepsilon}''(t_n) \Delta t + \theta \Delta t A_n \Delta \sigma.
\]

Thus,

\[
\Delta \sigma = E[\Delta \varepsilon - \Delta \varepsilon''] = [I + \theta \Delta t E A_n]^{-1} E [B \Delta u - \Delta \dot{\varepsilon}''(t_n)] = D [B \Delta u - \Delta \dot{\varepsilon}''(t_n)]. \tag{4.6}
\]

Finally, substituting this into the equilibrium equation gives

\[
\int_{\Omega} B^T D B \Delta u \, d\Omega = \int_{\Omega} B^T D \Delta \varepsilon''(t'') \, d\Omega + \Delta F. \tag{4.7}
\]

The computational algorithm then becomes

1. Initialize \( \sigma; Z_i \), set \( t_n = 0 \).
2. Calculate \( \dot{\varepsilon}''(t_n) = f(\sigma(t_n), Z_i(t_n)) \).
3. Solve (4.7) for \( \Delta u_n \).
4. Calculate \( \Delta \varepsilon = B \Delta u_n \).
5. Calculate \( \Delta \sigma \) from (4.6).
(6) Calculate \( \Delta Z \) from (4.5).

(7) Update \( \sigma_{t+1} = \sigma_t + \Delta \sigma \)

\[ Z_{t+1} = Z_t + \Delta Z_i \]

(8) If \( t + \Delta t < T \) set \( t_n = t + \Delta t \) go to (2).

(10) Stop.

**New algorithm**

A new computational method suggests itself, which is similar to the initial strain algorithm in that an incremental form of the equilibrium constraint condition is imposed as

\[
\int_{\omega} B^T E \Delta u \, d\omega = \int B^T E \Delta \varepsilon^a \, d\omega + \nabla F. \tag{4.8}
\]

We then proceed as follows:

(1) Initialize \( \sigma, Z_t \), set \( t_n = 0 \), and select \( \Delta T \).

(2) Estimate \( \Delta \varepsilon^a \) over \( \Delta T \) (using predictor method).

(3) Solve the equilibrium constraint for \( \Delta u_t \).

(4) Calculate \( \Delta \varepsilon = B \Delta u_t \).

(5) Assume \( \dot{\varepsilon} = \Delta \varepsilon / \Delta T \) is constant over \( \Delta T \).

(6) Subincrement and integrate \( \sigma, Z_t \) accurately over \( \Delta T \), neglecting the previous estimate of \( \Delta \varepsilon^a \).

(7) Calculate a new guess of \( \Delta \varepsilon^a = \Delta \varepsilon - E^{-1} \Delta \sigma \).

(8) Check for convergence of \( \Delta \varepsilon^a \) with \( \Delta \varepsilon^a \); if no convergence occurs set \( \Delta \varepsilon^a = \Delta \varepsilon^a \) and go to (3). otherwise set \( \Delta \sigma = E(\Delta \varepsilon - \Delta \varepsilon^a) \).

(9) If \( t_n + \Delta T = T \), Stop.

(10) Select a new \( \Delta T \) and go to (2).

This method possesses the following desirable characteristics:

(1) A constant elastic stiffness matrix is used throughout the solution process.

(2) Different time steps at different integration points allowed and the overall \( \Delta T \) is not restricted to be the smallest time step.

(3) Integration using subincrement requires the coordinates and geometry of each element to be loaded much less frequently in the finite element simulation.

(4) By selecting a larger overall time step \( \Delta T \), fewer total time steps may be taken and thus fewer right hand sides need be considered.

**5. NUMERICAL INTEGRATION**

This section presents techniques for integration of the constitutive equations which constitutes a critical step in the algorithms discussed in Section 4. To test the applicability of various integration techniques, a group of one-dimensional test problems has been selected which covers a variety of loading histories. These problems have been taken from examples in the literature to ensure that the constitutive models are used in the correct context. The constitutive models being integrated are those of Bodner and associates and Hart which are representative of the general exponential or power law form often assumed for the nonelastic strain rates.

