CONVERGENCE AND ERROR ESTIMATES FOR
FINITE ELEMENT SOLUTIONS OF
ELASTOHYDRODYNAMIC LUBRICATION

S. R. Wu and J. T. Oden
TICOM, University of Texas at Austin, Austin, TX 78712, U.S.A.

(Received 27 August 1986)

Abstract—By using a formulation of EHL problems which employs a nonlinear variational inequality and an exterior penalty method, convergence of finite element approximations to the EHL problem is studied. It is shown that the finite element method for the penalized problem is convergent. Then an a priori error estimate of the finite element solution is derived. Numerical experiments are performed for both line contact and point contact problems. In cases of light loads, first-order convergence in the $H^1$-norm for pressure approximation by linear elements and second-order convergence for quadratic elements are achieved as predicted. One order higher convergence in the $L^2$-norm is also observed as expected.

1. INTRODUCTION

Accuracy and error estimation is an important issue in determining the utility and acceptability of any numerical scheme. Since the early 1970s, considerable effort has been expended in developing theories concerning the numerical accuracy of finite element methods. Particularly for elliptical problems, Oden and Reddy [1] presented an introduction to the mathematical theory of finite elements and Ciarlet [2] summarized the important results of that period, including some nonlinear cases, see also Oden and Carey [3]. Some results of finite element methods for nonlinear problems characterized by strongly monotone operators were also obtained [4–6]. Oden and Reddy [7] made a substantial generalization of error estimation methods to problems in nonlinear elasticity which featured a pseudomonotone operator. While a great number of articles have been devoted to fluid dynamics, plates and shells, nonlinear elasticity etc., error estimates for finite element approximations of nonlinear lubrication problems do not appear to be available.

Li and Dai [8] made a significant analysis for gas thrust bearings, although elasticity of the bearing was not taken into account. Few other works on the mathematical properties of numerical methods for lubrication problems can be found, either on finite element methods or on finite difference methods.

Here error estimates are derived for approximations of the Reynolds-Hertz equations subject to some additional simplifying assumptions. We first demonstrate the convergence behavior of the approximate solutions in finite dimensional spaces. This provides a framework for the accuracy analysis and then the derivation of an error estimate.

2. PRELIMINARIES

For classical elastohydrodynamic lubrication (EHL) problems, the Reynolds-Hertz model of non-Newtonian flow through an elastic bearing is modeled by the following system of partial differential equations, an inequality constraint and boundary conditions:

\[
\begin{align*}
\nabla \cdot (h^3 e^{-xp} \nabla p) + 12 \mu_0 \partial (uh_1)/\partial x &= -12 \mu_0 \partial (uh_2)/\partial x, \\
p > 0 & \text{ in } \Omega_1 \\
p = 0 & \text{ in } \Omega_0, \quad p_{|\partial \Omega} = 0, \quad \Omega = \Omega_1 \cup \Omega_0,
\end{align*}
\]

where (referring to Fig. 1)

\[
h = h_1(p) + h_2.
\]
\[
\begin{align*}
\tilde{h}_1(p) &= h_1(p) = 2\pi E' \int p(\xi) \ln (x_0 - \xi/x - \xi)^2 d\xi \text{ (line contact)} \\
\tilde{h}_2 &= h_0 + h_z \\
h_z &= R - \sqrt{R^2 - x^2} \quad (\text{line contact}); \quad = R - \sqrt{R^2 - x^2 - y^2} \quad (\text{point contact}); \\
\end{align*}
\]

where \( h_0 \) is the reference thickness. One may also use the minimum film thickness \( h_m \) in the formulation, where

\[
\begin{align*}
\tilde{h}_1(p) &= h_1(p) - S(h) \\
\tilde{h}_2 &= h_m + h_z \\
S(h) &= \min(h(p) + h_z); \\
\end{align*}
\]

\( h_1(p) \) is the contribution of elastic deformation and \( E' \) is the effective elastic modulus, \( \mu_0 \) is the viscosity of the lubricant under atmospheric pressure, \( \alpha \) is the exponential parameter for the viscosity and \( u \) is the rolling velocity.

It is well known that this problem is a free boundary problem, with unknown interface \( \partial \Omega = \Omega \cap \Omega_0 \), where the pressure recovers the ambient value, assumed to be zero.

