A Finite Element Analysis of the General Rolling Contact Problem for a Viscoelastic Rubber Cylinder


ABSTRACT: Mathematical models of finite deformation of a rolling viscoelastic cylinder in contact with a rough foundation are developed in preparation for a general model for rolling tires. Variational principles and finite element models are derived. Numerical results are obtained for a variety of cases, including that of a pure elastic rubber cylinder, a viscoelastic cylinder, the development of standing waves, and frictional effects.

KEY WORDS: finite element, rolling contact, large deformation, viscoelasticity, friction, standing waves.

In a recent paper [1], we developed general models of rolling contact of elastic and viscoelastic cylinders that undergo very large deformations. The present paper presents a summary of that work and some preliminary results on extensions to rolling tires in three-dimensional situations.

Following this section, we consider a general class of rolling/cornering contact problems which includes the following features:

1. A three-dimensional, cylindrical, deformable body in steady-state rolling/cornering on a rough surface.
2. Unilateral contact of the cylinder with a surface which, without loss of generality, is assumed here to be rigid.
3. The cylinder/disk material is assumed to be a compressible or incompressible hyperelastic material or a hyperelastic/viscoelastic material characterized by constitutive equations valid for finite deformations.
4. Friction forces are developed on the contact surface.
5. The normal compliance of the contact interface is characterized by a nonlinear constitutive law that models experimentally observed characteristics of deformable interfaces, and
6. The presence of inertia terms in the formulation introduces, in addition to the usual load parameters, the parameter $\theta_3$ representing the...

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rolling angular velocity of the cylinder. At certain critical values of $\theta_3$, bifurcations of the solutions of the rolling contact problem occur; these are manifested as the emergence of standing waves in the cylinder/disk. Our formulation and algorithms capture these types of phenomena.

**Kinematics of Rolling/Cornering Contact**

The physical situation we wish to consider here is the steady state rolling/cornering of a deformable right-circular cylindrical/disk body of radius $R_0$ and thickness $2b$, which is rolling at a constant velocity $v_0$ along a fixed rough, rigid foundation, as shown in Fig. 1a. The cylinder may also be bonded to a rigid cylindrical axle of radius $R_i$ which is spinning at a constant angular velocity $\dot{\theta}_3$ about its own radius axis $X'_3$ and cornering at a constant

![FIG. 1a — Geometry of unloaded rolling or cornering cylinder.](image-url)
angular velocity $\dot{\theta}_2$ about the global $X_2$ axis. The inclination angle of the rolling cylinder to the foundation is $\beta$. The cases of a full cylinder ($R_i = 0$) or a hollow cylinder (with a center hole of radius $R_i$) also fall within the theory developed here. An axle load presses the cylinder down on the rigid foundation so that the distance $H$ from the axis of the cylinder to the foundation is always less than $R_o \cos \beta + b \sin \beta$ (in Fig. 1b).

Since the motion is steady state, an observer riding along with the cylinder will see the same deformed geometry at all times. To describe deformation of a continuum, we must compare this current deformed geometry with that of the body in some convenient arbitrary reference configuration, even if the body never actually assumes that configuration. In the case of a rolling/cornering deformable cylinder, the reference configuration is a natural one: the configuration assumed by a rigid (unstrained) cylinder/disk, not in contact with the foundation, spinning about the $X'_3$ axis at a constant angular velocity $\dot{\theta}_3$ and cornering about the $X_2$ axis at a constant angular velocity $\dot{\theta}_2$. In

![Diagram](image)

FIG. 1b — Geometry of loaded rolling or cornering cylinder.
other words, the deformation of the rolling/cornering cylinder at any time \( t \) (the actual value of \( t \) being irrelevant since the deformation is the same for all times) can be completely characterized by comparing the positions of material particles \( X \) in the body at time \( t \) with positions of \( X \) in a rigid cylinder at the same time \( t \). Thus, if \( x \) denotes a position vector emanating from an origin \( O \) on the axis of the cylinder and terminating at a particle \( X \) in the deformed cylinder, and if \( \rho(t) \) denotes a vector from \( O \) to the place that would have been occupied by \( X \) had the cylinder been rigid, then the deformed geometry of the rolling/cornering cylinder is defined by a relation of the form

\[
x = x(\rho(t))
\]

where \( x \) is a continuous invertible map from the reference configuration onto the deformed cylinder.

