AN ACCURATE AND EFFICIENT A POSTERIORI CONTROL OF HOURGLASS INSTABILITIES IN UNDERINTEGRATED LINEAR AND NONLINEAR ELASTICITY*

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In this paper, we present several ways to prevent and a posteriori eliminate hourglass instabilities from solutions obtained with underintegrated stiffness matrices. Several theoretical results previously obtained for a model problem are stated, then the excitation of spurious oscillations is analyzed for a broader class of problems. Finally, the analysis is extended to linear and nonlinear elasticity problems: the spurious modes are constructed and their a posteriori elimination is described. Numerical results show that this method is accurate and efficient.

0. Introduction

For the last several years, several finite element analysts have investigated the possibility to obtain accurate solutions from underintegrated stiffness matrices as a means to improve computational efficiency. However, it is known that while such underintegration can significantly reduce computational effort, the solutions obtained from resulting matrices may exhibit undesirable oscillations due to zero-energy (spurious) modes in the kernel of the underintegrated matrices.

In order to overcome this difficulty, two procedures are at the present time often discussed. The first one, due to Belytschko and co-workers [1–3, 5], consists in the addition of so-called stabilization matrices to the underintegrated matrix, thus eliminating spurious modes from the kernel of the stiffness matrix. The second one, due to Jacquotte and Oden [7–10], consists in computing the underintegrated solution and then a posteriori eliminating the instabilities present in this solution. In recent works [6, 7, 10] the accuracy and efficiency of such a method has been mathematically proved and numerically investigated for simple linear model problems. The purpose of this paper is to extend this a posteriori elimination idea to linear and nonlinear elasticity problems. In particular, we present a projection that efficiently eliminates oscillation, leading to an accurate solution. The results of several numerical experiments in linear and nonlinear elasticity are given and indicate that the method developed is effective for problems of this type.

This paper is divided in four sections. In Section 1, we review the a posteriori control of

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A posteriori spurious-mode control

In this section, we review procedures for obtaining a convergent solution from a certain class of underintegrated problems. The procedure, which consists of solving the problem up to within an arbitrary spurious mode and then eliminating this mode in a post-processing operation, is briefly described and several of its features are pointed out for further use in more complicated problems. Numerous details and examples can be found in several papers by Jacquotte and Oden [6, 7, 10].

A general discrete variational formulation of a boundary value problem is

\[ (P^h) \quad \text{find } u^h \in V^h \text{ such that} \]
\[ a(u^h, v^h) = f(v^h) \quad \forall v^h \in V^h. \quad (1.1) \]

In a finite element code, the matrix associated to the bilinear form \( a \) is computed using numerical integration. For a certain choice of element, a minimum number of integration points is required to obtain an accurate solution. When this number of points is reduced, the underintegrated formulation can be written:

\[ (\bar{P}^h) \quad \text{find } \bar{u}^h \in \bar{V}^h \text{ such that} \]
\[ \bar{a}(\bar{u}^h, v^h) = \bar{f}(v^h) \quad \forall v^h \in \bar{V}^h, \quad (1.2) \]

where
\[ \bar{V}^h = V^h/(\text{ker } \bar{a}/\text{ker } a). \quad (1.3) \]

Existence and uniqueness of \( \bar{P}^h \) have been discussed by Jacquotte and Oden [6] for the Poisson equation; it has also been proved that the projection of the solution of \( \bar{P}^h \) with respect to the spurious mode converges without any loss of rate of convergence towards the exact solution. For example, when
\[ \text{dim}(\text{ker } \bar{a}/\text{ker } a) = 1 \quad (1.4) \]
we have one spurious mode and the projection is given by
\[ \Pi: V^h \to V^h, \quad \bar{u}^h \to \bar{u}^h = \bar{u}^h - H[a(\bar{u}^h, H)/a(H, H)] \quad (1.5) \]
where \( H \) is one representative of the spurious mode. The case of several of these modes will be
discussed in Section 3. This procedure has been numerically tested with results corroborating the theory and illustrating its accuracy and efficiency.

Several remarks concerning this method must be made. First, note that this spurious-mode elimination is global: the solution on the entire domain is globally controlled and the spurious mode is subtracted from all the nodes at once. The same elimination applied element-by-element will be investigated in Section 3. Also, in order to apply the projection, we must a priori know exactly the spurious modes and their nodal values. As far as the Laplace equation with Neumann boundary conditions is concerned, one knows that these modes are the hourglass modes (4-node elements) and their 9-node generalizations, where values are independent of the geometry of the elements. When Dirichlet boundary conditions are applied on one part of the boundary, the underintegrated stiffness matrix is not rank-deficient and one can show that the solution obtained converges towards the exact solution without loss of order of convergence. However, oscillation can be excited when some very irregular loads (data) are applied. A study of such excitations is presented in the next section.