Take \( p \geq 0 \) in \( \Omega \) as a constraint and consider the fact that the Reynolds equation is valid only in the contact region \( \Omega_t \). This situation can be characterized by a nonlinear variational inequality.

First define the operator in Reynolds' equation as

\[
A: p \to -\nabla \cdot (h^t(p) e^{-up} \nabla p) + 12\mu_0 \partial (u \tilde{h}_1(p))/\partial x
\]

and a function

\[
f = -12\mu_0 \partial (u \tilde{h}_2)/\partial x.
\]

Then equation (1), reduces to \( Ap = f \) in \( \Omega_t \). We construct a weak statement for the problem in the formulation of a variational inequality:

\[
(P): \text{find } p \in K = \{ q \in V | q \geq 0 \text{ a.e. in } \Omega \}, \text{ such that } \\
\langle A(p) - f, \quad q - p \rangle_{V \times V} \geq 0, \quad \forall q \in K.
\]

Here we use the Sobolev space \( V = H^1_0(\Omega) \) as an appropriate domain for operator \( A: V \to V' = H^{-1}(\Omega) \), \( V' \) being the dual space of \( V \), and denote by \( \langle \cdot, \cdot \rangle \) the duality on \( V \times V' \).

It is easy to show the following (see Oden and Wu [9]).

**Lemma 2.1**

If \( p \) is a solution of the variational inequality (14), then \( p \) is a solution of equations (1) and (2) in a distributional sense, called a weak solution.

To further the study and to implement an effective numerical scheme, we introduce a penalty term to regularize the inequality constraint (1)_2. The penalty method has been used successfully in dealing with certain classes of contact problems and other kind of free boundary problems. For a general reference, see Oden and Kikuchi [10].

Let \( p^- = \min(p, 0) = (p - |p|)/2. \) Introduce a penalty operator \( \Phi: H^1_0(\Omega) \to H^{-1} \) as

\[
\Phi(p) = p^-/\epsilon \quad \text{with } \epsilon > 0.
\]

We then construct a penalty problem,

\[
(P_\epsilon): \text{for } \epsilon > 0, \text{ find } p_\epsilon \in V = H^1_0(\Omega), \text{ such that } \\
\langle A(p_\epsilon), q \rangle + \langle \Phi(p_\epsilon), q \rangle = \langle f, q \rangle, \quad \forall q \in V.
\]
The justification of the penalty method is established in the following convergence theorem (for the details of the proof, see Refs [11, 12]).

**Theorem 2.1**

Denote by \( p, \) the solutions of expression (9) corresponding to \( \epsilon > 0. \) As \( \epsilon \to 0, \) there exists a subsequence of \( \{ p_n \} \) which converges to some \( p \) in \( V \) weakly, where \( p \) is just a solution of the variational inequality (7), also a weak solution of equalities (1–4). Moreover this sequence actually converges to \( p \) strongly in \( V. \)

We cite here some properties of operator \( A \) for later reference.

**Lemma 2.2**

Operator \( A \) is bounded, coercive, pseudomonotone and continuous. There exists a constant \( C_4(\eta) \), for any \( p, q \in B_\eta = \{ q \in V \mid \| q \| \leq \eta \}, \) such that

\[
\| A(p) - A(q) \| \leq C_4 \| p - q \|_V. \tag{10}
\]

On the other hand, we have constants \( C_1 \) and \( C_n > 0 \), for any \( p, q \in B_\eta \), such that

\[
| \langle A(p) - A(q), p - q \rangle | \geq C_1 \| p - q \|_V^2 - C_n \| p - q \|_V \| p - q \|_\epsilon. \tag{11}
\]

## 3. APPROXIMATIONS OF PENALTY SOLUTIONS IN FINITE DIMENSIONAL SPACES

For approximating the functions in Sobolev space \( V, \) we introduce a family of finite dimensional subspaces \( \{ V_n \} : \)

\[
V_n \subset V_{n+1} \subset \cdots \subset V.
\]

Then

\[
\bigcup_{n \geq 1} V_n \text{ is everywhere dense in } V. \tag{12}
\]

We seek solutions of the approximate penalized problems in the subspaces \( V_n \):

\[
(P, N): \text{find } p^*_n \in V_n, \text{ such that } \langle A(p^*_n), q^n \rangle + \langle (p^*_n)^-, q^n \rangle / \epsilon = \langle f, q^n \rangle, \quad \forall q^n \in V_n. \tag{14}
\]

Denote for the penalized operator,

\[
A_\epsilon = A + \Phi. \quad A_\epsilon(q) = A(q) + q^-/\epsilon, \quad \forall q \in H^1_0(\Omega). \tag{15}
\]

Then we have (see Ref. [11]).