Henceforth, the following coordinates are introduced:

- \( R, \theta_3, Z \) = material cylindrical coordinates in the rigid spinning/cornering cylinder which labels a material particle \( X \) at time \( t = 0 \).
- \( r, \theta_3, z \) = referential coordinates: these are the cylindrical coordinates of positions in the rigid spinning/cornering cylinder which rotates about \( X_2 \) axis with angular velocity \( \theta_2 \) and are related to the material coordinates according to

\[
r = R, \quad \theta_3 = \theta_3 + \dot{\theta}_3 t, \quad z = Z
\]

- \( X_1, X_2, X_3 \) = global Cartesian referential coordinates that is fixed at time \( t \) as shown in Fig. 1a.
- \( X'_1, X'_2, X'_3 \) = local Cartesian referential coordinates that is fixed at time \( t \) as shown in Fig. 1b.

\[
X_1 = X'_1; \quad X_2 = X_2 \cos \beta + X_3 \sin \beta; \quad X_3 = -X_2 \sin \beta + X_3 \cos \beta
\]

- \( x_1, x_2, x_3 \) = global spatial Cartesian coordinates: these are the coordinates of points in space relative to a moving coordinate frame with origin fixed on the axes of the moving cylinder, with \( x_1-x_3 \) parallel to the foundation. \( x_2 \) normal to the foundation.

The deformation of the cylinder is defined by a twice differentiable invertible map \( \chi \) that takes the reference configuration of the cylinder (the places in the rigid cylinder at time \( t \)) onto the current (deformed) configuration at the same time \( t \). In particular, the motion carries points \( (r, \theta_3, z) \) in the reference configuration to points in the current configuration with spatial components \( x_i \) given by

\[
x_i = \tilde{x}_i(r, \theta_3, z) = x_i(X_1, X_2, X_3) = \tilde{x}_i(X'_1, X'_2, X'_3)
\]

Thus, time enters the description of motion only implicitly in terms of \( \dot{\theta}_2 \) and \( \theta_3 \).
Various other kinematical quantities of interest can now be computed from Eq. 1.

The Velocity Field

\[ \mathbf{v} = \mathbf{x}, \quad v_i = u_i(X) = \frac{\partial x_i}{\partial X_j} \dot{X}_j \]

where \( \dot{X}_j \) can be expressed most easily in terms of local \( X' \) coordinates

\[
\begin{align*}
\dot{X}_1 &= V_{01} + \dot{\theta}_2 X'_2 \cos \beta - (\dot{\theta}_3 - \dot{\theta}_2 \sin \beta) X'_2 \\
\dot{X}_2 &= \dot{\theta}_3 X'_1 \cos \beta \\
\dot{X}_3 &= (\dot{\theta}_3 \sin \beta - \dot{\theta}_2) X'_1
\end{align*}
\]

(2)

The Deformation Gradient

If \( \nabla \) denotes the gradient with respect to the reference configuration, the deformation gradient tensor \( \mathbf{F} \) is given by

\[
\mathbf{F} = \nabla X, \quad \text{or} \quad F_{ij} = \left( \frac{\partial x_i}{\partial r}, \frac{\partial x_i}{\partial \theta_3}, \frac{\partial x_i}{\partial z} \right)
\]

The right Cauchy-Green deformation tensor \( \mathbf{C} \) is then \( \mathbf{C} = \mathbf{F}^T \mathbf{F} \) and the Green-St. Venant strain tensor is \( \mathbf{E} = (\mathbf{C} - \mathbf{I})/2 \).