The first technique to be considered is the simple forward Euler method. This method is the simplest and easiest to use but suffers from being a first-order method and only conditionally stable. To use this method, it is necessary to select time steps so that the CFL conditions on stability are not violated and an acceptably small truncation error is introduced. This selection of a suitable time step for arbitrary loading histories, constitutive equations, and material types prohibits the method from being useful without some kind of automatic time step selection.

A first possibility of time step selection is that of calculating the maximum eigenvalue at each
point in time. This calculation is, however, too time consuming and is also generally unacceptable since the maximum eigenvalue may be changing over the time step.

A second possibility, proposed in [2, 3], consists of selecting a time step so that the increment of nonelastic strain over the step is some fraction of the total strain. Thus, a proposed step is given by $\Delta t = \tau \bar{\dot{e}} / \dot{e}^n$, where $\tau$ is a constant, generally around 0.1, and the bars represent equivalent values of the indicated quantities. Unfortunately, this method of time step selection is also generally unacceptable because, for stress states in the “elastic region”, $\dot{e}^n = 0$ and a very large time step is then indicated. Results obtained using this type of step selection, with a maximum allowable step size imposed, are similar to those shown in Fig. 4.

A third method of time step selection is presented in [12] for a single differential equation $y = F(y)$. This time step control is based on a comparison of a suitably defined error

$$e = \frac{\| \Delta t_k (\dot{y}(t_k) - \dot{y}(t_{k-1})) \|}{\| y(t_k) \|}$$

with prescribed error limits $e_{\text{max}}$ and $e_{\text{min}}$. The time step at the $k$th step, $\Delta t_k$, is then defined on the basis of its estimate $\tilde{\Delta} t_k$, according to

$$e_{\text{max}} < e: \text{ replace } \tilde{\Delta} t_k \text{ by } \tilde{\Delta} t_k/2 \text{ and recompute } e$$

$$e \leq e_{\text{max}}: \text{ set } \Delta t_k = \tilde{\Delta} t_k \text{ and compute } y(t_{k+1})$$

where the initial time step $\Delta t_1$ is prescribed. The next step size is then estimated by

$$e_{\text{min}} < e \leq e_{\text{max}}: \text{ set } \tilde{\Delta} t_{k+1} = \Delta t_k$$

$$e < e_{\text{min}}: \text{ set } \tilde{\Delta} t_{k+1} = 2 \Delta t_k.$$ 

An extension of this technique to the vector case $\dot{y} = F(y)$ is accomplished by introducing the infinity error norm and thus selecting the maximum value of $e$ from all vector components to determine an acceptable step size.

This *a priori* method of time step selection suffers from deficiencies similar to those of the second time step method, as can be seen from Fig. 5. Satisfactory results were obtained, however, for the constraint strain rate simulations (see Figs 7-11) the strain rate change tests (Figs 12 and 13) and the stress relaxation tests (see Fig. 15). Results obtained by using this method on the loading, unloading and reloading test, and also in cyclic testing situations, as is demonstrated in Fig. 5, were less than satisfactory. Note that the success of using this method depends strongly on the size of the initial step size prescribed.

For initial steps that are “too large”, stability difficulties or spurious loading paths may be encountered in any of the test problems. Thus it appears that the *forward Euler method with or without time step control is probably not a good choice.*

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**Fig. 4.** Calculated stress–strain curve for titanium with time steps selected from $\tau \bar{\bar{\dot{e}}} / \dot{e}^n$. 

Another group of integration techniques suggested in the literature are the higher-order explicit schemes. These techniques which include second- and fourth-order Runge-Kutta, and Adams-Bashford methods, and are suggested by some authors with the hope that higher order methods will allow larger time steps because of their inherently smaller truncation errors. Unfortunately, for such methods the issue of conditional stability is of central importance and dominates the issue of time-step selection. These methods have regions of absolute stability which are approximately equivalent to the stability region of the forward Euler method, with the result that often no appreciable difference can be seen between the results of these higher order methods and the forward Euler method.

