SOLUTION TO SIGNORINI-LIKE CONTACT PROBLEMS THROUGH INTERFACE MODELS—I. PRELIMINARIES AND FORMULATION OF A VARIATIONAL EQUALITY

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We present a study of contact problems usually modeled by Signorini's problem. Our approach differs in that we make use of constitutive relations for the normal response along the candidate contact surface. The form of these models is dictated by experimental evidence and they lead to a variational equality instead of an inequality. We have focused on the most delicate case of contact-traction boundary conditions for which we obtain existence and optimal uniqueness results under physically realistic assumptions. The other usual boundary conditions can be dealt with similarly with simplifications in the proofs. Signorini's problem is shown to be recovered as the limiting case of an infinite normal stiffness, while our model allows for perturbations describing friction phenomena, according to Coulomb's law or generalizations of it.

Serious mathematical difficulties arise from the fact that the most general type of interface models rules out the use of Sobolev's embedding theorem, without which the problem is no longer in the province of standard convex analysis but rather lies in the realm of the theory of hypermaximal monotone operators having a domain with empty interior. Several auxiliary results, including an apparently new property in Sobolev spaces, are proven which, together with the general method of proof, should be of interest in other problems. The contact-traction boundary conditions require compatibility conditions to be introduced. They have a somewhat more sophisticated form than the standard ones involved in Signorini's problem and further examination shows that their physical content agrees strikingly with common sense physical observations.

The work is divided into two principal parts: Part I, which focuses on preliminaries needed for the analysis of the traction problem, special Sobolev space properties, and the formulation of the weak form of the nonlinear boundary-value problem; and Part II [1] which addresses questions of existence and uniqueness of solutions.
1. INTRODUCTION

Let $\Omega$ be an open, bounded subset of $\mathbb{R}^N$ with a Lipschitz continuous boundary $\Gamma$, the disjoint union $\Gamma_C \cup \Gamma_F$ with $\Gamma_C$ and $\Gamma_F$ measurable and $\text{meas}(\Gamma_C) > 0$. For $N = 2$ or $N = 3$, $\Omega$ represents the reference configuration of a body in geometrical contact with another body along $\Gamma_C$. In other words, no external forces are present and the boundaries of the two bodies coincide along $\Gamma_C$. Suppose now that the body occupying the domain $\Omega$ is submitted to external forces $f$, consisting of body forces $b$ defined in $\Omega$ and tractions $t$ prescribed on $\Gamma_F$. Assuming that no point of $\Gamma_F$ may come in contact with the second body under the action of the forces $f$, i.e. that $\Gamma_C$ is the candidate contact surface for the deformed configuration, and that the material has a linear elastic behavior, the problem of contact with no friction is usually understood as the Signorini problem: minimize

$$\frac{1}{2} a(v, v) - \langle f, v \rangle_{\Omega},$$

over the closed convex subset of the Sobolev space $(H^1(\Omega))^N$,

$$K = \{ v \in (H^1(\Omega))^N, v_n \leq 0 \text{ on } \Gamma_C \}. \quad (1.2)$$

In (1.1) and (1.2) above, $a(\cdot, \cdot)$ denotes the symmetric bilinear form associated with the virtual work of stresses $\sigma(u)$ on strains $\varepsilon(v)(a(u,v) = \int_\Omega \sigma(u) : \varepsilon(v) \, dx)$, $\langle \cdot, \cdot \rangle_\Omega$ the duality pairing between the space $(H^1(\Omega))^N$ and its dual, and $v_n$ the component of the displacement $v$ along the outward normal vector $n$: $v_n = v \cdot n$ (euclidian inner product).

Problem (1.1)-(1.2) is a formal variational formulation of the equilibrium equations between the stresses $\sigma(u)$ and the external forces $b$ and $t$:

$$\text{div} \sigma(u) + b = 0 \text{ in } \Omega, \quad (1.3)$$

$$\sigma(u) \cdot n = t \text{ on } \Gamma_F, \quad (1.4)$$

and, on $\Gamma_C$,

$$u_n \leq 0, \quad (1.5)$$

$$\sigma(u) \cdot n = 0 \text{ if } u_n < 0, \quad (1.6)$$

$$\sigma(u) \cdot n = -\alpha n, \alpha \geq 0 \text{ if } u_n \neq 0. \quad (1.7)$$

A "qualification" of the formality of this interpretation can be found among the by-products of our approach (cf. remark 5.2).

Existence of solutions to problem (1.1)-(1.2) is known under a simple necessary and sufficient compatibility condition on the applied forces [5. see also 10]. The idea of using variational inequalities for solving contact problems goes back to [5, 10], but the theory has not been very successful in the more complicated problem with friction despite recent contributions by Necas, Jarusek and Haslinger [6, 13].

With the aim of analyzing problems of friction, Oden and Martins [14] have developed a different approach to contact problems. The key ingredient of their theory is the introduction of a model for the normal response at points of $\Gamma_C$ at which contact may occur. To do this, it is essential to remove the nonpenetration condition $u_n \leq 0$ on $\Gamma_C$. The normal response at a point $x \in \Gamma_C$ is then a function of $(u_n)_+(x)$. Contrary to a first natural reaction, removing the nonpenetration condition $u_n \leq 0$ is not physical nonsense. Indeed, in any mathematical model,
the boundary \( \Gamma_C \) is an idealized average candidate contact surface. The real candidate contact surface differing from \( \Gamma_C \) by a layer of asperities. How different the real surface is from \( \Gamma_C \) "measures" its roughness, a factor increasingly believed to originate friction phenomena (see, e.g., [14, 18, 20]). When contact occurs, the deformation of these asperities (incidentally of a nature totally different from the "visible" deformation of the body) allows small displacements of the boundary \( \Gamma_C \) towards the obstacle, violating the condition \( u_n \leq 0 \). Accordingly, when positive, the displacement \( u_n \) should nevertheless remain small. This point will be examined later on. In this view, removing the condition \( u_n \leq 0 \) does not amount to accepting actual penetration of the two bodies in contact but merely allows the average surface \( \Gamma_C \) to get closer to the obstacle. On the experimental side, these features of actual surfaces have been observed by many investigators, and the memoir [14] contains extensive arguments in support of such models. On the mathematical side, they present numerous advantages as follows. As we shall see, discrepancies between variational and boundary value problem formulations vanish, new compatibility conditions with precise physical interpretations are involved, perturbations allowing for friction phenomena become manageable, etc.

In such models, the normal response caused by the normal displacement \( u_n(x) \) at \( x \in \Gamma_C \) is then of the form \( \phi(x, u_n(x)) \) where \( \phi: \Gamma_C \times \mathbb{R} \rightarrow \mathbb{R} \) verifies \( \phi \geq 0 \) and \( \phi(x, t) = 0 \) for \( t \leq 0 \) (so that \( \phi(x, u_n(x)) \) actually depends only on \( (u_n)_t(x) \)). The function \( \phi \) depends on the interface condition and a few of its properties are dictated by common sense observations: it is intuitively clear that any positive normal displacement should produce a positive normal response, so that \( \phi(x, t) > 0 \) for \( t > 0 \). Next, an increase of the normal displacement must produce an increase of the normal response so that \( \phi(x, t) \) is increasing w.r.t. \( t \geq 0 \). Further, the resistance to penetration of the bodies suggests for a positive normal displacement \( u_n(x) \) that the ratio \( \phi(x, u_n(x))/u_n(x) \) (normal response vs normal displacement) be an increasing function of \( u_n(x) \), namely that \( \phi(x, t)/t \) is increasing w.r.t. \( t > 0 \).

Denoting by \( \phi(u_n) \) the function \( \phi(u_n)(x) = \phi(x, u_n(x)) \) for \( x \in \Gamma_C \), the contact problem (1.3)–(1.7) becomes, in this approach:

\[
\text{find } u \in (H^1(\Omega))^N \text{ such that }
\begin{align*}
\text{div } \sigma(u) + b &= 0 \quad \text{in } \Omega, \\
\sigma(u) \cdot n &= t \quad \text{on } \Gamma_F, \\
\sigma(u) \cdot n &= -\phi(u_n)n \quad \text{on } \Gamma_C.
\end{align*}
\]

(1.8)–(1.10)

In Section 3, we shall see that problem (1.8)–(1.10) has the equivalent formulation: find \( u \in (H^1(\Omega))^N \) such that

\[
a(u, v) + \int_{\Gamma_C} \phi(u_n)v_n \, ds = (f, v)_\Omega \quad \text{for every } v \in (H^1(\Omega))^N.
\]

(1.11)

Experimental evidence shows that the normal response \( \phi(x, t) \) has a power-like behavior for small values of \( t > 0 \), growing up to exponential as \( t \) is increased. Roughly speaking, the power zone ("light" normal loads) authorizes sliding, prohibited in the exponential zone ("heavy" normal loads): see [14, Fig. 48]. For this reason, in a study of similar static contact problems with friction (cf. [11]), we have limited ourselves to considering the choice \( \phi(x, t) = c_n(x)(t_+)^m \) with (experimentally justified) values of the exponent \( m_n \) allowing the use of Sobolev's embedding theorems and under convenient boundary conditions avoiding the need for compatibility conditions.
This paper is devoted to the study of problem (1.8)-(1.10) with a general \( \phi \). By comparison with the situation in [11], we face several new difficulties. First and foremost, the variational formulation (1.11) is a priori not well posed since Sobolev embedding theorems are not available without serious restrictions on the growth of the function \( t \to \phi(x, t) \) as \( t \) tends to \( +\infty \). In particular, exponential growth is prohibited when \( N \geq 3 \). This difficulty has been overcome by requiring \( \phi(u_n) \) to belong to the space \( L^1(\Gamma_C) \) as an additional condition to (1.8)-(1.10) and proving in this assumption that \( \phi(u_n)u_n \) belongs to \( L^1(\Omega) \) as soon as \( u \) is a solution to (1.8)-(1.10) (theorem 3.1). Once the variational formulation (1.11) has been justified, we show that it is equivalent to the minimization of a weakly sequentially lower semicontinuous convex functional over \( (H^1(\Omega))^N \) with values in \( \mathbb{R} = \mathbb{R} \cup \{+\infty\} \). In this process, other difficulties arise because the functional in question is nowhere continuous in the general case (i.e. its domain is empty), and the desired properties must be established by using convexity of the function \( t \to \int_0^t \phi(x, \tau) \, d\tau \) rather than that of the functional. Another technicality is to prove that the minimizers do verify equations (1.8)-(1.10) in the sense of distributions, and the condition \( \phi(u_n) \in L^1(\Gamma_C) \).