**Theorem 3.1**

Operator \( A_\epsilon \) is bounded, hemicontinuous, coercive, pseudomonotone, and continuous on the subspaces \( V_n (n = 1, 2, \ldots) \). And there exist solutions \( p^*_n \in V_n \subset H^1_0(\Omega) \) of expression (14) for every \( \epsilon > 0. \)

We also have the following orthogonality condition.

**Lemma 3.1**

For the approximate solution \( p^*_n \) of expression (14) in the approximation space \( V_n, \) we have the orthogonality condition

\[
\langle A_\epsilon(p_i) - A_\epsilon(p^*_n), q^* \rangle = 0, \quad \forall q^* \in V_n. \tag{16}
\]
Theorem 3.2
For the approximate solutions \( \{p^*_n\} \) in \( V_n \), there exists a subsequence of \( \{p^*_n\} \) as \( n \to \infty \), which converges weakly to a \( p \in V \), a solution of the original penalized problem (9) and \( A_i(p^*_n) \to f \) weakly in \( V' \). Furthermore, the subsequence, in fact, converges strongly to \( p \) in \( V \).

Proof. For the approximate solutions \( p^*_n \in V_n \),
\[
\langle f, p^*_n \rangle = \langle A_i(p^*_n), p^*_n \rangle = \langle (A + \Phi)(p^*_n), p^*_n \rangle, \quad n = 1, 2, \ldots ,
\]
Coercivity of \( A \) results in the boundedness of \( \{p^*_n\} \in V = H^1_0 \) and \( A_i(p^*_n) \) in \( V' = H^{-1} \). Hence we can find a subsequence, still denoted by \( \{p^*_n\} \), which converges weakly to some \( p \in V \), and \( A_i(p^*_n) \) converges weakly to some \( g \) in \( V' \).

We are to prove that \( p \) is a solution and \( g = f \).

For any \( q \in V \), we may find a sequence \( q^n \in V_n \) such that \( q^n \to q \) in \( V \). \( p^*_n \) is the solution in \( V_n \), so
\[
\langle A_i(p^*_n), q^n \rangle = \langle f, q^n \rangle \to \langle f, q \rangle,
\]
Thus \( \langle A_i(p^*_n), q \rangle = \langle A_i(p^*_n), q - q^n \rangle + \langle A_i(p^*_n), q^n \rangle \to \langle f, q \rangle \).

On the other hand, the fact that \( A_i(p^*_n) \to g \) weakly leads to \( \langle A_i(p^*_n), q \rangle \to \langle g, q \rangle \). This yields \( \langle g, q \rangle = \langle f, q \rangle \) for all \( q \in V \). Therefore
\[
\langle A_i(p^*_n), p^*_n - p \rangle = \langle f, p^*_n \rangle - \langle A_i(p^*_n), p \rangle \to \langle f, p \rangle - \langle g, p \rangle = 0.
\]
This means that the property \( P \) is satisfied by \( A \) with the sequence \( \{p^*_n\} \). \( A \), being pseudomonotone (for the definition of which and property \( P \), see Ref. [13]) gives
\[
\lim_{n \to \infty} \langle A_i(p^*_n), p^*_n - q \rangle \geq \langle A_i(p), p - q \rangle, \quad \forall q \in V.
\]
From
\[
\langle A_i(p^*_n), p^*_n \rangle = \langle f, p^*_n \rangle \to \langle f, p \rangle, \quad \langle A_i(p^*_n), q \rangle \to \langle g, q \rangle = \langle f, q \rangle
\]
we have
\[
\langle f, p - q \rangle \geq \langle A_i(p), p - q \rangle, \quad \forall q \in V.
\]

For any \( w \in V \), setting \( q = p_i - \theta w \) gives \( \langle f, \theta w \rangle \geq \langle A_i(p), \theta w \rangle \). Equivalently \( \langle A_i(p), w \rangle = \langle f, w \rangle \) since \( \theta \) can be chosen positive or negative. This establishes that \( p_i \) is a solution of problem (9).