A third group of integration techniques are the implicit schemes which employ direct or Jacobi iteration in the solution of the resulting nonlinear equations. These methods may be expressed, for a single equation \( \dot{y} = F(y) \), in a general form as

\[
y_{n+1} = \beta \Delta t F(y_{n+1}) + q
\]

where \( n + 1 \) indicates evaluation at time \( t_{n+1} \), \( \Delta t \) is the time step, \( \beta \) is a constant dependent on the numerical scheme, and \( q \) is a known function of the previously calculated values of \( y \) and \( F \). Using direct of Jacobi iteration implies that

\[
y^{(s+1)}_{n+1} = \beta \Delta t F(y^{(s+1)}_{n+1}) + q = \phi(y^{(s+1)}_{n+1})
\]
Fig. 7. Monotonic stress-strain curves for titanium at constant strain rates. (Bodner's equations in the original form.)

Fig. 8. Monotonic stress-strain curves for titanium at constant strain rates. (Bodner's equations in the deviatoric form.)

Fig. 9. Monotonic stress-strain curves for OFHC copper at constant strain rates. (Bodner's equations.)

Fig. 10. Monotonic stress-strain curves for 1100 aluminum at constant strain rates. (Hart's equations.)
Fig. 11. Monotonic stress–strain curves for 304 stainless steel at constant strain rates. (Hart's equations.)

Fig. 12. Stress-strain curves for titanium subjected to rapid changes in strain rate. (Bodner's equations.)

Fig. 13. Stress–strain curves for 304 stainless steel subjected to rapid changes in strain rate. (Hart's equations.)

Fig. 14. Stress–strain curves of titanium subjected to unloading and subsequent reloading at a faster rate. (Bodner's equations.)
Fig. 15. Stress relaxation curve for titanium after preloading to 2% strain at 1.6E - 5/s. (Bodner’s equations.)

and ($s$) indicating the iteration number and $y_{n+1}^0$ is provided by some predictor calculation. This sequence of approximations given by (5.2) will converge to the solution of (5.1) whenever $\phi(y_{n+1})$ satisfies a Lipschitz condition

$$|\phi(y_{n+1}) - \phi(y_{n+1})| < M |y_{n+1}^* - y_{n+1}^*|$$

for all $y_{n+1}$ and $y_{n+1}^*$, where the Lipschitz constant $M$ satisfies $0 \leq M \leq 1$. Then there exists a unique solution $\gamma$ of (5.1), and the sequence of approximations defined by (5.2) is such that $\lim y^{(s)} = \gamma$ (see [40]). A similar result also applies for systems of equations with the absolute values replaced by norms of corresponding vectors.

Integration techniques suggested in the literature which fall into this category include forward Euler predictor with trapezoidal corrector, forward Euler predictor with backward Euler corrector, and mid-step integration where rates at the midpoint of the time step are used in the integration. These methods are superior to those discussed in the proceeding paragraphs, in that they have large regions of absolute stability, which include the negative complex half plane and essentially eliminate the stiffness difficulty. They are, however, limited by the radius of convergence of the direct iteration technique. These methods are superior to those discussed in the proceeding paragraphs, in that they have large regions of absolute stability, which include the negative complex half plane and essentially eliminate the stiffness difficulty. They are, however, limited by the radius of convergence of the direct iteration technique. For example, if the partial derivatives of the Jacobian matrix $J = \partial F/\partial y$ are continuous and bounded in an appropriate region, then the Lipschitz constant of $F$ may be taken to $L = \|J\|$. Now for any matrix $A$, we have $\|A\| \geq \rho(A)$ where $\rho(A) = \max |\lambda|_i$ and $\lambda_i$ are the eigenvalues of $J$. This implies that the Lipschitz constant $L$, is greater than or equal to the magnitude of the maximum eigenvalue. Considering the general form given in (5.1), we can choose the Lipschitz constant $M$ to be $L \Delta t |\beta|$ which implies that (5.2) converges to the solution of (5.1) if $\Delta t < 1/(L |\beta|)$ or approximately $\Delta t < 1/(|\beta| \lambda_{\text{max}})$. This result suggests a time step restriction, again related to the maximum eigenvalue, which may be as restrictive as the stability requirements of the previous methods.