From these introductory comments, one might have guessed that the problem is, in some respects, pertaining to the theory of hypermaximal monotone operators (cf. [2] for an excellent account) rather than standard convex analysis. However, we have found no significant advantage in using the specialized vocabulary and the general results of this theory, while doing so might have caused some discomfort to the uninitiated reader. Nonetheless, it can be reasonably speculated on the basis of this relationship that our method of proof can be duplicated in other problems, thus ranging farther than the specific example for which it has been developed here.

Several general properties need to be established. Some of them, specifically related to integration theory, are collected in Section 2. In this respect, we note an interesting coincidence: most of the results of Section 2 hold under the apparently necessary condition that the mapping \( t \to \phi(x, t) \) has (at most) exponential growth at infinity. Other statements of general mathematical interest, related to a seemingly new property in Sobolev spaces, are proved in Section 3 (lemmas 3.3-3.6).

Coerciveness of the energy functional, hence existence of minimizers, is proved in Section 4 [1] over an appropriate quotient space (although the functional is not quadratic!) and under a compatibility condition on the applied external forces slightly stronger than that needed for solving problem (1.3)-(1.7). This compatibility condition is independent of the normal response \( \phi \) and, again, the necessary mathematical assumptions have significant physical counterparts. For instance, coerciveness for every external force \( f \) is obtained under a purely geometrical assumption bearing a striking interpretation. namely, that the body is “stuck” due to its contact along \( \Gamma_C \) in the reference configuration (in other words, contact along \( \Gamma_C \) in the reference configuration prevents the body from being moved without exerting external forces).

Uniqueness modulo the space \( \mathcal{N} \) of infinitesimal rigid motions \( \mathbb{R} \) verifying \( R_n=0 \) on \( \Gamma_C \), hence uniqueness if \( \mathcal{N} = \{0\} \), is obtained under another mild compatibility condition as soon as physical contact occurs. In any case, uniqueness of the area of physical contact \( \{x \in \Gamma_C; u_n(x) > 0\} \) is proved and it is shown that the same result is false in general if the area of physical contact is replaced by the area of geometrical contact \( \{x \in \Gamma_C; u_n(x) \geq 0\} \).

Considering problem (1.8)-(1.10) as a model for contact with no friction instead of Signorini’s problem thus allows elimination of several ambiguities in the latter. An important question is then to know how they relate to each other. In Section 5 [1], we present a very simple answer providing one more justification for the use of normal response models: the Signorini problem
coincides with the case of an infinite normal response along $\Gamma_C$. This takes us back to the natural requirement that, when positive, the displacement $u_n$ should be "small" along $\Gamma_C$. When $u$ is a solution to problem (1.8)-(1.10) and $\phi$ is a physically admissible normal response, a heuristic but strong argument in this direction is as follows. As a result of resistance to penetration, it is observed in physical experiments that $\phi(x, t)$ is very large for relatively small values of $t > 0$ (with $\phi(x, t) = c_n(x) t^m$, experiments have provided $c_n(x) = c_n$ in the range of $10^6-10^8$). On the other hand, by changing any normal response $\phi$ to $\lambda \phi$ and letting $\lambda$ tend to $+\infty$, we show in Section 5 that every solution $u(\lambda)$ is arbitrarily close to some solution to Signorini's problem in the strong topology of $(H^1(\Omega))^N$. Hence, $u_n(\lambda)$ is close to zero in, say, every space $L^p(\Gamma_C)$ such that $H^1(\Gamma) \hookrightarrow L^p(\Gamma)$. The interpretation of this result is that $u_n$ is close to zero on $\Gamma_C$ if the normal response $\phi(x, t)$ is large for relatively small values of $t > 0$, which is precisely the actual physical situation.

Another advantage of the formulation (1.8)-(1.10) is that it admits perturbations allowing consideration of friction phenomena according to Coulomb's law or generalizations of it. In this case, it suffices to consider a power-like normal response (since the exponential zone is characteristic of no sliding). The method of [11] is then available with appropriate modifications (see [1, Section 5]). The other aspects we discuss in Section 5 are the interpretations of our compatibility conditions for coerciveness and uniqueness in the simple case $N = 2$, the admissibility of an initial gap and the possibility of considering other boundary conditions. The use of interface models, such as those described here, together with nonquadratic energies of deformation, is the subject of current studies.

The work is naturally divided into two parts. Part I is composed of Sections 2 and 3, and is devoted to the establishment of special mathematical preliminaries and to the formulation of the variational equality which correctly characterizes the traction-contact problem with a nonlinear interface constitutive law. The major issues of existence and uniqueness of solutions are taken up in Part II [1].

2. TECHNICAL PRELiminaries

We shall begin with a review of some general results. Hypotheses will later be complemented according to the applications we have in mind and further properties will be established.

Let $\omega$ be an open subset of $\mathbb{R}^n$, $m \geq 1$, and $\phi: \omega \times \mathbb{R} \rightarrow \mathbb{R}$, a Carathéodory function, i.e.

$$\phi(x, \cdot): \mathbb{R} \rightarrow \mathbb{R} \text{ is continuous for almost all } x \in \omega.$$  \hspace{1cm} (2.1)

$$\phi(\cdot, t): \Omega \rightarrow \mathbb{R} \text{ is measurable for every } t \in \mathbb{R}.$$  \hspace{1cm} (2.1)

The Nemytskii operator $\hat{\phi}$ associated with the function $\phi$ is defined for every measurable function $\xi: \omega \rightarrow \mathbb{R}$ by

$$\hat{\phi}(\xi)(x) = \phi(x, \xi(x)) \text{ for almost all } x \in \omega.$$  \hspace{1cm} (2.2)

It can be shown that $\hat{\phi}(\xi)$ is a measurable function. Krasnoselskii [9] has given necessary and sufficient conditions for the operator $\hat{\phi}$ to act continuously from $L^q(\omega)$ into $L^r(\omega)$ when $q$ and $r$ verify the condition $1 \leq q, r < +\infty$. Besides, setting

$$\Phi(x, t) = \int_0^t \phi(x, r) \, dr$$  \hspace{1cm} (2.3)
and assuming \( q > 1 \) and \( r = q^* = q/(q - 1) \) (Hölder conjugate of \( q \)), he has shown under the same assumptions that the functional

\[
j(\xi) = \int_\omega \Phi(\xi) \, dx.
\]

(2.4)

is continuously differentiable on \( L^q(\omega) \) with derivative \( \dot{\phi} \) in the sense that

\[
j'(\xi) \cdot h = \int_\omega \dot{\phi}(\xi) h \, dx.
\]

(2.5)

for every pair \((\xi, h) \in (L^q(\omega))^2\). For \( q > 2 \), these results are complemented in [16] by showing, under the additional assumptions that (1) the mapping \( \phi \) is continuously differentiable with respect to \( t \) for almost all \( x \in \omega \) and (2) its derivative \( \dot{\phi} \) verifies an appropriate growth condition (ensuring that \( \dot{\phi} \in C^0(L^q(\omega), L^{\frac{q(q-2)}{2}}(\omega)) \)), that the functional \( j(2.5) \) is twice continuously differentiable on \( L^q(\omega) \) with

\[
j''(\xi) \cdot (h, k) = \int_\omega \ddot{\phi}(\xi) hk \, dx.
\]

for every triple \((\xi, h, k) \in (L^q(\omega))^3\). The same conclusion is false for \( q = 2 \), except in the trivial case when \( \phi(x, t) = a(x)t \) and \( a \in L^1(\omega) \), but it can be extended to the case \( q = +\infty \), of special interest to us in this paper. Proposition 2.1 below summarizes the various statements for \( q = +\infty \) in a form which will be suitable for our later purposes. Details and extensions can be found in [17].

**Proposition 2.1.** Assume that the Carathéodory function \( \phi \) is of class \( C^1 \) with respect to \( t \) for almost all \( x \in \omega \) and that \( \dot{\phi} \) verifies the following growth condition. For every \( t_0 > 0 \), there is a function \( a_{t_0} \in L^1(\omega) \) such that

\[
|\phi_t(x, t)| \leq a_{t_0} \quad \text{for } |t| \leq t_0.
\]

(2.6)

Then

(i) \( \dot{\phi} \in C^0(L^q(\omega), L^1(\omega)) \)

(ii) \( \dot{\phi} \in C^1(L^q(\omega), L^1(\omega)) \) as soon as \( \phi(\cdot, 0) \in L^1(\omega) \), with

\[
D\dot{\phi}(\xi) \cdot h = \ddot{\phi}(\xi) h.
\]

(2.7)

for every pair \((\xi, h) \in (L^q(\omega))^2\).

(iii) Setting

\[
\Phi(x, t) = \int_0^t \phi(x, \tau) \, d\tau.
\]

(2.8)

the functional

\[
j(\xi) = \int_\omega \Phi(\xi) \, dx
\]

(2.9)

\[\text{No additional assumption is needed to show that } \phi, \text{ is a Carathéodory function.}\]
is twice continuously differentiable on $L^r(\omega)$ and
\[
j'(\xi) \cdot h = \int_\omega \phi(\xi) h \, dx.
\] (2.10)
for every pair $(\xi, h) \in (L^r(\omega))^2$ (note that the combination of (i) and (ii) yields
\[
j''(\xi) \cdot (h, k) = \int_\omega \phi(\xi) hk \, dx
\] (2.11)
for every triple $(\xi, h, k) \in (L^r(\omega))^3$.