In order to solve the underintegrated system

$$\bar{A}\bar{U} = \bar{F}$$

we must have the orthogonality condition

$$\bar{F} \in (\operatorname{ker} \bar{A})^\perp.$$  

(1.7)

From a practical point of view, the applied load must be orthogonal to the spurious mode; if this is not the case, concentrated spurious forces may be induced at the nodes that have been fixed to solve modulo spurious modes.

This is clearly understood on the following example where $\Omega$ is a square partitioned into four or nine 4-node elements. For the Laplace equation, we consider two loads $F_A$ and $F_B$, concentrated at two opposite points and satisfying the equilibrium condition as shown in Fig. 1. When the number of elements is even (Fig. 1(a)), the system of forces is orthogonal to the hourglass mode $H$ and therefore the reactions at the fixed points are zero as desired. Conversely, when the number of elements is odd (Fig. 1(b)), the system of forces is not orthogonal to $H$ and two reactions $R_1$ and $R_2$ appear at the fixed points; the system of forces $F_A$, $F_B$, $R_1$ and $R_2$ is then orthogonal to both the translation and the hourglass mode. Such considerations can explain the peculiar rates of convergence obtained for the data of the unit square discretized with $N \times N$ square elements:

$$f = \begin{cases} -6x(y^2 - 1)^2 + 4x^3(1 - 3y^2), & \text{if } x \leq \frac{1}{2}, \\ \frac{1}{2}(y^2 - 1)^2 + \frac{1}{3}(-12x^2 + 24x - 7)(1 - 3y^2), & \text{if } x > \frac{1}{2}. \end{cases}$$

(1.8)

Fig. 2(a) shows the rates of convergence of $\bar{u}^h$ towards $u^h$ in the $L^2$- and $H^1$-norms, where $\bar{u}^h$ is obtained from the underintegrated problem and the projection, and $u^h$ is the solution of the fully integrated problem. We can clearly see the good (more than optimal) behavior of the rates when the mesh is even; in this case, the discontinuity line ($x = \frac{1}{2}$) corresponds to a mesh line. However, the quality of the solution deteriorates when the mesh is odd or when the
discontinuity line is across the mesh and coincides with the integration points. Note that when $f$ is averaged on $x = \frac{1}{2}$,

$$f\left(\frac{1}{2}, y\right) = \frac{1}{2}(f\left(\frac{1}{2}^+, y\right) + f\left(\frac{1}{2}^-, y\right)).$$

(1.9)

then all the points on both plots lay on one straight line. Also, note that the knowledge of the exact solution

$$u = \begin{cases} x^3(y^2 - 1)^2, & \text{if } x \leq \frac{1}{2}, \\ \frac{1}{16}(-12x^2 + 24x - 7)(y^2 - 1), & \text{if } x > \frac{1}{2} \end{cases}$$

(1.10)
has allowed the computation of the rates of convergence of $u^h$ towards $u$ and slopes 2 and 1 have been obtained.

The interpretation of (1.7) has been possible when Neumann boundary conditions are applied. When Dirichlet conditions are applied, any right-hand-side vector can produce a solution, but practically, similar behavior is observed (Fig. 2(b)). These results suggest that for any problem one must pay attention to the data and make sure it is orthogonal, in a certain sense, to the spurious modes. This observation will again emerge from the discussion in the next section and provide a basis for the elimination of spurious oscillations for the linear elasticity problem.

Fig. 2. Rates of convergence obtained with discontinuous data: (a) Neumann boundary conditions, (b) Dirichlet boundary conditions.
2. Excitation of spurious oscillations

The existence of spurious oscillations in underintegrated problems is not only encountered when Neumann boundary conditions are applied. In this section we analyze precisely how modes that oscillate with wavelengths of order $h$ are excited when underintegration is used, whereas they are damped when the integration is exact.

For this discussion, we consider the unit square

$$\Omega = [0, 1] \times [0, 1]$$

discretized into $N \times N$ elements. As a model problem, we again consider the Laplace equation on $\Omega$:

$$\begin{align*}
-\Delta u &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega \cap \{x = 0\}, \\
\frac{\partial u}{\partial n} &= g \quad \text{on } \partial \Omega \setminus \{x = 0\}.
\end{align*}$$

We include two kinds of loads: body forces and surface loads, and we will observe separately the effects of each.