Histories of the Motion

The particular method of description of the deformation of a rolling/cornering cylinder employed here makes it possible to also define histories of the motion of particles. To illustrate the idea, consider position \( \rho(t) \) of a particle on the rigid spinning/cornering reference configuration at three different instants of time, \( t_1 < t_2 < t \), \( t \) being the current time (see Fig. 2). Denote \( \rho(t_i) = X_i \), \( i = 1, 2 \) and \( x_i = \chi(X_i) \) (\( x = \chi(\rho(t)) \)). Since the motion is steady-state, we must have

\[
\begin{align*}
x_1(t_1) &= x_1(t_2) = x_1(t) = \chi(X_1) \\
x_2(t_1) &= x_2(t_2) = x_2(t) = \chi(X_2), \quad \text{etc.}
\end{align*}
\]

(4)

Thus, to define the history of the deformation, we need only to determine current positions of points on a line of constant radius \( R \) and constant \( Z \) in the reference configuration. Thus, for example, to compute past values of a deformation measure such as \( \mathbf{F} \) at times \( t_1 < t_2 < t \), we compute \( X_1 = \rho(t_1)_{(R,Z)}, X_2 = \rho(t_2)_{(R,Z)} \); then \( \mathbf{F}_1 = \nabla \chi(X_1), \mathbf{F}_2 = \nabla \chi(X_2) \). Thus:

\[ \{ \mathbf{F}((r, \theta_3, z), \tau), -\infty < \tau < t \} = \{ \mathbf{F}((r, \psi_3, z), t), -\infty < \psi_3 \leq \theta_3 = \Theta_3 + \dot{\theta}_3 t \}. \]

A similar description is given in [2].
Constitutive Equations

While many aspects of our formulation given here and in the next section are valid for arbitrary materials, the principal focus of the present analysis is on rolling/cornering cylinders composed of homogeneous isotropic rubber-like materials. We shall, therefore, focus on classes of compressible or incompressible hyperelastic materials and on a restricted class of viscoelastic materials for which viscous effects can be characterized by linear perturbations of a hyperelastic material.

Toward this end, let $T$ denote the Cauchy stress tensor and $S$ the second Piola-Kirchhoff stress tensor, related to one another according to

$$S = (\det F)F^{-1}TF^{-T}. \quad (5)$$

The material is *homogeneous hyperelastic* whenever there exists a differentiable, scalar-valued stored energy density function $\sigma = \sigma(C)$, representing the strain energy per unit mass in the current configuration, such that
\[ T = 2\rho F(\partial \sigma / \partial C)(F)^T \]

or

\[ S = 2\partial W(C)/\partial C \] (6)

where

\[ \sigma(C) = W(C)/\rho_0 \]

with \( \rho_0 \) the mass density in the reference configuration. Thus, \( W \) is the strain energy density function per unit volume in the reference configuration.

If the material is incompressible (which is often assumed to be the case when analyzing many rubber-like materials), all motions are subject to the incompressibility constraint.

\[ \det F = 1. \]

For the isotropic case, \( W \) then reduces to a function of only principal invariants \( I_c \) and \( II_c \) of \( C \) and determines \( S \) only up to within an arbitrary hydrostatic pressure \( p \):

\[ W = W(I_c, II_c); \quad S = \partial W/\partial C - \rho_0 C^{-1}. \]

Real rubber-like materials are rarely perfectly elastic and usually exhibit some degree of strain-history dependence. For a large class of elastomers, it is sufficient to model the viscoelastic aspects of the response by appending to the hyperelastic law a suitably defined linear strain-history dependent term; i.e.

\[ S(t) = S^e + \int_{r=-\infty}^{r} \mathcal{H}(\tau). \]

In some of the calculations to follow, we use a simple viscoelastic constitutive law [3], which, together with the method for computing histories described earlier, leads to a constitutive equation of the form:

\[ S(\theta_3) = \frac{\partial W}{\partial E} + \nu \left\{ E(\theta_3) - \frac{1}{\theta_3 T_e} \int_{-\infty}^{\theta_3} e^{-(\theta_3 - \Psi)/\theta_3 T_e} E(\psi_2) d\psi_2 \right\} - \rho_0 C^{-1}, \] (8)

where \( \theta_3 \) is the angle traversed in the time \( t \) along an arc in the reference configuration.