Suppose now that
\[
\Phi(\cdot, t) \geq 0 \quad \text{for every} \quad t \in \mathbb{R}.
\]
(2.12)
Then, for every measurable function $\xi: \omega \rightarrow \mathbb{R}$, $\Phi(\xi)$ is a nonnegative measurable function. The functional $j$ of (2.9) can thus be extended to all measurable real-valued functions as a
mapping (still denoted by $j$) with values in $\mathbb{R}^* = \mathbb{R} \cup \{+\infty\}$ by setting
\[
j(\xi) = \begin{cases} 
\int_\omega \Phi(\xi) \, dx & \text{if} \quad \Phi(\xi) \in L^1(\omega) \\
+\infty & \text{otherwise}.
\end{cases}
\] (2.13)

**Proposition 2.2.** The extended functional $j$ is lower semicontinuous on $L^1(\omega)$.

**Proof.** The result follows by combining [4, p. 218, proposition 1.1 and p. 222, corollary 1.2]. \blackslug

**Remark 2.1.** The same argument shows that the extended functional $j$ is lower semicontinuous on $L^p(\omega)$ for every $1 \leq p \leq +\infty$.

We shall consider a particular case when assumption (2.12) is satisfied, namely
\[
\phi_1(\cdot, t) \geq 0 \quad \text{for} \quad t > 0,
\]
\[
\phi_1(\cdot, t) = 0 \quad \text{for} \quad t \leq 0
\]
(2.15)
and
\[
\phi(x, t) = \int_0^t \phi_1(x, \tau) \, d\tau \quad (\Leftrightarrow \phi(\cdot, 0) = 0).
\] (2.16)

In this case, the three functions $\phi_1(\cdot, t), \phi(\cdot, t)$ and $\Phi(\cdot, t)$ are nonnegative for every $t \in \mathbb{R}$
and vanish for $t \leq 0$. Together with an appropriate condition limiting the growth of the function $\phi(x, t)$ as $t$ tends to $+\infty$, the above assumptions will now allow us to complement proposition 2.1 as follows.

**Proposition 2.3.** Assume that (2.15) and (2.16) hold and suppose further that the function $\phi_1(x, \cdot)$ is nondecreasing for almost all $x \in \omega$ and that there are constants $T > 0$ and $\mu > 0$
such that
\[
\phi_1(\cdot, t) \leq \mu \phi(\cdot, t) \quad \text{for} \quad t \geq T
\] (2.17)
with \( \phi(\cdot, T) = \Phi(T) \in L^1(\omega) \) and either \( \phi(x, T) > 0 \) or \( \phi(x, \cdot) \equiv 0 \) for almost all \( x \in \omega \). Then, for every measurable function \( \xi \) such that \( \Phi(\xi) \in L^1(\omega) \) and every function \( \eta \in L^\infty(\omega) \) one has

\[
\Phi(\xi + \eta) \in L^1(\omega), \quad \dot{\Phi}(\xi + \eta) \in L^1(\omega)
\]

and the functional

\[
\eta \in L^\infty(\omega) \rightarrow j^*_\xi(\eta) = j(\xi + \eta)
\]

is real-valued and twice continuously differentiable with

\[
j^*_\xi(\eta) \cdot h = \int_\omega \dot{\Phi}(\xi + \eta) h \, dx \tag{2.19}
\]

\[
j^*_\xi(\eta) \cdot (h, k) = \int_\omega \dot{\Phi}(\xi + \eta) hk \, dx \tag{2.20}
\]

for every triple \((\eta, h, k) \in (L^\infty(\omega))^3\).

**Proof.** As a first step, we show that \( \dot{\Phi}(\xi) \) belongs to \( L^1(\omega) \) as follows. Integrating both sides of inequality (2.17) we obtain

\[
\phi(\cdot, t) \preceq \phi(\cdot, T) + \mu \Phi(\cdot, t) - \mu \Phi(\cdot, T)
\]

\[
\preceq \phi(\cdot, T) + \mu \Phi(\cdot, t) \quad \text{for} \quad t \preceq T. \tag{2.21}
\]

Let then \( \xi : \omega \rightarrow \mathbb{R} \) be a measurable function. From (2.21),

\[
\xi(x) \preceq T \Rightarrow \phi(x, \xi(x)) \preceq \phi(x, T) + \mu \Phi(x, \xi(x)).
\]

On the other hand, it follows from (2.15) that the function \( \phi(x, \cdot) \) is nondecreasing for almost all \( x \in \omega \). Hence,

\[
\xi(x) \preceq T \Rightarrow \phi(x, \xi(x)) \preceq \phi(x, T) \preceq \phi(x, T) + \mu \Phi(x, \xi(x)).
\]

This shows that

\[
0 \preceq \dot{\Phi}(\xi) \preceq \phi(T) + \mu \Phi(\xi) \in L^1(\omega),
\]

proving the relation \( \dot{\Phi}(\xi) \in L^1(\omega) \).

Setting

\[
\phi_\xi(x, t) = \phi(x, \xi(x) + t). \tag{2.22}
\]

we now note that \( \phi_\xi \) is a Carathéodory function, and the mapping \( t \rightarrow \phi_\xi(x, t) \) is continuously differentiable for almost all \( x \in \omega \) with

\[
(\phi_\xi)_t(x, t) = \phi_t(x, \xi(x) + t). \tag{2.23}
\]

The sole nontrivial part of this assertion is the measurability of the function \( \phi_\xi(\cdot, t) \) for every \( t \in \mathbb{R} \). However, it suffices to notice that the function \( (x, t) \in \omega \times \mathbb{R} \rightarrow \phi(x, t + \tau) \) is (obviously) a Carathéodory function and to apply the measurability result used at the beginning of this section after replacing \( \tau \) by \( \xi(x) \).
At this stage, we see that the properties to be established follow from proposition 2.1, provided we can prove that the function $\phi_\xi$ fulfills the required hypotheses. As $\phi_\xi(\cdot, 0) = \phi(\cdot, \xi(\cdot)) = \hat{\phi}(\xi)$ is in $L^1(\omega)$, as we have just seen, we need only prove for every $t_0 > 0$ that there is a function $b_{t_0} \in L^1(\omega)$ such that

$$0 \leq (\phi_\xi)_t(x, t) \leq b_{t_0}(x) \quad \text{for almost all } x \in \omega \quad \text{and} \quad |t| \leq t_0. \tag{2.24}$$

We begin by observing that an equivalent formulation of assumption (2.17) is

$$\phi(\cdot, \tau + t) \equiv e^{\mu t} \phi(\cdot, \tau) \quad \text{for} \quad t \geq 0 \quad \text{and} \quad \tau \geq T. \tag{2.25}$$

the proof of which reduces to a simple verification. In particular,

$$\phi(\cdot, t) \equiv e^{\mu(t - T)} \phi(\cdot, T) \quad \text{for} \quad t \geq T. \tag{2.26}$$

Substituting into (2.17), we arrive at

$$\phi_t(\cdot, t) \equiv \mu e^{\mu(t - T)} \phi(\cdot, T) = \mu e^{\mu(t - T)} \dot{\phi}(T) \quad \text{for} \quad t \geq T. \tag{2.27}$$

Meanwhile, from the monotonicity of almost all functions $\phi_t(x, \cdot)$, one has

$$\phi_t(\cdot, t) \leq \phi_t(\cdot, T) \leq \mu \phi(\cdot, T) = \mu \dot{\phi}(T) \quad \text{for} \quad t \geq T. \tag{2.28}$$

Hence, given $t_0 > 0$ and setting

$$a_{t_0} = \mu \sup(1, e^{\mu(t_0 - T)}) \dot{\phi}(T) \in L^1(\omega).$$

we obtain the estimate

$$0 \leq \phi_t(\cdot, t) \leq a_{t_0} \quad \text{for} \quad |t| \leq t_0. \tag{2.29}$$

To prove (2.24), we may assume $t_0 \equiv T$, since the same function $b_T$ can be taken as $b_{t_0}$ when $t_0 \equiv T$. We shall find an appropriate choice for $b_{t_0}$ by considering the three cases: $\xi(x) \leq T - t_0$; $T - t_0 < \xi(x) < T$; and $\xi(x) \geq T$. Assume first $\xi(x) \leq T - t_0$. Then, for $|t| \leq t_0$, one has $\dot{\xi}(x) + t \leq T$. From (2.23) and (2.26), we deduce

$$0 \leq (\phi_\xi)_t(x, t) \leq \mu \dot{\phi}(T)(x). \tag{2.30}$$

Next, suppose $T - t_0 < \xi(x) < T$. Since $t_0 \equiv T > 0$ by hypothesis and for $|t| \leq t_0$, we see that $-2t_0 < \xi(x) + t < 2t_0$. Applying (2.27), we get

$$0 \leq (\phi_\xi)_t(x, t) \leq a_{2t_0}(x). \tag{2.31}$$

Finally, assume that $\xi(x) \geq T$. For $0 \leq t \leq t_0$, relation (2.25) is available with $\tau = \xi(x)$ and we obtain

$$\phi(x, \xi(x) + t) \equiv e^{\mu t} \phi(x, \xi(x)) = e^{\mu t} \dot{\phi}(\xi)(x) \leq e^{\mu t_0} \dot{\phi}(\xi)(x).$$

More generally, as soon as $\xi(x) + t \equiv T$, relations (2.17) and (2.23) yield

$$0 \leq (\phi_\xi)_t(x, t) \leq \mu \phi(x, \xi(x) + t).$$

Together with the previous inequality, we find

$$0 \leq (\phi_\xi)_t(x, t) \leq \mu e^{\mu t_0} \dot{\phi}(\xi)(x). \tag{2.32}$$
for $0 \leq t \leq t_0$. Next, due to the monotonicity of almost all functions $\phi(x, \cdot)$ and from (2.17):

$$0 \leq (\phi_{\xi})_t(x, t) \leq \mu \phi(x, \xi(x)) = \mu \dot{\phi}(\xi)(x)$$

when $t < 0$ and $\xi(x) + t \geq T$. This shows that inequality (2.30) is valid whenever $\xi(x) \geq T, |t| \leq t_0$ and $\xi(x) + t \geq T$. It remains to examine the case when $\xi(x) \geq T$ and $\xi(x) + t < T$. If so, $t$ is negative and

$$-t_0 < T - t_0 \leq \xi(x) + t < T \leq t_0,$$

whence, from (2.23) and (2.27),

$$0 \leq (\phi_{\xi})_t(x, t) \leq a_{t_0}(x). \quad (2.31)$$

According to the estimates (2.28)–(2.31) and since $\dot{\phi}(T), \dot{\phi}(\xi), a_{t_0}$ and $a_{2t_0}$ belong to the space $L^1(\omega)$, we can take

$$b_{t_0} = \sup(\mu \dot{\phi}(T), \mu e^{\mu a_{t_0}} \dot{\phi}(\xi), a_{t_0}, a_{2t_0}) \in L^1(\omega)$$

in (2.24) and the proof is complete. \(\blacksquare\)

By localization and partition of unity, the results of this section easily carry over to the case in which the open set $\omega$ is replaced by the Lipschitz continuous boundary $\Gamma$ of a (bounded) open subset $\Omega$ of $\mathbb{R}^N$. Indeed, this merely introduces positive measurable bounded weights, which does not affect the form of the required hypotheses. Further, $\Gamma$ can also be replaced by any measurable subset $\Gamma_C$ for it is immediately seen that the assumptions are not affected by extending all the data by zero for values of the variable $x$ in $\Gamma \setminus \Gamma_C$. Theorem 2.1 below summarizes the conclusions in this new context.