A discrete Fourier analysis can be carried out for this simple model problem and one can exhibit the eigenfunctions and eigenvalues of fully and underintegrated stiffness matrices. For the particular mixed boundary conditions of (2.1), we can construct the basis of functions:

$$\psi^i(k/N) = \sin(ik\pi/N), \quad \phi^j(k/N) = \cos(jk\pi/N), \quad 0 \leq k \leq N, \quad 0 \leq j \leq N,$$

These functions are defined with sine and cosine functions and they therefore oscillate. Among them we distinguish 'smooth' modes with longer wavelengths ($O(1)$) from 'irregular' modes with shorter wavelengths ($O(h)$). Smooth (respectively, irregular) modes correspond to smaller (respectively, larger) values of $i$ or $j$. Examples of each extreme are shown in Fig. 3 for $N = 10$.

The resolution of the fully integrated problem

$$(P^h) \quad \begin{align*}
\text{find } u^h \in V^h \text{ such that } \\
(u^h, v^h)_h &= (f, v^h)_0 + (g, v^h)_{\partial \Omega} \quad \forall v^h \in V^h
\end{align*}$$

leads to the search for coefficients $u_{ij}$ such that
Fig. 3. Examples of 'smooth' or 'irregular' eigenfunctions.

\[ u^h = \sum_{\substack{0 \leq i \leq N \atop 0 \leq j \leq N}} u_{ij} X^i_j. \]  

(2.5)

The basis \( \{X^i_j\} \) being an eigenbasis, we have

\[ u_{ij} = A_{ij} (f, X^i_j)_0 + (g, X^i_j)_{0, \beta_i} \]  

(2.6)

where

\[ A_{ij} = \frac{1}{\beta_i' + \beta_i} \]  

(2.7)

with

\[ \beta_i' = \frac{6}{h^2} \frac{1 - \cos(i\pi/N - \pi/2N)}{2 + \cos(i\pi/N - \pi/2N)} \]  

and

\[ \beta_i = \frac{6}{h^2} \frac{1 - \cos(j\pi/N)}{2 + \cos(j\pi/N)} \]  

(2.8)
The values \( A_i \) have been calculated exactly with these formulae and their values are reported in Table 1 for \( N = 10 \). The 20 highest values are in the dashed zone. We clearly can observe that:

(i) these values range from the highest value to 1\% of this value;

(ii) these values are associated with smooth modes (tensor products of smooth modes).

On the other hand, the eigenvalues of irregular modes are smaller and because of this, these modes will be damped: only smooth modes will contribute in (2.5).

When underintegration is used and when \( g \) is zero, the problem can be written as

\[
(P^h) \quad \text{find } \tilde{u}^h \in V^h \text{ such that } \\
(\tilde{u}^h, v^h)_{1,h} = (f, v^h)_{0,h}.
\]

The solution \( \tilde{u}^h \) is

\[
\tilde{u}^h = \sum_{1 \leq i \leq N} \tilde{u}_{i,j} X^{i,j},
\]

with

\[
\tilde{u}_{i,j} = \tilde{A}_{ij}(f, X^{i,j})_0,
\]

where

\[
\tilde{A}_{ij} = \frac{\alpha_i \alpha_j}{\alpha_i \beta_i + \alpha_i \beta_j},
\]

and

\[
\alpha_i = \frac{3 \left(1 + \cos \left(\frac{i \pi}{N} - \frac{\pi}{2N}\right)\right)}{2 \left(2 + \cos \left(\frac{i \pi}{N} - \frac{\pi}{2N}\right)\right)}, \quad \alpha_j = \frac{3 \left(1 + \cos \left(\frac{j \pi}{N}\right)\right)}{2 \left(2 + \cos \left(\frac{j \pi}{N}\right)\right)}.
\]

Table 1
Array of eigenvalues \( A_i \)

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Again, the values of \( \tilde{A}_{ij} \) have been calculated exactly and they are reported in Table 2. The 20 highest values are in the dashed zone. The comparison between Tables 1 and 2 shows that these 20 values are approximately the same and they are associated with the same smooth modes. In this case, irregular modes will still be damped, and one can predict that no oscillation will occur.