If these predictor corrector methods with direct iteration are to be used, some type of time step control is again necessary. One possible time step selection technique for use with the forward Euler predictor, trapezoidal corrector is given in [12] for a single differential equation $y = F(y)$. This automatic control is implicit in nature in that an initial time step size is estimated from the previous time step and then adjusted after the corrector calculations have
been performed. This time step control is basically the same as the third technique described above for use with the forward Euler method, except that for this case the error is defined by

$$e = \frac{|\Delta t_k(F_{k+1}^p - F_k)|}{2 |y_{k+1}^c|}$$

where $y_{k+1}^c$ is the corrector value of $y_{k+1}$ and $F_{k+1}^p = F(y_{k+1})$.

This combination of predictor-corrector with time step control performed satisfactorily on all the test problems, but was not efficient in test situations when the constitutive equations were fairly stiff. Results for the constant rate simulations, the strain rate change test, the stress relaxation tests, creep test, and the stress change test are the same as those obtained by the forward Euler method with time step control, with plotting accuracy. Use of this combination on the loading, unloading, reloading test is shown in Fig. 14, on the cyclic tension compression tests is shown in Figs 19–22, on the cyclic relaxation test in Fig. 23, and on the cyclic creep test in Fig. 24.

We also mention a very popular group of integration techniques: predictor-corrector methods which use Newton iteration in the correction process. For a system of $m$-equations in $m$-unknowns such as $H(y) = 0$, the Newton method can be written as

$$y^{(s+1)} = y^{(s)} - J^{-1}(y^{(s)})H(y^{(s)}), \quad s = 0, 1, 2 \cdots$$

where $(s)$ is the iteration number and $J^{-1}$ is the inverse of the Jacobian matrix $\partial H/\partial y$. If this method is applied to eqn (5.1), we obtain the sequence of approximations

$$y_{n+1}^{(s+1)} = y_{n+1}^{(s)} - \left[I - \Delta t \frac{\partial F(y_{n+1}^{(s)})}{\partial y}\right]^{-1} (y_{n+1}^{(s)} - \Delta t \frac{\partial F(y_{n+1}^{(s)})}{\partial y} - q), \quad s = 0, 1, 2 \cdots$$

where $I$ is the identity matrix.

Proposed integration techniques to be used in conjunction with Newton iteration include the implicit methods mentioned above and stiffy stable methods discussed by Gear [41]. These methods are presented in the predictor–corrector format with a $p$th order predictor formula of

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Fig. 16. Stress relaxation curves for 1100 aluminium with different initial hardness values. (Hart's equations.)

Fig. 17. Creep curves of 1100 aluminum with variable initial hardness and applied stress. (Hart's equations.)
Fig. 18. Calculated strain–time curve for 304 stainless steel subjected to a stress change test. (Hart's equations.)

Fig. 19. Cyclic stress–strain curves for titanium for ±1% strain. (Bodner's equations.)

Fig. 20. Cyclic stress–strain curves for 1100 aluminum for ±1% strain. (Bodner's equations.)
Fig. 21. Cyclic stress-strain curves for OFHC copper for ±1% strain. (Bodner's equations.)

Fig. 22. Cyclic stress-strain curves for 304 stainless steel for ±0.5% strain. (Hart's equations.)