**Theorem 2.1.** Let $\Omega$ be an open bounded subset of $\mathbb{R}^N$ with Lipschitz continuous boundary $\Gamma$ and surface measure $d\sigma$ and let $\Gamma_C \subset \Gamma$ be a measurable subset. Let $\phi: \Gamma_C \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function and assume for almost all $x \in \Gamma_C$ that the function $\phi(x, \cdot)$ is continuously differentiable with $\phi_t(x, \cdot)$ nondecreasing. Assume further

$$\phi_t(\cdot, t) \geq 0 \quad \text{for} \quad t > 0, \quad (2.32)$$

$$\phi_t(\cdot, t) = 0 \quad \text{for} \quad t \leq 0. \quad (2.33)$$

$$\phi(x, t) = \int_0^t \phi_t(x, \tau) d\tau \quad (\Leftrightarrow \phi(\cdot, 0) = 0). \quad (2.34)$$

and that there are constants $T > 0$ and $\mu > 0$ such that

$$\phi_t(\cdot, t) \leq \mu \phi_t(\cdot, t) \quad \text{for} \quad t \geq T, \quad (2.35)$$

with $\phi(\cdot, T) \in L^1(\Gamma_C)$ and either $\phi(x, T) > 0$ or $\phi(x, \cdot) = 0$ for almost all $x \in \Gamma_C$. Setting

$$\Phi(x, t) = \int_0^t \phi(x, \tau) d\tau \quad (2.36)$$

and denoting by $\phi_t, \dot{\phi}$ and $\dot{\Phi}$ the Nemytskii operators associated with $\phi_t, \phi$ and $\Phi$ respectively, the following conclusions hold:
(i) the functional
\[ f(\xi) = \begin{cases} \int_{\Gamma_c} \dot{\phi}(\xi) \, ds & \text{if } \dot{\phi}(\xi) \in L^1(\Gamma_c) \\ + \infty & \text{otherwise.} \end{cases} \] (2.37)
defined for every measurable function \( \xi : \Gamma_c \rightarrow \mathbb{R} \) is lower semicontinuous on the space \( L^1(\Gamma_c) \).

(ii) for every measurable function \( \xi \) such that \( \dot{\phi}(\xi) \in L^1(\Gamma_c) \) and every function \( \eta \in L^\infty(\Gamma_c) \) one has
\[ \dot{\phi}(\xi + \eta) \in L^1(\Gamma_c), \dot{\phi}_t(\xi + \eta) \in L^1(\Gamma_c) \]
and the functional
\[ \eta \in L^\infty(\Gamma_c) \rightarrow j_z(\eta) = f(\xi + \eta) \] (2.38)
is real-valued and twice continuously differentiable with
\[ j_\xi^z(\eta) \cdot h = \int_{\Gamma_c} \dot{\phi}(\xi + \eta) h \, ds, \] (2.39)
\[ j_\xi^{zz}(\eta) \cdot (h, k) = \int_{\Gamma_c} \dot{\phi}_t(\xi + \eta) hk \, ds. \] (2.40)
for every triple \( (\eta, h, k) \in (L^\infty(\Gamma_c))^3 \).

Remark 2.2. Functions verifying the assumptions of theorem 2.1 generate a convex cone containing all functions of the form
\[ \phi(x, t) = c(x)(t_+)^m, \]
where \( m > 1 \) is an arbitrary real number, \( t_+ \) is the function
\[ t_+(t) = \begin{cases} t & \text{if } t \geq 0, \\ 0 & \text{if } t < 0, \end{cases} \]
and \( c \in L^1(\Gamma_c) \) is an arbitrary nonnegative given function. Moreover, condition (2.35) allows for a modification of the growth of (convex combinations of) such functions as \( t \) tends to \( + \infty \), up to and including exponential order.

The assumptions on the function \( \phi \) made in theorem 2.1, and in particular limitation of the growth in \( t \) to exponential order, will later allow us to characterize the solutions to the problem considered in Section 1 as minimizers of a convex functional with values in \( \mathbb{R}^+ \). As usual, existence of minimizers will be obtained by establishing a coerciveness property, relying on complementary (and, of course, compatible) specification of the growth of the mapping \( \phi(x, \cdot) \) for almost all \( x \in \Gamma_c \). The suitable hypothesis and its main consequences regarding the problem under consideration are examined as the next objective of this section.

As an additional assumption, we shall require that
\[ 0 \leq \frac{\phi(x, t)}{t} \leq \phi_t(x, t) \quad \text{for } t > 0 \quad \text{and almost all } x \in \Gamma_c. \] (2.41)
Note since the function \( \phi(x, \cdot) \) is continuously differentiable that this new assumption implies \( \phi(\cdot, 0) = 0 \) and hence makes condition (2.34) superfluous. Besides, it is easily checked that (2.41) amounts to saying that the function \( t \mapsto \phi(x, t)/t \) is nondecreasing on \((0, + \infty)\) for almost all \( x \in \Gamma_C \). This function is strictly increasing on \((0, + \infty)\) under the mildly stronger condition
\[
0 < \frac{\phi(x, t)}{t} < \phi_t(x, t) \quad \text{for} \quad t > 0.
\]

Monotonicity and nonnegativity properties allow us to set for almost all \( x \in \Gamma_C \)
\[
l(x) = \lim_{t \to +\infty} \frac{\phi(x, t)}{t} \geq 0 \quad \text{(possibly} + \infty) \tag{2.43}
\]
and \( l(x) \) is equivalently defined through any sequence \( (\phi(x, t_k)/t_k) \) with \( \lim t_k = + \infty \) and verifies
\[
l(x) \geq \frac{\phi(x, t)}{t} \quad \text{for every} \quad t > 0 \quad \text{and almost all} \quad x \in \Gamma_C. \tag{2.44}
\]

As a result of Egorov’s theorem, \( l \) is a measurable function on \( \Gamma_C \).

Remark 2.3. It is obvious that functions verifying property (2.41) form a convex cone and that this assumption does not restrict the class described in remark 2.2. This shows that adding condition (2.41) is compatible with the assumptions of theorem 2.1, which is due to the fact that \( T > 0 \) can be taken arbitrarily large in (2.35). Roughly speaking, the combination of (2.35) and (2.41) means that the growth of \( \phi(x, \cdot) \) is superlinear and at most of exponential order, monotonicity being imposed by (2.32).

Theorem 2.2. In addition to the hypotheses of theorem 2.1, assume that (2.41) holds and let \( \xi : \Gamma_C \to \mathbb{R} \) be a measurable function. Then, the mapping
\[
t \mapsto \frac{1}{t^2} j(t \xi) \in \mathbb{R}.
\]
where \( j \) denotes the functional (2.37), is nondecreasing on \((0, + \infty)\) and
\[
\lim_{t \to +\infty} \frac{1}{t^2} j(t \xi) = \frac{1}{2} \int_{\Gamma_C} l \xi_+^2 \, ds,
\]
where \( \xi_+ = \sup(\xi, 0) \) and with the (usual) convention that
\[
l(x) \xi_+^2(x) = 0 \quad \text{when} \quad l(x) = + \infty \quad \text{and} \quad \xi_+(x) = 0.
\]

Proof. A preliminary observation is that for almost all \( x \in \Gamma_C \), the function \( t \mapsto \Phi(x, t)/t^2 \) is nondecreasing on \((0, + \infty)\): To see this, it suffices to compute
\[
\frac{d}{dt} \left( \frac{\Phi(x, t)}{t^2} \right) = \frac{1}{t} \left( t \phi(x, t) - 2 \Phi(x, t) \right)
\]

† Although assumption (2.35) can be omitted in this statement.

‡ Allowing, of course, the value \(+\infty\) for the right-hand side.
and note that \( t\Phi(x, t) - 2\Phi(x, t) \geq 0 \) for \( t > 0 \) as it follows by multiplying (2.41) by \( t \) and integrating. This yields the monotonicity of the function (2.45) from the relation (for \( t > 0 \))

\[
\frac{1}{t^2} \Phi(x, t\xi(x)) = \begin{cases} 
\frac{\Phi(x, t\xi(x))}{t^2 \xi^2(x)} & \text{for } \xi(x) > 0, \\
0 & \text{for } \xi(x) \leq 0.
\end{cases}
\]  

(2.47)

A second conclusion from the above observation is that \( \lim_{t \to +\infty} \Phi(x, t)/t^2 \) exists (possibly \( +\infty \)) for almost all \( x \in \Gamma_C \). Actually, a more precise result is true, namely.

\[
\lim_{t \to +\infty} \frac{\Phi(x, t)}{t^2} = \frac{1}{2} \lim_{t \to +\infty} \frac{\Phi(x, t)}{t} = \frac{1}{2} l(x).
\]  

(2.48)

Although it holds in a much more general context, this relation is easy to prove under our assumptions. Indeed, it is a simple exercise to check that (2.48) follows from de l'Hôpital's rule.