Contact, Interface, and Friction Conditions

Let \( \Gamma_c \) denote the candidate contact surface in the reference configuration (\( \Gamma_c \) could be taken as the entire outside surface of the cylinder) and let \( n_0 \) denote a unit exterior normal to this surface and \( n \) denote a unit outward normal to the deformed surface as shown in Fig. 3. The traction vector \( t \) at a point on \( \Gamma_c \) is then given by

\[ t = T_0^T n_0 \]
while the normal and tangential stress components are
\[ t_n = t \cdot n, \quad t_n = t_n n, \quad t = t - t_n \]

The *slip velocity* \( w_1 \) is defined as the velocity difference between the particle velocity \( v \) on \( \Gamma_c \) and the foundation velocity \( V_0 \):
\[ w_1 = v - V_0 \]

The *unilateral contact condition* on \( \Gamma_c \) is the constraint such that [4]
\[ x_2 \leq H \quad \text{on} \quad \Gamma_c \]
when the rigid foundation is flat and horizontal. We also have the corresponding conditions.
\[ x_2 < H \Rightarrow t = 0 \quad \text{on} \quad \Gamma_c \]
\[ x_2 = H \Rightarrow t_n < 0 \quad \text{on} \quad \Gamma_c \]

which collectively imply the condition
\[ t_n(x_2 - H) = 0 \]

This latter *complementarity condition* characterizes a free boundary on \( \Gamma_c \) defined by the contact interface.
If the contact interface (the thin layer of compressed material on the contact surface) is essentially rigid and the classical Coulomb's law of friction is assumed to hold, then we must add to the contact condition the friction-slip conditions:

\[ |t_t| < \mu |t_{nn}| = w_t = 0 \]

\[ |t_t| = \mu |t_{nn}| = w_t \neq 0, \quad \text{and} \quad w_t = -\lambda t_t \text{ for some positive real number } \lambda \]

Here \( \mu \) is the coefficient of friction, a constant.

The relationship between the frictional stress \( t_t \) and the slip velocity \( w_t \) is illustrated in Fig. 4. The jump discontinuity in \( t_t \) at zero slip velocity is sometimes the source of technical (and numerical) difficulties, and it is not uncommon to replace this discontinuous relation with regularized law:

\[ I W t t = -\mu I t_{nn} t p, I W t / I w t (9) \]

where \( t p, \) is a sufficiently smooth function of the positive number \( |w_t| \) as indicated by the smooth dashed line in Fig. 4 [5].

Recent studies [6] on dynamic friction suggest that a satisfactory model of contact and friction, particularly in the case of a moving surface in contact, should include a rational model of the contact interface itself, including a constitutive equation for the interface which simulates its normal compliance; e.g.

\[ I n = f(x_2, \dot{x}_2) \]

A detailed study of such interface models for dynamic friction phenomena on dry metallic surfaces was recently completed [6] and suggests that for many materials a power-law relation between the approach \( a = (x_2 - H)_+ \) and the contact pressure \( I n \) is consistent with a large volume of experimental work. In the present case, such a nonlinear interface law would assume the form

\[ I n = f(x_2) = -c_n(x_2 - H)_+^m \quad (10) \]

where coefficients \( c_n \) and \( m_n \) are material constants which depend upon the constitution and roughness of the actual contact surface. It is interesting to note that an interface law of this type is very similar in form to a contact pressure computed from a penalty approximation of the constraint [7–10].

Power laws such as (Eq 10) lead to frictional stress formulas of the form suggested by both theoretical and experimental studies of rubber sliding on rough rigid surfaces [11, 12].

**Steady Spinning/Cornering of a Deformable Cylinder**

To establish notation and conventions, we first describe the simplest case of a spinning/cornering deformable cylinder which is not in contact with another body. All admissible motions \( \eta \) of the cylinder belong to the space...
$V$, the precise form of which depends upon the form of the strain energy density function $W$. If

$$V = \left\{ \eta = (\eta_1, \eta_2, \eta_3) \middle| \int_{0}^{\infty} W(\nabla \eta) d\nu_0 < \infty \right\}$$

(11)
where \( \Omega_0 \) is the interior of the cylinder in the reference configuration and \( dv_0 \) is an element of volume in the reference configuration. The space of admissible motions is defined by

\[
\mathcal{V} = \{ \eta \in \mathcal{V} | \eta r_{\Omega_x} = R_j \text{ a.e. on } \Gamma_D \}
\]

where \( R_j \) is the position vector in the reference configuration.