Going a step further, if we taken a sequence \( r^* \in V_n, r^* \to p_i \) in \( V \), then from boundedness, we have
\[
|\langle A_i(p_i) - A_i(p^*_n), p_i - r^* \rangle| \leq \|A_i(p_i) - A_i(p^*_n)\|_V \|p_i - r^*\|_V \to 0.
\]
By virtue of Lemma 2.2 and the monotonicity of \( \Phi \), we have
\[
\langle A_i(p_i), p_i - p^*_n \rangle \geq C_1 \|p_i - p^*_n\|_{L^2} - C_q \|p_i - p^*_n\|_{L^q} \|p_i - p^*_n\|_{H^1}.
\]
Orthogonality (16) gives
\[
\langle A_i(p_i) - A_i(p^*_n), p_i - r^* \rangle = \langle A_i(p_i) - A_i(p^*_n), p_i - p^*_n \rangle.
\]
Again the compact embedding of \( H^1 \) in \( L^q \) results in \( p^*_n \to p_i \) strongly in \( L^q \). Finally we obtain \( \|p_i - p^*_n\|_{H^1} \to 0 \).

Next, we may construct a finite dimensional approximation for the variational inequality. Denote
\[
K_n = K \cap V_n = \{q \in V_n | q \geq 0 \text{ a.e. in } \Omega\}
\]
\[
K_n \subset K_{n+1} \subset \cdots \subset K, \quad \overline{K_n} = K
\]
We also have the approximation property:
\[
\forall q \in K, \exists \text{ a sequence } q^n \in K_n, q^n \to q \text{ in the } V\text{-norm}.
\]
The $K_n$ are closed and convex subsets of $V_n$. Our approximation is characterized as the problem
\begin{equation}
(P^n): \text{find } p^n \in K_n, \text{ such that } \langle A(p^n) - f, q^n - p^n \rangle \geq 0, \quad \forall q^n \in K_n.
\end{equation}

Similar to Theorem 2.1, we have directly the convergence property.

**Theorem 3.3**

In the approximate spaces $V_n$, when $\epsilon \to 0$, there exists a subsequence of $\{p^n\}$, the solutions of the penalized problem (14), which converges to some $p^n \in K_n$ in $V$, the solution of expression (19).

Recall that in finite dimensional spaces, weak convergence is equivalent to strong convergence. On the other hand, we have the following for the approximation for expression (7).

**Theorem 3.4**

There exists a subsequence of $\{p^n\}$, the approximate solutions of the variational inequality in finite dimensional spaces (19), which converges to some $p \in K$ in the $V$ norm as $n \to \infty$. Moreover, $p$ is a solution of problem (P), the variational inequality (7).

The proof is omitted here, which may be found in Wu [12].

Up to now we have constructed the finite dimensional approximations for both the variational inequality and the penalized problem. Also we have established the convergence relations (all in the $H^1$ norm), summarized in the following diagram.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{diagram.png}
\caption{Convergence Diagram}
\end{figure}

4. AN A PRIORI ERROR ESTIMATE

Making use of inequality (8) in Lemma 2.2, along with the monotonicity of the operator $\Phi$ and the orthogonality (16), we have

\begin{align*}
C_1(\|p_n - p^n\| - \|q_n - q^n\|) - C_n\|p_n - p^n\|_{L^p} \|q_n - p^n\|_m & \leq \langle A(p_n) - A(p^n), p_n - p^n \rangle \\
& \leq \langle A_n(p_n) - A_n(p^n), p_n - p^n \rangle = \langle A_n(p_n) - A_n(p^n), p_n - r^n \rangle, \quad \forall r^n \in V_n.
\end{align*}

By the continuity of $A$,
\begin{equation}
\|A(p_n) - A(p^n)\| \leq C_A \|p_n - p^n\|_m.
\end{equation}