We are now in position to prove relation (2.46) as follows. Suppose first that the left-hand side of (2.46) is a (nonnegative) real number \( I \). For every sequence \( t_k \) tending to \( +\infty \), one has

\[
I = \lim_{t \to +\infty} \frac{1}{t_k} \int_{\Gamma_C} \Phi(t_k, \xi) \, ds
\]  

(2.49)

and

\[
\frac{1}{t_k} \int_{\Gamma_C} \Phi(t_k, \xi) \, ds \leq I \quad \text{for every } k \in \mathbb{N}.
\]  

(2.50)

On the other hand, it follows from (2.47) and (2.48) (and the convention \( l(x)\xi_+(x) = 0 \) when \( l(x) = +\infty \) and \( \xi_+(x) = 0 \)) that the nondecreasing sequence \( \Phi(x, t_k \xi(x))/t_k^2 \) tends to \( l(x)\xi_+(x)/2 \) almost everywhere on \( \Gamma_C \). With (2.49), (2.50) and the monotone convergence theorem, it follows that

\[
\frac{1}{2} \int_{\Gamma_C} l\xi_+^2 \, ds = I.
\]

To complete the proof, we shall use the inequality

\[
\frac{\Phi(x, t)}{t^2} \leq \frac{1}{2} l(x) \quad \text{for } t > 0,
\]

which follows from (2.48) and the monotonicity of \( \Phi(x, t)/t^2 \). With (2.47), this yields

\[
\frac{1}{2} l\xi_+^2 \geq \frac{1}{t^2} \Phi(t\xi) \quad \text{for } t > 0.
\]  

(2.51)
Assume next that the right-hand side of (2.46) is a (nonnegative) real number, namely \( l \xi_2 \in L^1(\Gamma_C) \). If so, relation (2.46) follows from Lebesgue's dominated convergence theorem by using a sequence \( i_k > 0 \) tending to \( +\infty \) in (2.51). As both sides of (2.46) coincide when either one is finite, they coincide when either one is \( +\infty \) as well, and the proof is complete. ■

3. VARIATIONAL FORMULATION

This section is intended to show the equivalence of the contact problem described in Section 1 with a minimization problem over the space \((H^1(\Omega))^N\). We shall begin with a review of some classical results and introduce a few notations to be used throughout the remainder of this paper.

Given a bounded domain \( \Omega \) with a Lipschitz continuous boundary \( \Gamma \), the outer normal \( n \) is defined almost everywhere (see, e.g., [12]) and is a measurable function. Componentwise, we then have

\[
n_i \in L^\infty(\Gamma), 1 \leq i \leq N.
\]

\[
\sum_{i=1}^N n_i^2 = 1 \quad \text{on} \quad \Gamma.
\]  

(3.1)

The space \( H^1(\Omega) \) is the usual Sobolev space of distributions with partial derivatives of order \( \leq 1 \) in \( L^2(\Omega) \). The space \((H^1(\Omega))^N\) is endowed with the usual inner product inducing the norm

\[
\|v\|_{1,\Omega} = \left( \int_\Omega (v_i v_i + v_{ij} v_{ij}) \, dx \right)^{1/2}.
\]  

(3.2)

The trace operator maps the space \((H^1(\Omega))^N\) linearly and continuously onto the space \((H^{1/2}(\Gamma))^N\) with topological dual

\[
[(H^{1/2}(\Gamma))^N]' = (H^{-1/2}(\Gamma))^N.
\]  

(3.3)

From (3.1) and for every \( 1 \leq p \leq +\infty \), a given element \( \xi \in (L^p(\Gamma))^N \) has a decomposition of the form

\[
\xi = \xi_T + \xi_n n.
\]  

(3.4)

with

\[
\xi_n = \xi \cdot n = \xi_i n_i \in L^p(\Gamma),
\]

\[
\xi_T = \xi - \xi_n n \in (L^p(\Gamma))^N.
\]  

(3.5)

The components \( \xi_n \) and \( \xi_T \) will be referred to as the normal and tangential components of \( \xi \) respectively.
We next make precise the assumptions on the data of the problem as follows. We shall assume that the elasticity coefficients $E_{ijkl}$ verify

$$E_{ijkl} \in L^\infty(\Omega)$$

and that the uniform ellipticity condition

$$E_{ijkl}(x)A_{kl}A_{ij} \geq \alpha A_{ij}A_{ij}$$

holds with some constant $\alpha$ for almost all $x \in \Omega$ and every $N \times N$ symmetric array $A_{ij}$.

The body forces are chosen so that

$$b \in (L^2(\Omega))^N,$$

while the prescribed tractions $t$ on $\Gamma_f = \Gamma \setminus \Gamma_c$ are submitted to the condition

$$t \in (L^2(\Gamma_f))^N.$$

The function $\phi: \Gamma_c \times \mathbb{R} \to \mathbb{R}$ characterizing the normal response along $\Gamma_c$ is supposed to fulfill the assumptions of theorem 2.2, namely $\phi$ is a Carathéodory function such that the mapping $\phi(x, \cdot)$ is continuously differentiable for almost all $x \in \Gamma_c$ and

$$0 \leq \frac{\phi(\cdot, t)}{t} \leq \phi_i(\cdot, t) \quad \text{for } t > 0 \dagger$$

$$\phi_i(x, \cdot) \text{ is nondecreasing for almost all } x \in \Gamma_c.$$  \hfill (3.10)

$$\phi_i(\cdot, t) = 0 \quad \text{for } t \leq 0.$$ \hfill (3.11)

and there are constants $T > 0$ and $\mu > 0$ such that

$$\phi_i(\cdot, t) \leq \mu \phi(\cdot, t) \quad \text{for } t \geq T.$$ \hfill (3.12)

with $\phi(\cdot, T) \in L^1(\Gamma_c)$ and either $\phi(x, T) > 0$ or $\phi(x, \cdot) \equiv 0$ for almost all $x \in \Gamma_c$.

Physical justification of these assumptions is provided by remarks 2.2 and 2.3 and the related comments of Section 1.

In what follows, we shall denote by $a(\cdot, \cdot)$ the continuous bilinear form over $(H^1(\Omega))^N$ defined by

$$a(u, v) = \text{virtual work produced by the action of the stresses}$$

$$\sigma(u) \text{ on the strains } \varepsilon(v)$$

$$= \int_{\Omega} \sigma(u) : \varepsilon(v) \, dx = \int_{\Omega} E_{ijkl} \partial_j u_k \partial_i v_l \, dx.$$

The external forces (body forces $b$ and prescribed tractions $t$) are described by an element $f \in [(H^1(\Omega))^N]'$ through the relation

$$\langle f, v \rangle_{\Omega} = \int_{\Omega} b \cdot v \, dx + \int_{\Gamma_f} t \cdot v \, ds.$$ \hfill (3.14)

\dagger Recall that this condition implicitly contains (2.34).
a formula in which \( \langle \cdot, \cdot \rangle_{\Omega} \) stands for the duality pairing between \( (H^1(\Omega))^N \) and its topological dual.

Our aim is to examine how the problem (1.8)-(1.10) relates to the minimization problem:

\[
\min_{v \in (H^1(\Omega))^N} \frac{1}{2} a(v, v) + \int_{\Gamma_C} \Phi(v_n) \, ds - \langle f, v \rangle_{\Omega}
\]

where, as in Section 2, \( \Phi \) is the Nemytskii operator associated with the function \( \Phi(x, t) = \int_0^t \phi(x, \tau) \, d\tau \). However, as the integral \( \int_{\Gamma_C} \Phi(v_n) \, ds \) is not defined in general, the correct formulation of the above problem is: minimize

\[
\frac{1}{2} a(v, v) + j(v_n) - \langle f, v \rangle_{\Omega}
\]

where, for every measurable function \( \xi : \Gamma_C \to \mathbb{R} \), we have set

\[
j(\xi) = \begin{cases} 
\int_{\Gamma_C} \Phi(\xi) \, ds & \text{if } \Phi(\xi) \in L^1(\Gamma_C), \\
+\infty & \text{otherwise}.
\end{cases}
\]

The first step of our approach is classical and relies upon a generalization of Green's formula, given in Lemma 3.1 below, whose proof can, for instance, be found in [7, theorems 5.8 and 5.9]. We denote by \( \mathcal{D}(\Omega) \) the space of indefinitely differentiable functions with compact support in \( \Omega \), equipped with the usual inductive limit topology, and by \( \mathcal{D}'(\Omega) \) its topological dual, the space of distributions over \( \Omega \). We shall also make use of the spaces \( \mathcal{D}(\overline{\Omega}) \) and \( \mathcal{D}(\Gamma) \) of restriction to \( \overline{\Omega} \) and \( \Gamma \), respectively, of indefinitely differentiable functions on \( \mathbb{R}^N \).

Consider the operator

\[
div \sigma : (H^1(\Omega))^N \to (\mathcal{D}'(\Omega))^N
\]

\[
[\text{div } \sigma(v)]_i = \partial_j \sigma_{ij}(v) = \partial_j (E_{ijk} \partial_k v_k)
\]

and define the subspace \( H_{\text{div } \sigma}(\Omega) \) of \( (H^1(\Omega))^N \) by

\[
H_{\text{div } \sigma}(\Omega) = \{ v \in (H^1(\Omega))^N ; \text{div } \sigma(v) \in (L^2(\Omega))^N \}.
\]

**Lemma 3.1. (generalized Green's formula).** There is a unique linear continuous operator

\[
\pi : H_{\text{div } \sigma}(\Omega) \to (H^{-1/2}(\Omega))^N,
\]

satisfying

\[
[\pi(u)]_i = \sigma_{ij}(u) n_j \quad \text{on } \Gamma
\]

for every \( u \in (H^1(\Omega))^N \) such that \( \sigma_{ij}(u) = E_{ijk} \partial_k \mu_k \in \mathcal{C}^1(\overline{\Omega}) \), \( 1 \leq i, j, k, l \leq N \) and

\[
a(u, v) + \int_{\Omega} \langle \text{div } \sigma(u) \cdot v \rangle \, dx = \langle \pi(u), v \rangle_{\Gamma}
\]

for every \( u \in H_{\text{div } \sigma}(\Omega) \) and every \( v \in (H^1(\Omega))^N \), where \( \langle \cdot, \cdot \rangle_{\Gamma} \) denotes the duality pairing between \( (H^{1/2}(\Omega))^N \) and \( (H^{-1/2}(\Omega))^N \).
Due to the above lemma, a generalized form of problem (1.8)–(1.10) is as follows. Find \( u \in (H^1(\Omega))^N \) such that

\[
\text{div} \sigma(u) + b = 0 \quad \text{in} \ \Omega.
\]

(3.20)

with the boundary conditions

\[
\pi(u) = t \quad \text{on} \ \Gamma_F, \\
\pi(u) = -\phi(u_n)n \quad \text{on} \ \Gamma_C.
\]

(3.21) (3.22)

Note, however, that this formulation contains some ambiguity. Indeed, \( \pi(u) \) is merely in the space \((H^{-1/2}(\Omega))^N\) while \( \phi(u_n) \)—hence, \( \phi(u_n)n \)—is only a measurable function. As \( \pi(u) \) is not a function in general, and a measurable function does not induce a distribution in a canonical sense, it is not clear how relation (3.22) must be understood. To circumvent this difficulty, we shall include, as a part of the problem, the condition

\[
\pi(u) \in (L^1(\Gamma))^N.
\]

(3.23)

Such a condition does make sense in \((\mathcal{D}'(\Gamma))^N\): it means that the action of the distribution \( \pi(u) \) on elements \( \xi \in (\mathcal{D}(\Gamma))^N \) is represented (in a necessarily unique way) by some element of \((L^1(\Gamma))^N\). As usual, the notation \( \pi(u) \) stands for both the distribution and its representative in \((L^1(\Gamma))^N\). Conditions (3.21) and (3.22) can then be understood in the sense of measurable functions, namely, almost everywhere. We now give a first characterization of solutions to problem (3.20)–(3.23).