The total energy of the spinning/cornering cylinder is defined by the functional.

\[
\pi : \mathcal{V} \rightarrow \mathbb{R}, \quad \pi(\eta) = \mathcal{H}(\eta) - \mathcal{E}(\eta)
\]

where \( \mathcal{H} \) is the total kinetic energy and \( \mathcal{E} \) is the total potential energy:

\[
\mathcal{H}(\eta) = \frac{1}{2} \int_{\Omega_0} \rho_0 \mathbf{v} \cdot \mathbf{v} dv_0 = \frac{1}{2} B(\eta, \eta) \quad \forall \eta \in \mathcal{V}
\]

\[
\mathcal{E}(\eta) = \int_{\Omega_0} W(\nabla \eta) dv_0 - \int_{\Omega_0} \rho_0 b_0 \cdot \eta dv_0 \quad \forall \eta \in \mathcal{V}
\]

Here, \( B(\cdot, \cdot) \) is the symmetric bilinear form.

\[
B : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}, \quad B(\mathbf{v}, \eta) = \int_{\Omega_0} \rho_0 \mathbf{v} \cdot \nabla \eta dv_0
\]

with the obvious notation, \( \mathbf{v} \) and \( \eta \) being given in Eq 2.

The spinning/cornering cylinder is in a steady state motion whenever \( \delta \pi(\mathbf{v}, \eta) = 0 \) for all admissible motions \( \eta \). Thus, we led to our first variational problem:

Find a motion \( \mathbf{v} \in \mathcal{V} \) such that

\[
A(\mathbf{v}, \eta) = B(\mathbf{v}, \eta) + f(\eta) \quad \forall \eta \in \mathcal{V}
\]

Where

\[
A(\mathbf{v}, \eta) = \int_{\Omega_0} T_0^T(\mathbf{v}) : \nabla \eta dv_0
\]

and

\[
T_0 = \nabla \mathbf{v} S(\nabla \mathbf{v}).
\]

A standard calculation shows that whenever sufficiently smooth solutions \( \mathbf{v} \) exist, they are also solutions of the classical problem.

\[
\text{Div} \ T_0^T(\mathbf{v}) + \rho_0 b_0 = \rho_0 \mathbf{a} \quad \text{in } \Omega_0
\]

\[
\mathbf{v} = \mathbf{R}_i \quad \text{on } \Gamma_D
\]

\[
T_0^T(\mathbf{v}) n_0 = 0 \quad \text{on } \Gamma / \Gamma_D = \Gamma_C.
\]

Rolling Contact without Friction

To pass from Eq 13 to the case of frictionless rolling contact, we introduce the unilateral constraint set.
\[ \mathcal{H} = \{ \eta \in \mathcal{V} | \eta_2 \leq H(X_1, X_3) \text{ on } \Gamma_c \} \]

where "\( \leq \)" now denotes an order relation on traces of elements of \( \mathcal{V} \) [13].

Steady rolling contact is now characterized by the variational inequality. Find \( x \in \mathcal{H} \) such that

\[ A(x, \eta - x) - B(x, \eta - x, \dot{\theta}_2, \dot{\theta}_3) \geq f(\eta - x) \quad \forall \eta \in \mathcal{H} \quad (14) \]

This variational inequality also applies to the case of incompressible materials if we replace \( \mathcal{H} \) by the set

\[ \mathcal{H}^* = \{ \eta \in \mathcal{V} | \eta_2 \leq H \text{ on } \Gamma_c, \det \eta = 1 \text{ a.e. in } \Omega_0 \} \].

Alternatively, we can handle the incompressibility constraint by the method of Lagrange multipliers. Then, we introduce a space \( Q \) of multipliers. E.g., if \( \det \eta \in L^p(\Omega_0) \), then \( Q = L^p(\Omega_0) \) where \( (1/p) + (1/p') = 1 \). We then have.