When a load is only applied on the boundary \( (f = 0, g \neq 0) \), \( \tilde{u}_i \) is now

\[
\tilde{u}_i = \tilde{A}_i (g, \chi^i)_{0, \alpha \beta}.
\]

where

\[
\tilde{A}_i = \frac{1}{\alpha_i \beta_j + \alpha_j \beta_i}.
\]

Again the values of \( \tilde{A}_{ij} \) are reported in Table 3 and the 20 highest values are in the dashed zone. Among these 20 values, three correspond to very irregular modes. In particular, the third value is associated with \( \chi^{10,10} \). Therefore, we can predict a strong contribution of irregular modes within the solution \( \tilde{u}_i \), which will show oscillations.

Finally, the possibility of calculating the boundary integral in such a way that \( (g, \chi^i)_{0, \alpha \beta} \) is damped for large \( i \) and \( j \) arises. Unfortunately, nothing general can be proved in this regard. In particular, if the load \( g \) is concentrated at \( (x_0, y_0) \), then

\[
(g, \chi^i)_{0, \alpha \beta} = g \chi^i (x_0, y_0)
\]

and this value is not necessarily zero. However, if the concentrated load \( g \) is split into several terms, such that the sum of each contribution is zero:

\[
g = \sum_{k=1}^{\kappa} g_k, \quad 0 = \sum_{k=1}^{\kappa} g_k \chi^i (x_k, y_k).
\]
Table 3
Array of eigenvalues $\tilde{A}_{ij}$

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<td>0.44</td>
<td>0.37</td>
<td>0.29</td>
<td>0.23</td>
<td>0.17</td>
<td>0.14</td>
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</tr>
<tr>
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<td>0.30</td>
<td>0.27</td>
<td>0.24</td>
<td>0.20</td>
<td>0.17</td>
<td>0.14</td>
<td>0.12</td>
<td>0.11</td>
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<tr>
<td>6</td>
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<td>0.21</td>
<td>0.21</td>
<td>0.20</td>
<td>0.19</td>
<td>0.18</td>
<td>0.17</td>
<td>0.16</td>
<td>0.14</td>
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<td>0.10</td>
<td>0.12</td>
<td>0.16</td>
<td>0.25</td>
<td>0.57</td>
<td>4.57</td>
</tr>
</tbody>
</table>

Then the mode $\chi^{ij}$ will not be excited. This treatment of the right-hand side of the system of equations can efficiently prevent the excitation of oscillations; it can also be interpreted as a generalization of (1.7) as mentioned at the end of the previous section.

Finally, note that the previous study has been carried out explicitly for a simple case ($-\Delta$, 4-node elements). More complicated cases have been studied numerically [4].

3. Underintegration in linear elasticity

In this section, we would like to discuss the effects of underintegration in elastostatics problems. We first generalize the ideas previously discussed to the linear elasticity problem, then we describe numerical results obtained. In this discussion, we concentrate our theoretical and numerical efforts on the 9-node element. For purpose of simplicity, we limit ourselves to the bidimensional, isotropic, plane-strain linear elasticity problem defined by the operator

$$
\mathbf{A} = \mathbf{\beta}' \mathbf{C} \mathbf{\beta},
$$

where

$$
\mathbf{\beta}' = \begin{pmatrix}
\frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\
0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x}
\end{pmatrix},
\mathbf{C} = \begin{pmatrix}
\lambda + 2\mu & \lambda & 0 \\
\lambda & \lambda + 2\mu & 0 \\
0 & 0 & \mu
\end{pmatrix}
$$

3.1. The spurious modes of the underintegrated matrix

We first intend to recall the construction of the spurious mode associated to the 9-node, 4-integration point bidimensional element. It is well known that this element has 3 spurious modes; for an arbitrary geometry, it can be proved [6] that two of these modes are the 9-node generalized hourglass modes in both $x$- and $y$-directions.
We note that the third spurious mode is also present in the underintegrated 8-node element, but the existence of the ninth node allows neighboring elements to share this mode. For example, on a square mesh, if the nodal displacement vector is \( \mathbf{W} \) in \( \Omega_0 \), where

\[
\mathbf{W} = \begin{pmatrix} -2 & 2 & 2 & -2 & 0 & -1 & 0 & 1 & 0 \\ 2 & 2 & -2 & -2 & -1 & 0 & 1 & 0 & 0 \end{pmatrix}.
\]  

(3.4)

then the displacements \( \mathbf{W} + 3\mathbf{H}_x + t_y \) and \( \mathbf{W} - 3\mathbf{H}_y - t_x \) on the right of \( \Omega_0 \) and above \( \Omega_0 \) allow the construction of a continuous global displacement, also denoted \( \mathbf{W} \), on the 3 elements considered and, furthermore, on the entire mesh as shown in Fig. 4. We finally have

\[
\ker \mathbf{K}_{\text{under}} = \text{span}\{t_x, t_y, r, \mathbf{H}_x, \mathbf{H}_y, \mathbf{W}\}.
\]  

(3.5)

The usual node numbering is used: corner—1 through 4; then midside—5 through 8; and finally centroid—node 9.