**Lemma 3.2.** An element \( u \in (H^1(\Omega))^N \) is a solution to problem (3.20)–(3.23) if and only if \( \phi(u_n) \in L^1(\Gamma_C) \) and

\[
a(u, v) + \int_{\Gamma_C} \phi(u_n)n \cdot v \, ds = \langle f, v \rangle_\Omega \quad \text{for every} \quad v \in (\mathcal{D}(\Omega))^N.
\]

(3.24)

**Proof.** Let \( u \in (H^1(\Omega))^N \) be a solution to problem (3.20)–(3.23). From (3.22) and (3.23), one has \( \phi(u_n)n \in (L^1(\Gamma_C))^N \) and hence \( \phi(u_n) = \phi(u_n)n \cdot n \in L^1(\Gamma_C) \). Multiplying (3.20) by \( v \in (\mathcal{D}(\Omega))^N \) and integrating over \( \Omega \), the generalized Green's formula (3.19) of Lemma 3.1 yields

\[
0 = \int_\Omega (\text{div} \sigma(u) + b) \cdot v \, dx = (\pi(u), v)_\Gamma - a(u, v) + \int_\Omega b \cdot v \, dx.
\]

(3.25)

The expression \( (\pi(u), v)_\Gamma \) depends on the restriction of \( v \) to the boundary \( \Gamma \) only, an element of \((\mathcal{D}(\Gamma))^N\). Thus, applying (3.23), we get

\[
(\pi(u), v)_\Gamma = \int_\Gamma \pi(u) \cdot v \, ds.
\]

(3.26)

Using a decomposition of the integral over \( \Gamma \) into integrals over \( \Gamma_C \) and \( \Gamma_F \) and from (3.21) and (3.22), (3.26) reads

\[
(\pi(u), v)_\Gamma = \int_{\Gamma_F} t \cdot v \, ds + \int_{\Gamma_C} \phi(u_n)n \cdot v \, ds.
\]

(3.27)
As \( \hat{\phi}(u_n) \cdot v = \tilde{\phi}(u_n)v_n \) by definition of \( v_n \), it suffices to combine (3.25) and (3.27) and use the definition (3.14) of the external forces \( f \) to see that \( u \) is a solution to equation (3.24).

Conversely, let \( u \in (H^1(\Omega))^N \) be a solution to equation (3.24) such that \( \hat{\phi}(u_n) \in L^1(\Gamma_C) \). Taking \( v \) arbitrary in \((\mathcal{B}(\Omega))^N \) and by definition of \( f \) (cf. (3.14)) it is immediate that \( \text{div} \ \sigma(u) = b \) in \((\mathcal{B}'(\Omega))^N \) and hence \( u \) belongs to the space \( H_{\text{div}, \sigma}^1 \) (3.18). Together with the generalized Green's formula (3.19) of lemma 3.1, we see for an arbitrary \( v \in (\mathcal{B}(\Omega))^N \) that

\[
\int_{\Gamma_C} \hat{\phi}(u_n)v_n \, ds + \langle \pi(u), v \rangle_{\Gamma} = \int_{\Gamma_F} t \cdot v \, ds.
\]

or, equivalently,

\[
\langle \pi(u), v \rangle_{\Gamma} = -\int_{\Gamma_C} \hat{\phi}(u_n)v_n \, ds + \int_{\Gamma_F} t \cdot v \, ds.
\]

This relation involves restrictions of elements of \((\mathcal{B}(\Omega))^N\) to the boundary \( \Gamma \) only, and hence can be equivalently stated for an arbitrary \( v \in (\mathcal{B}(\Omega))^N \). As \( \hat{\phi}(u_n) \in L^1(\Gamma_C) \), one has \( \hat{\phi}(u_n) \in (L^1(\Gamma))^N \) and it follows that the distribution \( \sigma(u) \) is represented by the element of \((L^1(\Gamma))^N\) defined by \( t \) on \( \Gamma_F \) and by \( \hat{\phi}(u_n) \) on \( \Gamma_C \). Hence, \( u \) is a solution to problem (3.20)–(3.23).

Let \( u \in (H^1(\Omega))^N \) be a solution to problem (3.20)–(3.23). From relation (3.24), it is clear that the mapping \( v \in (\mathcal{B}(\Omega))^N \mapsto \int_{\Gamma_C} \hat{\phi}(u_n)v_n \, ds \) extends as a linear continuous form over \((H^1(\Omega))^N\). say \( \langle \hat{\phi}(u_n)v_n \rangle_\Omega \) (depending on the trace of \( v \) on \( \Gamma \) only) and relation (3.24) remains valid with \( \langle \hat{\phi}(u_n)v_n \rangle_\Omega \) replacing \( \int_{\Gamma_C} \hat{\phi}(u_n)v_n \, ds \) for an arbitrary \( v \in (H^1(\Omega))^N \). On the other hand, from theorems 2.1 and 2.2, the expression \( \langle \hat{\phi}(u_n)v_n \rangle_\Omega \) strongly resembles the derivative at \( u \) of the functional

\[
v \in (H^1(\Omega))^N \mapsto j(v_n).
\]

where \( j \) is the functional (3.16). Strictly speaking, this is not true since the functional (3.28) is not differentiable (for instance, because it may take the value \( +\infty \)). In some cases, lack of differentiability for convex functionals is not too serious a problem, but a standard assumption is that the functional at least have a domain with nonempty interior, i.e. be continuous on a nonempty open subset. Here, this condition is not satisfied in general: for choices of \( N \) and \( \phi \) such that the Sobolev embedding theorems are not available, the functional (3.28) may perfectly be continuous at no point of \((H^1(\Omega))^N\); since it may take the value \( +\infty \) near any point \( v \) with \( j(v_n) < +\infty \). The idea of using Orlicz-like spaces is complicated by the fact that the function \( \phi \) is allowed to depend on the point \( x \in \Gamma_C \). Besides, it is not clear at all that changing the space \((H^1(\Omega))^N\) into a smaller one, over which the functional (3.28) would have nicer properties, would not lead to later problems (as far as coerciveness is concerned for instance). Despite the fact that the functional (3.28) is convex, we have then no standard way of justifying the arguments with which it is formally easy to deduce that solutions to problem (3.20)–(3.23) are minimizers of the functional (3.15).

To make up for the lack of regularity of the functional (3.28), we shall adopt an approach based on convexity properties of the function \( \Phi \) instead of convexity of the functional (3.28) directly. But, to do this, it is essential to obtain further information on the term \( \langle \tilde{\phi}(u_n)n, v \rangle_\Omega \) for a general \( v \in (H^1(\Omega))^N \). The next few results are devoted to proving (in lemma 3.6) for
every \( v \in (H^1(\Omega))^N \) and provided that \( u \) is a solution to (3.20)-(3.23) that \( \Phi(u_n)v \in L^1(\Gamma_c) \) and \( \langle \Phi(u_n)v \rangle \Omega = \int_{\Gamma_c} \Phi(u_n)v \, ds \), exactly as when \( v \in (\mathcal{D}(\tilde{\Omega})) \). A somewhat surprising assertion in which nonnegativity of the function \( \Phi \) is one of the two keys.

**Lemma 3.3.** Let \( T \geq 0 \) be an element of \( L^1(\Gamma_c) \) and suppose that the mapping

\[
v \in \mathcal{D}(\tilde{\Omega}) \to \int_{\Gamma_c} T v \, ds
\]

extends as a linear continuous form \( \langle T, v \rangle_\Omega \) over the space \( H^1(\Omega) \). Then, for every \( v \in H^1(\Omega) \), one has \( Tv \in L^1(\Gamma_c) \) and

\[
\langle T, v \rangle_\Omega = \int_{\Gamma_c} T v \, ds \quad \text{for every } v \in H^1(\Omega).
\]

**Proof.** Let \( v \in H^1(\Omega) \) be given. Suppose first \( 0 \leq v \leq M \) for some constant \( M \). Using the classical procedure of extension to \( H^1(\mathbb{R}^N) \) and regularization, it is easy to find a sequence \((v^{(k)})\) of elements of \( \mathcal{D}(\tilde{\Omega}) \) tending to \( v \) in \( H^1(\Omega) \) and verifying \( 0 \leq v^{(k)} \leq M \). After extracting a sub-sequence, we may assume that \( v^{(k)} \) tends to \( v \) almost everywhere on \( \Gamma_c \). From the continuity of \( \langle T, \cdot \rangle_\Omega \) on the one hand and Lebesgue's dominated convergence theorem on the other hand. we get \( \langle T, v \rangle_\Omega = \lim \int_{\Gamma_c} T v \, ds \) and \( Tv \in L^1(\Gamma_c) \) with \( \int_{\Gamma_c} Tv \, ds = \lim \int_{\Gamma_c} T v^{(k)} \, ds \). These relations prove (3.30) when \( 0 \leq v \leq M \).