Fig. 23. Computed stress-strain curves for titanium for strain controlled cycling with positive strain limits showing creep relaxation. (Bodner's equations.)
the form

\[ y_{n+1} = \alpha_1 y_n + \cdots + \alpha_p y_{n+1-p} + \eta \Delta t F(y_n) \]

and a corrector

\[ y_{n+1}^{(s+1)} = \alpha_1^* y_n^{(s)} + \cdots + \alpha_p^* y_{n+1-p}^{(s)} + \eta^* \Delta t F(y_{n+1}^{(s)}) \]

with \( \alpha_1, \eta, \alpha_1^*, \) and \( \eta^* \) constants depending on the order \( p \). This format differs significantly from the more conventional methods in that only one rate term is used with several previous solution values rather than several rate terms with one previous solution value.

All of these predictor–corrector methods also have infinite regions of absolute stability (for linear dynamical systems) and the use of Newton iteration provides for better convergence limits than standard direct iteration techniques. A drawback to these methods, however, is that they generally require the calculation and "inversion" of the Jacobian matrix, for each time step or, at least, periodically during the solution process. For only mildly stiff problems, such calculations may be more time consuming than simply using a smaller time step size and direct iteration. For the test problems considered here, Gear stiffly stable methods performed better than the other predictor–corrector methods, with numerical results being the same, to within plotting accuracy, as those obtained using the forward Euler predictor–trapezoidal corrector with time step control.

6. TEST CASES

The computational methods and integration techniques discussed earlier were tested on a series of one-dimensional problems to determine which techniques performed best and some of these results are given in Figs 4–24. These problems were selected from results reported in the literature, so that comparisons of the numerical results could be made with published data. Also, along these test cases are problems which involve many of the complex loading histories which internal-state-variable constitution equations are capable of modeling. While only two sets of constitutive equations were considered, the Bodner and Partoms equations in both the original and deviatoric form were used in these calculations. The following shorthand notation is used:

- **BPO**—Bodner–Partom equations in the original form
- **BPN**—Bodner–Partom equations in the deviatoric form
- **H**—Harts equations
TI—titanium
AL—1100 aluminum
CU—OFHC copper
SS—304 stainless steel

In all problems using Bodner's equations the reference temperature is room temperature, while the problems dealing with Harte's equations were at temperatures of 250 and 400°C for the 1100 aluminum and 304 stainless steel, respectively. The other material data may be found in Refs [15–27].

Test problem #1. Constant strain rate tests

Several constant strain rate simulations were performed under the following conditions:

<table>
<thead>
<tr>
<th>Eqn type</th>
<th>Mat type</th>
<th>Strain rate</th>
<th>total strain</th>
</tr>
</thead>
<tbody>
<tr>
<td>BPO/BPN</td>
<td>TI</td>
<td>3.2E-3/s</td>
<td>2%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.6E-3/s</td>
<td>2%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.6E-4/s</td>
<td>2%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.6E-5/s</td>
<td>2%</td>
</tr>
<tr>
<td>BPN</td>
<td>CU</td>
<td>2.0E-3/s</td>
<td>1%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.0E-4/s</td>
<td>1%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.0E-5/s</td>
<td>1%</td>
</tr>
<tr>
<td>H</td>
<td>AL/SS</td>
<td>3.33E-3/s</td>
<td>1-2%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3.33E-4/s</td>
<td>1-2%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3.33E-5/s</td>
<td>1-2%</td>
</tr>
</tbody>
</table>

This large number of constant strain rate tests were performed due to the large changes in stiffness of the constitutive equations for different material types and strain rates. Note the large differences in the results for the TI material using the original and deviatoric form of Bodner's equations, see Figs 7 and 8. Also note the strain rate insensitivity of the copper and stainless steel specimen, Figs 9–11, and non-hardening nature of the aluminum specimen under these conditions, Fig. 10.