Find \( (x, p) \in \mathcal{H} \times Q \) such that

\[ \text{FIG. 5 — Dimensions of the rolling cylinder.} \]
\[ A(\chi, \eta - \chi) - B(\chi, \eta - \chi, \dot{\theta}_2, \dot{\theta}_3) + I(p, \chi, \eta - \chi) \geq f(\eta - \chi) \quad \forall \eta \in \mathcal{H} \]

\[ (q, (\det \nabla \chi - 1)) = 0 \quad \forall q \in Q \]

Here \( I(\cdot, \cdot, \cdot) \) is the trilinear form.

\[ I(p, \chi, \eta) = \int_{\mathcal{V}_0} p \text{ adj } \nabla \chi : \nabla \eta d\mathcal{V}_0 \]

+ NEAR CONTACT SURFACE \( \mathcal{V}_0 = 18 \)

○ NEAR BOND SURFACE

FIG. 6 — Radial stress for standard example.
where \( \text{adj} \nabla_X \) is the transpose of the matrix of cofactors of \( \nabla_X \) and

\[
(q, (\det \nabla_X - 1)) = \int_{\Omega} q(\det \nabla_X - 1) dv_0.
\]

Rolling Contact with Friction

The addition of friction presents a significant complication to the variational formulation. Owing to the presence of slip velocities in the friction law, admissible motions may belong to a special velocity-motion space.
\[ \mathcal{U} = \{ \eta = (\dot{x}_t, x_n) \in \mathcal{V} \mid \forall \chi \in \mathcal{V} \} \]

where \( \dot{x}_t \) is the tangential velocity component and \( x_n \) is the motion in the normal direction.

The frictional effects enter the formulation through a functional \( j \) which is used to represent the virtual work (power) produced by frictional stresses:

\[
j : \mathcal{V} \times \mathcal{U} \rightarrow \mathbb{R}
\]

\[
j(\chi, \eta) = \int_{\Gamma_e} \mu |T_0^T(\chi)n_0 \cdot n| ||\eta - v_0|| ds_0
\]

**Fig. 8** — Shear stress for standard example.
The rolling/cornering contact problem for a cylinder of compressible material for which Coulomb's law of friction holds on the contact surface, is characterized by the variational inequality governing the following variational boundary-value problem:

Find $x \in U \cap H$ such that

$$A(x, \eta - \nabla \xi) - B(x, \eta - \nabla \xi, \dot{\theta}_2, \dot{\theta}_3) + j(x, \eta) - j(x, \nabla \xi) \geq f(\eta - \nabla \xi)$$

$\forall \eta \in U \cap H$

+ NEAR CONTACT SURFACE $V_0 = 18$
O NEAR BOND SURFACE

**FIG. 9** — Radial strain for standard example.
Here we have used the notation
\[ \nabla \delta \chi = (\dot{x}_1, x_n). \]

Numerical Examples and Discussions
In this section, we discuss numerical results obtained using the models and methods discussed previously. We begin with the analysis of the motion of the rolling cylinder shown in Fig. 5. The dimensions of the cylinder are \( R_1 = 1 \) in. and \( R_0 = 2 \) in. As noted earlier, the cylinder is assumed to be in a

![Diagram](image)

**FIG. 10 — Circumferential strain for standard example.**
state of plane strain and to be constructed of a homogeneous isotropic incompressible hyperelastic material with properties close to those of rubber. The hyperelastic material is characterized by the standard Mooney-Rivlin law with coefficients of the stored energy function of $C_1 = 80$ psi (551.5 kN) and $C_2 = 20$ psi (137.9 kN). The mass density is taken to be $\rho_0 = 0.036$ lb·sec$^2$/in$^4$ (0.618 kg·s$^2$/m$^4$) and the angular velocity of the rolling cylinder is $\omega = 10$ rad/s. The cylinder is supported on the portion $\Gamma_D$ of its boundary and is compressed along a frictional interface $\Gamma_c$ by a flat "rigid" foundation, the vertical downward displacement of which is prescribed: \textit{i.e.}

\begin{center}
\begin{tabular}{c}
\text{FIG. 11} \quad \text{Shear strain for standard example.}
\end{tabular}
\end{center}
DISP = R₀ - H = 0.2 in (5.1 mm). The tangential velocity of the rigid foundation is prescribed as \( v₀ = \omega H \). The normal contact properties of the interface are characterized by \( c_n = 10^3 \) and \( m_n = 1 \). The coefficient of friction along \( \Gamma_c \) is taken to be \( \mu = 0.3 \) and we take \( b_n = 0 \). The regularization parameter \( \epsilon \) is set equal to 0.25 in/s (6.35 mm/s). Our plan here is to analyze this representative example and to then compare results obtained in several series of calculations in which key parameters of the rolling contact problem are varied.