Suppose next that \( v \geq 0 \). For every \( k \in \mathbb{N} \setminus \{0\} \) set

\[
v^{(k)} = \inf(v, k).
\]

Clearly, \( 0 \leq v^{(k)} \leq k \) and \( v^{(k)} \in H^1(\Omega) \) for every \( k \) (cf. [12, lemma 1.1, p. 313]). Arguing as in [12], it is easily seen that \( \|v^{(k)}\|_{L^1(\Omega)} \leq \|v\|_{L^1(\Omega)} \). As \( v^{(k)} \) tends to \( v \) in \( L^1(\Omega) \), it follows that \( v \) is the unique cluster point of the sequence \( (v^{(k)}) \) in the weak topology of \( H^1(\Omega) \) and, hence, \( v^{(k)} \rightharpoonup v \) in \( H^1(\Omega) \). As a result, \( \langle T, v \rangle_\Omega = \lim \langle T, v^{(k)} \rangle_\Omega \). But \( \langle T, v^{(k)} \rangle_\Omega = \int_{\Gamma_c} T v^{(k)} \, ds \) from the first part of the proof, so that

\[
\langle T, v \rangle_\Omega = \lim \int_{\Gamma_c} T v^{(k)} \, ds.
\]

From the nonnegativity of \( T \), the sequence \( (Tv^{(k)}) \) is nonnegative and nondecreasing and tends to \( Tv \) almost everywhere. Applying the monotone convergence theorem, we find

\[
\int_{\Gamma_c} Tv \, ds = \lim \int_{\Gamma_c} T v^{(k)} \, ds.
\]

The combination of (3.31) and (3.32) shows that \( Tv \in L^1(\Gamma_c) \) and (3.30) holds when \( v \geq 0 \).

Finally, let \( v \) be arbitrary in \( H^1(\Omega) \) and write \( v = v_+ - v_- \) with \( v_+ = \sup(v, 0), v_- = -\inf(v, 0) \). From [12, lemma 1.1, p. 313], we know that \( v_+ \) and \( v_- \) belong to \( H^1(\Omega) \). Thus,

\[
\langle T, v \rangle_\Omega = \langle T, v_+ \rangle_\Omega - \langle T, v_- \rangle_\Omega.
\]
From the above and since $v_+$ and $v_-$ are nonnegative, $Tv_+$ and $Tv_-$ are in $L^1(\Gamma_C)$ and this relation reads

$$\langle T, v \rangle_{\Omega} = \int_{\Gamma_C} Tv_+ \, ds - \int_{\Gamma_C} Tv_- \, ds = \int_{\Gamma_C} T v \, ds,$$

which completes the proof. ■

The following result is a first extension of lemma 3.3 to vector-valued functions.

**Lemma 3.4.** Let $T \in (L^1(\Gamma_C))^N$ be given and suppose that the components $T_i$ of $T$ in the canonical basis of $\mathbb{R}^N$ verify $T_i \geq 0, 1 \leq i \leq N$. Suppose also that the mapping

$$v \in (H^1(\Omega))^N \rightarrow \int_{\Gamma_C} T \cdot v \, ds$$

extends as a linear continuous form $\langle T, v \rangle_{\Omega}$ over the space $(H^1(\Omega))^N$. Then, for every $v \in (H^1(\Omega))^N$, one has $T \cdot v \in L^1(\Gamma_C)$ and

$$\langle T, v \rangle_{\Omega} = \int_{\Gamma_C} T \cdot v \, ds.$$

**Proof.** Let $1 \leq i \leq N$ be fixed and take $v \in H^1(\Omega)$. Denote by $v \in (H^1(\Omega))^N$ the vector-valued function whose components of order $j \neq i$ are 0 and whose $i$th component is $v$. For $v \in \mathfrak{D}(\bar{\Omega})$, one has $v \in (\mathfrak{D}(\Omega))^N$ and

$$\langle T, v \rangle_{\Omega} = \int_{\Gamma_C} T \cdot v \, ds = \int_{\Gamma_C} T_i v \, ds.$$

As the mapping $v \in H^1(\Omega) \rightarrow v \in (H^1(\Omega))^N$ is obviously continuous, this shows that the mapping

$$v \in \mathfrak{D}(\bar{\Omega}) \rightarrow \int_{\Gamma_C} T_i v \, ds$$

extends as a linear continuous form $\langle T_i, v \rangle_{\Omega}$ over $H^1(\Omega)$. Applying lemma 3.3, it follows that $T_i v \in L^1(\Gamma_C)$ for every $v \in H^1(\Omega)$ and

$$\langle T_i, v \rangle = \int_{\Gamma_C} T_i v \, ds.$$

From the denseness of $(\mathfrak{D}(\bar{\Omega}))^N$ in $(H^1(\Omega))^N$, the identity

$$\langle T, v \rangle_{\Omega} = \sum_{i=1}^N \langle T_i, v_i \rangle_{\Omega}$$

for $v = (v_i) \in (\mathfrak{D}(\bar{\Omega}))^N$ remains valid for $v = (v_i) \in (H^1(\Omega))^N$ and thus reads

$$\langle T, v \rangle_{\Omega} = \sum_{i=1}^N \int_{\Gamma_C} T_i v_i \, ds = \int_{\Gamma_C} T \cdot v \, ds. \quad ■$$

Lemma 3.4 can be considerably generalized as follows.
Lemma 3.5. Let $T \in (L^1(\Gamma_C))^N$ be given and suppose that the mapping

$$v \in (\mathfrak{D}(\Omega))^N \rightarrow \int_{\Gamma_C} T \cdot v \, ds$$

extends as a linear continuous form $\langle T, v \rangle_\Omega$ over the space $(H^1(\Omega))^N$. Let $(\Gamma_k)$ be a finite covering of $\Gamma$ by open subsets and suppose for every $k$ that there is a system of coordinates in $\mathbb{R}^N$ in which $T_i \equiv 0$ on $\Gamma_C \cap \Gamma_k$, $1 \leq i \leq N$. Then, for every $v \in (H^1(\Omega))^N$, one has $T \cdot v \in L^1(\Gamma_C)$ and

$$\langle T, v \rangle_\Omega = \int_{\Gamma_C} T \cdot v \, ds.$$

Proof. Each open subset $\Gamma_k$ of $\Gamma$ is the intersection $\bigcap U_k \cap \Gamma$ of $\Gamma$ with an open subset $U_k$ of $\Omega$. Let $(\theta, \theta_k)$ be a partition of unity associated with the covering $(\Omega, U_k)$ of $\Omega$. For $v \in (\mathfrak{D}(\Omega))^N$, one has $\theta_k v \in (\mathfrak{D}(\Omega))^N$ and

$$\int_{\Gamma_C} T \cdot \theta_k v \, ds = \int_{\Gamma_C} \theta_k T \cdot v \, ds.$$

The multiplication by $\theta_k$ being continuous from $(H^1(\Omega))^N$ into itself, this shows that the mapping

$$v \in (\mathfrak{D}(\Omega))^N \rightarrow \int_{\Gamma_C} \theta_k T \cdot v \, ds$$

extends as a linear continuous form $\langle \theta_k T, v \rangle_\Omega$ over the space $(H^1(\Omega))^N$.

Let $\gamma_1, \ldots, \gamma_N$ be a basis of $\mathbb{R}^N$ such that $T = \sum_{i=1}^N T_i \gamma_i$, with $T_i \equiv 0$, $1 \leq i \leq N$, on $\Gamma_C \cap \Gamma_k$ (such a basis exists by hypothesis). Denote by $e_1, \ldots, e_N$ the canonical basis of $\mathbb{R}^N$ and by $A \in \text{Isom}(\mathbb{R}^N)$ the linear mapping defined by

$$A \gamma_i = e_i, 1 \leq i \leq N.$$  

Set

$$S_k = \sum_{i=1}^N \theta_k T_i e_i \in (L^1(\Gamma_C))^N.$$  

For $v \in (H^1(\Omega))^N$, one has

$$S_k \cdot v = \sum_{i=1}^N \theta_k T_i e_i \cdot v = A(\theta_k T) \cdot v = \theta_k T \cdot A^* v.$$  

where $A^*$ is the adjoint of $A$. Obviously, the mapping $v \rightarrow A^* v$ is an isomorphism of $(H^1(\Omega))^N$ and $A^* (\mathfrak{D}(\Omega))^N = (\mathfrak{D}(\Omega))^N$. From (3.34), it then follows that the mapping

$$v \in (\mathfrak{D}(\Omega))^N \rightarrow \int_{\Gamma_C} S_k \cdot v \, ds \left( = \int_{\Gamma_C} \theta_k T \cdot A^* v \, ds \right)$$

† A condition obviously fulfilled whenever $\Gamma_C \cap \Gamma_k = \emptyset$.  

extends as a linear continuous form $\langle S_k, v \rangle_\Omega$ over the space $(H^1(\Omega))^N$. In addition

$$\langle S_k, v \rangle_\Omega = \langle \theta_k T, A^* v \rangle_\Omega$$  \hfill (3.35)$$

for every $v \in (H^1(\Omega))^N$ since equality holds for $v \in (\mathfrak{B}(\Omega))^N$. On the other hand, the components $\theta_k T_i$, $1 \leq i \leq N$, of $S_k$ in the canonical basis of $\mathbb{R}^N$ verify $\theta_k T_i \geq 0$ on $\Gamma_k \cap \Gamma_c$ since $\theta_k \geq 0$. As supp $\theta_k \subseteq \Gamma_k$, one has $\theta_k T_i \geq 0$ on all of $\Gamma_c$ and lemma 3.4 ensures for every $v \in (H^1(\Omega))^N$ that $S_k \cdot v \in L^1(\Gamma_c)$ with

$$\langle S_k \cdot v \rangle_\Omega = \int_{\Gamma_c} S_k \cdot v \, ds.$$  

Using this result in conjunction with (3.34) and (3.35) in which $(A^*)^{-1} v$ replaces $v$ and since the mapping $v \mapsto (A^*)^{-1} v$ is continuous from $(H^1(\Omega))^N$ into itself, we deduce for every $v \in (H^1(\Omega))^N$ that $\theta_k T \cdot v \in L^1(\Gamma_c)$ with

$$\langle \theta_k T \cdot v \rangle_\Omega = \int_{\Gamma_c} \theta_k T \cdot v \, ds.$$ \hfill (3.36)$$

Since $\sum_k \theta_k = 1$ on $\Gamma$, one has $T \cdot v = \sum_k \theta_k T \cdot v \in L^1(\Gamma_c)$ and the relation

$$\langle T \cdot v \rangle_\Omega = \sum_k \langle \theta_k T \cdot v \rangle_\Omega.$$ \hfill (3.37)$$

obvious for $v \in (\mathfrak{B}(\Omega))^N$. remains valid for $v \in (H^1(\Omega))^N$ by denseness and continuity. The combination of (3.36) and (3.37) yields

$$\langle T \cdot v \rangle_\Omega = \int_{\Gamma_c} T \cdot v \, ds,$$

for an arbitrary $v \in (H^1(\Omega))^N$ and the proof is complete. \hfill \blacksquare

We are finally in position to prove the following lemma.