Test problem #2. Strain rate change tests

Several strain rate change tests were simulated under the following conditions:

<table>
<thead>
<tr>
<th>Eqn type</th>
<th>Mat type</th>
<th>History of loading</th>
</tr>
</thead>
<tbody>
<tr>
<td>BPO</td>
<td>TI</td>
<td>Strain at 1.6E-5/s to 1% strain then change to 3.2E-3/s and strain to 2%</td>
</tr>
<tr>
<td>BPO</td>
<td>TI</td>
<td>Strain at 3.2E-3/s to 1% strain then change to 1.6E-5/s and strain to 2%</td>
</tr>
<tr>
<td>H</td>
<td>SS</td>
<td>Strain to 3.33E-3/s for 0-30 s then change to 3.33E-4/s for 30-60 s</td>
</tr>
</tbody>
</table>

Results for this type of testing may be seen in Figs 12 and 13. Generally, this type of loading did not lead to numerical difficulties when changing from a high strain rate to lower rate, but caused some problems when changing from a low to high. This was due to the larger time steps allowed in the initial lower strain rate simulations. Also note the asymptotic approach of the results to the constant loading situation for Bodner's equations, while Harte's equations predict an almost immediate jump in the response to the constant loading case.

Test problem #3. Loading, unloading, and reloading test

A single example of loading at one strain rate, unloading into the "elastic region", and reloading at a different strain rate was performed under the following conditions:

<table>
<thead>
<tr>
<th>Eqn type</th>
<th>Mat type</th>
<th>History of loading</th>
</tr>
</thead>
<tbody>
<tr>
<td>BPO</td>
<td>TI</td>
<td>Strain at 1.65E-5/s up to 1% strain, unload into the &quot;elastic region&quot; and then reload at 3.2E-3/s up to 2% total strain</td>
</tr>
</tbody>
</table>
Results for this test problem using the forward Euler predictor–trapezoidal corrector are shown in Fig. 14. The forward Euler predictor with time step control from [12] experienced difficulties in the reloading part of this problem. For this problem using the old form of the equations and the TI specimen, the predictor-corrector with direct iteration performed the best.

**Test problem #4. Stress relaxation tests**

Stress relaxation simulations were conducted under the following conditions:

<table>
<thead>
<tr>
<th>Eqn type</th>
<th>Mat type</th>
<th>History of loading</th>
</tr>
</thead>
<tbody>
<tr>
<td>BPO</td>
<td>TI</td>
<td>Strain to 2% at 1.65E - 5/s and hold for 70 min</td>
</tr>
<tr>
<td>H</td>
<td>AL</td>
<td>Strain to 0.03% total strain at 3.33E - 4/s and hold for 10 h</td>
</tr>
</tbody>
</table>

Results of this testing are shown in Figs 15 and 16. For this problem there were no rapid load changes or serious stiffness problems and thus the forward Euler method with time step control performed as well as the other methods.

**Test problem #5. Creep test**

Two creep simulations were performed under the following conditions:

<table>
<thead>
<tr>
<th>Eqn type</th>
<th>Mat type</th>
<th>History of loading</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>AL</td>
<td>Rapidly stress to 1500 psi and then hold constant over 0-100 h</td>
</tr>
<tr>
<td>H</td>
<td>AL</td>
<td>Rapidly stress to 1000 psi and then hold constant over 0-1000 h</td>
</tr>
</tbody>
</table>

Results are reported in Fig. 17 for a 1-in. material specimen. This problem also posed no stiffness difficulties due to the low level of initial stressing, and thus the simpler methods are more efficient for this case.

**Test problem #6. Stress change test**

A single stress change test was simulated under the following conditions:

<table>
<thead>
<tr>
<th>Eqn type</th>
<th>Mat type</th>
<th>History of loading</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>SS</td>
<td>Rapidly stress to 20000 psi and hold constant over 0-10 h, then rapidly increase the load to 30000 psi and hold constant over 10-100 h</td>
</tr>
</tbody>
</table>

Results of the numerical integration for this problem are shown in Fig. 18. This problem having an imposed stress history and relatively large hold times poses no large stiffness problem and therefore the simpler methods are more efficient here also.