Note that,
\begin{align*}
\forall q_1, q_2 \in V \|\Phi(q_1) - \Phi(q_2)\|_{-1} = \sup_{\|r\|_m = 1} \langle q_1 - q_2, r \rangle / \|r\|_m.
\end{align*}

It is easy to prove that $\|q_1 - q_2\|_0 \leq \|q_1 - q_2\|$. Therefore
\begin{align*}
\|\Phi(q_1) - \Phi(q_2)\|_{-1} \leq \|q_1 - q_2\| / \epsilon \leq \|q_1 - q_2\| / \epsilon.
\end{align*}

Then we arrive at an inequality.

**Lemma 4.1**

For the approximate solutions $\{p^n\}$ of problem $(P^n)$ (14) for $\epsilon > 0$, there exists a constant $C_r > 0$, such that
\begin{equation}
C_1\|p_n - p^n\|_m - C_n\|p_n - p^n\|_{L^p} \leq C_n\|p_n - r^n\|_H, \quad \forall r^n \in V_n.
\end{equation}
By the compactness of $L^s$ in $H^1$, there is a $C_q > 0$ such that $\| \cdot \|_{L^s} \leq C_q \| \cdot \|_{H^1}$. So we have
\[
(C_1 - C_q C_q) \| p - p^* \| \leq C_1 \| p - r^* \|_{H^1}, \quad \forall r^* \in V_n.
\] 
(21)

Now we use parameter $h$ traditionally to indicate the mesh size in the finite element method, and we construct a family of finite element approximation spaces with mesh parameter $h_n$ as follows:
\[
V_{h_1} \subset V_{h_2} \subset \cdots \subset V, \quad 1 > h_1 > h_2 > \cdots > 0
\]
\[
U V_{h_n} \text{ is everywhere dense in } V \text{ with } h_n \to 0.
\] 
(22)

According to finite element interpolation theory [1–3], we are able to establish an a priori error estimate for the approximate solutions of penalized problem in finite dimensional spaces $V_n$.

**Theorem 4.1**

If the penalty solution $p, \in H^r(\Omega), r \geq 1$, and the shape functions of finite elements contain $P_k$, the complete polynomials of degrees \( \leq k, k \geq 1 \); then under the assumption $C_1 - C_q C_q > 0$, and for a regular affine family of finite elements, there exists a constant $C > 0$, such that for the approximate solutions $p^*_n$ on $V$, we have the following a priori error estimate:
\[
\| p - p^*_n \|_{H^r} \leq C h^\mu \| p \|_{H^{r+k}},
\]
\[
\mu = \min(k, r - 1).
\] 
(23)

**Corollary 4.1**

If the penalty solution $p, \in H^r(\Omega), r \geq k + 1$, under the conditions in Theorem 4.1, we have the error estimate
\[
\| p - p^*_n \|_{H^r} \leq C h^\mu \| p \|_{H^r}, \text{ (linear elements)}
\] 
(24)

and
\[
\| p - p^*_n \|_{H^r} \leq C h^2 \| p \|_{H^r}, \text{ (quadratic elements)}.\]
(25)

**Remark 4.1**

For 1-D problems, we have proved the regularity of solutions [11], that $p, \in H^2(\Omega)$, so at least expression (24) is realistic. And we may obtain
\[
h(p, - h(p^*_n) = \frac{2}{\pi E} \int_a^b \ln \left(\frac{\xi - x}{\xi - x_0}\right)^2 \left[p_r(\xi) - p^*_r(\xi)\right] \, d\xi
\]
\[
h'(p, - h'(p^*_n) = \frac{2}{\pi E} \int_a^b \ln (\xi - x)^2 \left[p_r(\xi) - (p^*_r)'(\xi)\right] \, d\xi.
\]

Thus we have
\[
\| h - h^*_n \|_{H^s} \leq C \| p - p^*_n \|_{H^s}, \quad s = 0, 1.
\] 
(26)

This means the film thickness has at least the same rate of convergence as the pressure.

Moreover, for the load $w = \int p \, d\Omega$, we have
\[
\| w - w^*_n \| \leq \int p \, d\Omega - \int p^*_n \, d\Omega \leq C \| p - p^*_n \|_{L^2}.
\] 
(27)

Thus the convergence of the load is at least as fast as the pressure in the $L^2$-norm.