**Lemma 3.6.** Let $T \geq 0$ be an element of $L^1(\Gamma_c)$ and suppose that the mapping

$$v \in (\mathfrak{B}(\Omega))^N \rightarrow \int_{\Gamma_c} T v_n \, ds = \int_{\Gamma_c} T n \cdot v \, ds$$

extends as a linear continuous form $\langle T n, v \rangle_\Omega$ over the space $(H^1(\Omega))^N$. Then, for every $v \in (H^1(\Omega))^N$, one has $T v_n \in L^1(\Gamma_c)$ and

$$\langle T v_n \rangle_\Omega = \int_{\Gamma_c} T v_n \, ds.$$
is a system of coordinates in which \( n_i \geq 0 \) on \( \Gamma_k \) (see Fig. 1 when \( N = 2 \); all the normal vectors have nonnegative components in the basis \( \gamma_1, \gamma_2 \).

As \( T \in L^1(\Gamma_C) \) one has \( Tn \in (L^1(\Gamma_C))^N \) and, in the appropriate system of coordinates exhibited above, \( Tn_i \geq 0, 1 \leq i \leq N \) on \( \Gamma_C \cap \Gamma_k \) since \( T \geq 0 \). Our assertion is then a simple application of lemma 3.5 with \( T = Tn \).

**THEOREM 3.1.** An element \( u \in (H^1(\Omega))^N \) is a solution to (3.20)-(3.23) if and only if

\[
\phi(u_n) v_n \in L^1(\Gamma_C) \quad \text{for every } v \in (H^1(\Omega))^N \quad \text{and}
\]

\[
a(u, v) + \int_{\Gamma_C} \phi(u_n) v_n \, ds = (f, v)_\Omega \quad \text{for every } v \in (H^1(\Omega))^N. \tag{3.38}
\]

**Proof.** Suppose that \( \phi(u_n) v_n \in L^1(\Gamma_C) \) for every \( v \in (H^1(\Omega))^N \). Taking \( v \) as the constant function equal to the \( i \)-th vector of the canonical basis of \( \mathbb{R}^N \), we find

\[
\phi(u_n) n_i \in L^1(\Gamma_C), \quad 1 \leq i \leq N.
\]

Thus,

\[
\phi(u_n) \sup_{1 \leq i \leq N} |n_i| \in L^1(\Gamma_C).
\]

As \( \sup_{1 \leq i \leq N} |n_i(x)| \geq 1/\sqrt{N} \) for almost all \( x \in \Gamma_C \) (cf. 3.1) we deduce

\[
0 \leq \phi(u_n) \leq \sqrt{N} \phi(u_n) \sup_{1 \leq i \leq N} |n_i|
\]

and hence \( \phi(u_n) \in L^1(\Gamma_C) \). If, in addition, (3.38) holds, it is obvious from lemma 3.2 that \( u \) is a solution to problem (3.20)-(3.23).

Conversely, if \( u \) is a solution to problem (3.20)-(3.23), lemma 3.2 ensures that \( \phi(u_n) \in L^1(\Gamma_C) \) and the mapping

\[
v \in (\mathbb{D}(\tilde{\Omega}))^N \rightarrow \int_{\Gamma_C} \phi(u_n) v_n \, ds
\]
extends as the linear continuous form over the space \((H^1(\Omega))^N\)

\[ v \in (H^1(\Omega))^N \rightarrow \langle f, v \rangle_\Omega - a(u, v). \]

The conclusion follows from lemma 3.6 with \(T = \hat{\phi}(u_n)\).

With theorem 3.1 as a starting point, we shall now be able to prove the following theorem.

**Theorem 3.2.** Let \(u \in (H^1(\Omega))^N\) be a solution to problem (3.20)-(3.23). Then, \(u\) is a minimizer of the functional

\[ v \in (H^1(\Omega))^N \rightarrow J(v) = \frac{1}{2} a(v, v) + j(v_n) - \langle f, v \rangle_\Omega \in \mathbb{R} \]  

where \(j\) denotes the functional (3.16).

**Proof.** Let \(u \in (H^1(\Omega))^N\) be a solution to problem (3.20)-(3.23). We first show that \(J(u) < +\infty\), or, equivalently, that \(j(u_n) < +\infty\). From (3.9), it follows that the mapping \(\Phi(x, t) = \int_0^t \phi(x, \tau) \, d\tau\) verifies

\[ 0 \leq \Phi(x, t) \leq \frac{t}{2} \phi(x, t) \]

for \(t > 0\) (a relation that was already used in the proof of theorem 2.2). This inequality extends to \(t \in \mathbb{R}\) since \(\Phi\) and \(\phi\) vanish for nonpositive values of \(t\). In particular, taking \(t = u_n(x)\), we get

\[ 0 \leq \Phi(u_n) \leq \frac{1}{2} \phi(u_n)u_n. \]  

As \(\hat{\phi}(u_n)u_n \in L^1(\Gamma_C)\) from theorem 3.1, we find \(\hat{\phi}(u_n) \in L^1(\Gamma_C)\), namely, \(j(u_n) < +\infty\) (cf. (3.16)).

Now, let \(v \in (H^1(\Omega))^N\) be given. We must prove that \(J(v) \geq J(u)\). Of course, this is satisfied if \(j(v_n) = +\infty\). Assume then \(j(v_n) < +\infty\), i.e. \(\Phi(u_n) \in L^1(\Gamma_C)\). One has

\[ J(v) - J(u) = \frac{1}{2} a(v - u, v - u) + a(u, v - u) + j(v_n) - j(u_n) - \langle f, v - u \rangle_\Omega. \]  

From our assumptions, the mapping \(\Phi(x, \cdot)\) is convex for almost all \(x \in \Gamma_C\) so that the inequality

\[ \Phi(x, t) \geq \Phi(x, \tau) + \phi(x, \tau)(t - \tau) \]

holds for every pair \((t, \tau) \in \mathbb{R} \times \mathbb{R}\) and almost all \(x \in \Gamma_C\). Choosing \(t = v_n(x)\) and \(\tau = u_n(x)\) yields

\[ \Phi(v_n) \geq \Phi(u_n) + \hat{\phi}(u_n)(v_n - u_n). \]

As the three terms \(\Phi(v_n), \hat{\phi}(u_n)\) and \(\hat{\phi}(u_n)(v_n - u_n)\) are in \(L^1(\Gamma_C)\) (the latter from theorem 3.1 with \(v - u\) replacing \(v\)), integrating both sides of the above inequality provides

\[ j(v_n) \geq j(u_n) + \int_{\Gamma_C} \hat{\phi}(u_n)(v_n - u_n) \, ds. \]
Applying (3.38) with $v - u$ replacing $v$, we obtain

$$j(v_n) = j(u_n) + \langle f, v - u \rangle_\Omega - a(u, v - u).$$

Substituting into (3.41), we deduce

$$J(v) - J(u) \geq \frac{1}{2} a(v - u, v - u) \geq 0,$$

and the proof is complete. 

**Remark 3.1.** From the proof of theorem 3.2, it also follows if $u$ and $v$ are two solutions of problem (3.20)-(3.23), and hence two minimizers of the functional (3.39), that $a(v - u, v - u) = 0$, which characterizes the difference $v - u$ as an infinitesimal rigid motion (cf. Section 4).

The converse of theorem 3.2 is essentially based on the results of Section 2.

**Theorem 3.3.** Let $u \in (H^1(\Omega))^N$ be a minimizer of the functional $J$ (3.39). Then, $u$ is a solution to problem (3.20)-(3.23).

**Proof.** Since the functional $j$ (3.16) takes finite values for $\xi = v_n$ and some $v \in (H^1(\Omega))^N$ (for instance, if $v \in (\mathcal{D}(\bar{\Omega}))^N$), because $v_n \in L^2(\Gamma_C)$ so that $\Phi(v_n) \in L^1(\Gamma_C)$, as it follows from theorem 2.1 (ii) with $\xi = 0$ and $\eta = v_n$) one has $\Phi(u_n) \in L^1(\Gamma_C)$ when $u$ is a minimizer of the functional (3.39). From theorem 2.1 (ii) in which $\xi = u_n$ and $\eta = 0$, we deduce that $\Phi(u_n) \in L^1(\Gamma_C)$. Next, for any given $v \in (\mathcal{D}(\bar{\Omega}))^N$, the normal component $v_n$ belongs to $L^2(\Gamma_C)$. Again, from theorem 2.1 (ii) in which $\xi = u_n$ and $\eta = tv_n$, the function

$$t \in \mathbb{R} \rightarrow j(u_n + tv_n)$$

is real-valued and differentiable at the origin with

$$\frac{d}{dt} j(u_n + tv_n)|_{t=0} = \int_{\Gamma_C} \Phi(u_n)v_n \, ds.$$

Hence, the function

$$t \in \mathbb{R} \rightarrow J(u + tv)$$

is real-valued and differentiable at the origin with

$$\frac{d}{dt} J(u + tv)|_{t=0} = a(u, v) + \int_{\Gamma_C} \Phi(u_n)v_n \, ds - \langle f, v \rangle_\Omega.$$

Since $t = 0$ is a minimum for the function (3.42), we obtain

$$a(u, v) + \int_{\Gamma_C} \Phi(u_n)v_n \, ds - \langle f, v \rangle_\Omega = 0$$

for every $v \in (\mathcal{D}(\bar{\Omega}))^N$ and the result follows from lemma 3.2.
The next section is devoted to proving the existence of minimizers of the functional (3.39) under suitable compatibility conditions between the applied forces and the geometry of $\Gamma_c$ and to the study of uniqueness or nonuniqueness of solutions to problem (3.20)–(3.23).

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