**Test problem #7. Cyclic tension-compression tests**

Several cyclic tension-compression simulations were performed for:

<table>
<thead>
<tr>
<th>Eqn type</th>
<th>Mat type</th>
<th>Strain rate (=)</th>
<th>#Cycles</th>
<th>± % Strain</th>
</tr>
</thead>
<tbody>
<tr>
<td>BPN</td>
<td>TI</td>
<td>3.2E - 3/s</td>
<td>10</td>
<td>1%</td>
</tr>
<tr>
<td>BPN</td>
<td>AL</td>
<td>2.0E - 3/s</td>
<td>5</td>
<td>1%</td>
</tr>
<tr>
<td>BPN</td>
<td>CU</td>
<td>2.0E - 4/s</td>
<td>1</td>
<td>1%</td>
</tr>
<tr>
<td>H</td>
<td>SS</td>
<td>1.0E - 3/s</td>
<td>5</td>
<td>0.5%</td>
</tr>
</tbody>
</table>
(See Figs 19–22 for results of this simulation.) In this group of problems, the constitutive equations were significantly stiffer for AL, CU and SS specimens. Also, the region of time over which the equations were stiff was large in relation to the total time and therefore the stiff stable methods Gear performed best in these cases. Note that the number of cycles and/or percent strain was in some cases limited by the stiffness of the equations, so that the predictor–corrector methods using direct iteration were competitive in total computing time.

**Test problem #8. Cyclic relaxation**

A single cyclic relaxation simulation was performed under the following conditions:

<table>
<thead>
<tr>
<th>Eqn type</th>
<th>Mat type</th>
<th>History of loading</th>
</tr>
</thead>
<tbody>
<tr>
<td>BPO</td>
<td>TI</td>
<td>Strain at 2.5E - 3/s between 1.25% total strain and 0.25% total strain over 5 cycles</td>
</tr>
</tbody>
</table>

Results of the numerical integration for this problem are shown in Fig. 23. Five cycles were selected here because at this point an almost steady state was reached. For this problem the constitutive equations were only moderately stiff and the predictor corrector with direct iteration and Gears methods performed approximately the same.

**Test problem #9. Cyclic creep test**

A single cyclic test was simulated under the following conditions:

<table>
<thead>
<tr>
<th>Eqn type</th>
<th>Mat type</th>
<th>History of loading</th>
</tr>
</thead>
<tbody>
<tr>
<td>BPO</td>
<td>TI</td>
<td>Stress to 325 MPa at 32.5 MPa and then to -225 MPa at -32.5 MPa and continue for 5 cycles</td>
</tr>
</tbody>
</table>

Results for this history of loading shown in Fig. 24. This problem also has only a prescribed stress history and the predictor methods with direct iteration performed well here.

In summary, we list for the test problems considered:

1. The computational methods in Section 4 produced identical results with all the time integration techniques discussed in Section 5, with no particular advantage of one algorithm over the other.
2. Gears integration package of [14] showed some inadequacies in strain control situations. These difficulties were overcome by a slight modification of his methods of time step selection.
3. The forward gradient scheme presented in Section 4 showed much the same type of deficiencies as the other forward type of integration methods. The results for a cyclic loading situation may be seen in Fig. 6 where the time steps have been selected as a fraction of the total strain.
4. For reasons mentioned in Section 5 and from results of the test problems, no method suggested in all of the literature examined is completely satisfactory. Of those actually tested, the two that performed best were forward Euler predictor–trapezoidal corrector with direct iteration and step control of [12] in conjunction with either the initial strain of the initial strain method, and Gears stiff stable methods with Newton iteration corrections in conjunction with the new algorithm of Section 4.