Results are shown in Figs. 6–10. If viscous effects are negligible and the contact interface is frictionless, the stresses are symmetric or anti-symmetric with respect to the central point of the contact region as expected. The normal compliance and the tangential component of the velocity on the contact interface are also symmetric. The symmetry of the contact pressure profile can also be observed from the radial stress distribution curve. On the other hand, the introduction of viscous effects or friction destroys this symmetry.

![Graph](image)

**FIG. 12** — Pressure distribution on contact surface with different foundation velocities.
The stress distribution along the surface $\Gamma_D$ is smoother than that of the outer surface. Furthermore, the distribution curve shifts toward the trailing edge due to the shear effects. In an ideal case (no viscous effects or friction), the shear curve is anti-symmetric. In the more general case, it behaves similar to a sine wave near the bonded (hub) surface (see Fig. 8). The shear distribution near the contact surface is quite different. The most critical shear stress is, in general, developed on the bonded surface which corresponds to the leading and trailing edges of the contact surface. Furthermore, the computed shear force changes sign inside the contact region.

Figures 9, 10, and 11 show the computed variation in Green strain. The strains near the contact surface are on the order of 20 percent vertical displacement to radial thickness ratio. It is important to note that Figs. 6–11 also represent the history (stress or strain) of a particle during a rotating time cycle.

FIG. 13 — Velocity distribution on contact surface with different foundation velocities.
FIG. 14 — Stress contour for $v_0 = 20$ in./s (508 mm/s).

FIG. 15 — Stress contour for $v_0 = 22$ in./s (559 mm/s).
FIG. 16 — Stress contour for \( v_0 = 24 \) in./s (586 mm/s).

FIG. 17 — Pressure distribution on contact surface with different hub radii.
Some small oscillations in radial stress are observed on the leading and trailing edge of the contact interface as shown in Fig. 6.

Effects of Foundation Tangential Velocity with Different Regularization Parameters

Changes in the magnitude of the foundation tangential velocity significantly influence the contact pressure and the particle velocity distribution on the contact surface. For a stiff material like steel, the change of the particle velocity distribution on the contact surface is generally small and smooth. In general, its distribution on the contact interface will correspond to a non-slip state from the leading edge to a slip state behind the leading edge of the contact surface. However, for a soft material like rubber, the change of the particle velocity distribution on the contact surface is large.
and irregular and only a small portion of the contact surface may, in general, undergo non-slip.

Consider a hyperelastic cylinder without viscous terms and suppose that the angular velocity is $\omega = 10 \text{ rad/s}$ and the vertical displacement is $\text{DISP} = 0.2 \text{ in} (5.1 \text{ mm})$. Thus, the rigid velocity on the contact surface $\omega H$ is $18 \text{ in/s} (457 \text{ mm/s})$. Figures 12-16 show stress contours obtained for different ground velocities. The full slip case (i.e. $18 \text{ in/s} (457 \text{ mm/s})$) is easily solved by setting the foundation tangential velocity equal to $18 \text{ in/s} (457 \text{ mm/s})$. Very fast convergence is attained (one step with 7 iterations). The solution scheme diverges immediately for foundation velocities greater than $v_0 = 18 \text{ in/s}$.

**Effects of Changes of Hub Radius**

As another example, we show the effects of changes of the hub radius of the cylinder, keeping $R_0$ fixed. In other words, we maintain the same prescribed displacements but decrease the thickness of the rubber case. As expected, we obtain significant increases in the stresses and strains for fixed $R$ and $\omega$. Results are given in Figs. 17-20.

In summary, an increase in angular velocity and rubber thickness generally decreases the total contact force. On the other hand, an increase in displacement and frictional forces will increase the total contact force.

![Stress contour for $R_1 = 0.89 \text{ in.} (21.4 \text{ mm})$.](image)
Lastly, when we compare the maximum extensional and shear strain, the value of maximum shear strain is almost twice that of the extensional strain for the examples considered.

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