## 5. NUMERICAL EXPERIMENTS ON ERROR ESTIMATES FOR FINITE ELEMENT SOLUTIONS

A series of tests on finite element solutions have been performed. For simplicity, uniform meshes are employed. A very fine mesh is taken for obtaining a solution $p^*_n$ in place of the exact solution. And then the analysis is based on the comparisons between this solution and the solutions obtained from the coarse meshes.
5.1. Line contact problems

Figures 2 and 3 show the computed error behavior for light load cases of line contact, computed with the linear elements. It is evident that the logarithm relation between \( \| p^*_e - p^*_h \|_{H^1} \) and \( h \), the mesh parameter, is linear with slope 1. That is, first order convergence is obtained, and the prediction of expression (24) is verified. In fact, the result holds for a quite wide range of loads.

Figures 4 and 5 show the analysis with quadratic elements. For a certain range of loads, the rates of convergence for pressure in the \( H^1 \)-norm are \( O(h^2) \), as predicted in expression (25).

For classical linear elliptic problems, such results are optimal. Since the operator of the governing equation is only pseudomonotone, the convergent rates are almost optimal only for a certain range of loads.

5.2. Point contact problems

For 2-D point contact problems, Figs 6 and 7 show that expression (24) also holds for certain cases with bilinear elements, but a deterioration in the rate of convergence is observed for slightly heavier loads as shown in Fig. 8. A similar situation appears for the solutions computed with biquadratic elements, shown in Figs 9-11. This suggests that condition \( C_1 - C_q < 0 \) does not hold all the time or the solutions are not smooth enough to be in the Sobolev spaces of higher order.

Remark 5.1

For linear elliptic problems, the Aubin–Nitsche method [3] provides higher order convergence when the error is measured in the Sobolev spaces of lower orders. Here it is interesting to notice that the similar behavior is observed. The plots here show that in many cases, the pressure exhibits
Fig. 3. Error analysis (1-D linear elements, $H_0 = -0.2\times10^{-4}$), line contact ($W = 0.515\times10^{-5}$).

Fig. 4. Error analysis (1-D quadratic elements, $H_0 = 0$), line contact ($W = 0.305\times10^{-5}$).

Fig. 5. Error analysis (1-D quadratic elements, $H_0 = -0.1\times10^{-4}$), line contact ($W = 0.940\times10^{-5}$).

Fig. 6. Error analysis (2-D bilinear elements, $H_0 = 0.5\times10^{-5}$), point contact ($W = 0.680\times10^{-8}$).
Fig. 7. Error analysis (2-D bilinear elements, \( H_0 = 0.1 \times 10^{-5} \)), point contact (\( W = 0.126 \times 10^{-7} \)).

Fig. 8. Error analysis (2-D bilinear elements, \( H_0 = 0.5 \times 10^{-6} \)), point contact (\( W = 0.142 \times 10^{-7} \)).

Fig. 9. Error analysis (2-D biquadratic elements, \( H_0 = 0.9 \times 10^{-5} \)), point contact (\( W = 0.0463 \times 10^{-8} \)).

Fig. 10. Error analysis (2-D biquadratic elements, \( H_0 = 0.75 \times 10^{-5} \)), point contact (\( W = 0.528 \times 10^{-8} \)).
second-order convergence with linear elements, and third-order convergence with quadratic elements, when measured in the $L^2$-norm, just one order higher than those in $H^1$-norm as the case of linear elliptic problems.

Remark 5.2
The predictions (26) and (27) are also verified for a certain range of loads, as shown in the plots here. When calculated with linear elements, the film thickness gains first order convergence in the $H^1$-norm and second order convergence in the $L^2$-norm, just the same as the pressure. The load converges with a rate of $O(h^2)$, which is the same as the pressure measured in the $L^2$-norm. When calculated with quadratic elements, the rates of convergence are one order higher than the corresponding rates with linear elements, as expected.

Acknowledgement—Support from the Air Force Office of Scientific Research (AFSC) under Contract No. F49620-84-C-0024 is gratefully acknowledged.

REFERENCES