7. OTHER APPLICATIONS

In this final section, we apply two of the better techniques described earlier to some specific engineering problems found in the literature. The methods used are:

1. The forward Euler predictor–trapezoidal corrector with direct iteration and time step
Example problem #1 (composite sheet using Bodner's equations)

This problem, taken from [2], is essentially one-dimensional in nature. It consists of a composite material strip half titanium and half copper, supported by two walls (see Fig. 25). The specimen is rapidly loaded at the material interface with a uniform pressure of 150 MPa, which is approximately six times the "yield stress" of the copper, and held constant over a period of 100 h. The finite element model and 100 times the deformed configuration at time = 100 h is shown in Fig. 26. The one-dimensional elastic bar elements which have been

control of [23] in conjunction with the initial strain method or the initial strain rate method, and

(2) Gears stiffly stable method in conjunction with the new algorithm of Section 4.

Example problem #1 (composite sheet using Bodner's equations)
Damage and rate-dependent plasticity

Fig. 29. Variation of radial stress in a creeping cylinder in plane strain.

Fig. 30. Variation of axial stress in a creeping cylinder in plane strain.

Fig. 31. Variation of circumferential stress in a creeping cylinder in plane strain.

Fig. 32. Perforated tension strip in uniaxial tension.

included are essentially rigid and are used here to allow a homogeneous mode of deformation to be modeled. The stress in the copper and titanium components is plotted versus time in Fig. 27. This plot shows the rapid change in stress from the initial elastic solution to almost steady state values. Figure 28 shows the relative displacement-time curve for a point on the material interface. In this figure, \( U_{el} \) is the elastic displacement of the interface. These results compare well with those in [2], even though a different constitutive model, Bodner's model, was used.
Fig. 33. Normal stress distributions along section A–A in a perforated tension strip.

Fig. 34. Time dependent regions of nonelastic deformation for a perforated tension strip (applied stress = 100 MPa).

Fig. 35. Regions of equivalent nonelastic strain at 30 s for a perforated tension strip (applied stress = 100 MPa).

Fig. 36. Time dependent regions of nonelastic deformation for a perforated tension strip (applied stress = 125 MPa).
For this problem, the first method of solution required approximately 3 min 20 s of CPU time, while the second method required 5 min 20 s (on a Harris 800 II).

Example problem #2 (pressure cylinder using Hatrs equations)

A hollow circular aluminum cylinder, with an internal radius of 5 in. and external radius of 10 in., is loaded under plane strain conditions with an internal pressure of 750 psi. The pressure is rapidly applied, so that an initial elastic stress distribution is present, and held constant over 100 h. The finite element discretization for this problem consists of five, eight-node quadratic elements along the cylinder thickness.

Plots of the radial stress, axial stress, and circumferential stress distributions at various times are shown in Figs 29–31. These results are essentially the same as those given in [5].

For this problem, method 1 required 11 min 30 s of CPU time, while method 2 required 22 min 10 s.

Example problem #3 (perforated torsion strip using Bodners equations)

A rectangular perforated tension strip is loaded with a uniform tensile stress as is shown in Fig. 32. The stress is applied rapidly, so that an initial elastic stress distribution is present, and held constant for a period of 30 s.

For a stress of 100 MPa, plots of the axial stress versus time along section A–A are shown in Fig. 33. The advancement of the "plastic zone" from 0.5 to 30 s is also shown in Fig. 34, with regions of equivalent nonelastic strain at time = 30 s shown in Fig. 35.

This tension strip was also subjected to a stress of 125 MPa and held constant for 30 s as above. The corresponding growth of the "plastic zone" is shown in Fig. 36. Comparing this figure with Fig. 34, we observe a large increase in size of the nonelastic region for an increased loading.

For loading up to 100 MPa, the first integration method required 100 min 20 s of CPU time, while the second algorithm used 49 min 20 s. For the 125 MPa loading, the first solution method completed in 226 min 10 s and the second method required 131 min 40 s (on a Harris 800 II).

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