**REMARK 3.1.** For a regular mesh, the mode \( \mathbf{W} \) is characterized by the amplification of the contribution of the mode \( 3\mathbf{H} + t \) in either direction. This is theoretically exact for only Neumann (traction) boundary conditions, but also practically true and observed for the oscillations appearing with any conditions. Therefore, according to the previous interpretations of (1.7) and (2.17), a way to prevent these oscillations is to consider boundary loads orthogonal to the mode \( 3\mathbf{H} + t \).

**REMARK 3.2.** Contrary to the \( \mathbf{H} \)-modes, the explicit value of \( \mathbf{W} \) depends upon the geometry of the element. Even though no general expressions have been obtained for an arbitrary geometry, one can prove \[6\] that when the element is quadrilateral with nodes at the corners, midside and centroid, a representant of \( \mathbf{W} \) can be constructed as follows:

(i) The displacement of midside node is normal to the side, alternatively inwardly and outwardly oriented with magnitude proportional to the length of side.

(ii) The displacement of a corner is obtained by multiplication by \(-2\) of the sum of the two displacements of the closest midside nodes.

(iii) The displacement of the centroid is zero.

An example of \( \mathbf{W} \) is shown in Fig. 5.

**REMARK 3.3.** Although only Neumann boundary conditions are considered, certain other conditions of symmetry can lower the number of spurious modes. In particular if both \( x \)- and \( y \)-axes are axes of symmetry, one spurious mode still remains.

3.2. A global control of the spurious modes

The first series of numerical examples is designed to show that a global control such as the control described in Section 1 leads to solutions converging with optimal rate of convergence.
Since a knowledge of the spurious modes is required, the domain will be discretized with square subdomains.

When several spurious modes $H_i$ ($i = 1, \ldots, I$) are present, we refine until the projection defining the control satisfies

$$a(H\tilde{u}^h, H_i) = 0, \quad i = 1, \ldots, I. \quad (3.6)$$

where $a(\cdot, \cdot)$ denotes the bilinear elasticity form associated to the operator $A$ of (3.1). This leads to the calculations of $\{\lambda_i\}_{i=1, \ldots, I}$ such that

$$\sum_{j=1}^{I} a(H_i, H_j)\lambda_j = a(\tilde{u}^h, H_j), \quad i = 1, \ldots, I, \quad (3.7)$$
then, the projection is given by

\[
\tilde{u}^h = \Pi u^h = \tilde{u}^h - \sum_{i=1}^{I} \lambda_i H_i.
\]  

(3.8)

The \( \lambda \) are given by the resolution of a system of \( I \) equations of \( I \) unknowns; the coefficients of this system involve the computation of \( a(H_i, \cdot) \). They can be written as

\[
a(H_i, v^h) = \tilde{a}(H_i, v^h) + a'(H_i, v^h) = a'(H_i, v^h).
\]

(3.9)

where \( \tilde{a} \) is the underintegrated bilinear form and \( a' \) the residual form that can also be interpreted as the form associated to a stabilization matrix [1–3]. In order to calculate \( a'(H_i, v^h) \), we use explicit expressions of \( K_{\text{stab}} \); a form only using \( \gamma \cdot \gamma \) terms introduced by Belytschko et al. [1, 2] was implemented and the optimal rates of convergence were not obtained. However, the exact expression for \( K_{\text{stab}} \) (for a regular mesh) is

\[
K_{\text{stab}} = \frac{4\Omega}{135} 2(\lambda + \mu) \begin{pmatrix} s_\gamma s_\gamma^t & 0 \\ 0 & s_\gamma s_\gamma^t \end{pmatrix} + \frac{\Omega_{\varepsilon}}{45} \begin{pmatrix} (\lambda + 2\mu) s_\gamma s_\gamma^t + \lambda s_\delta s_\delta^t \\ 0 & \lambda s_\delta s_\delta^t + (\lambda + 2\mu) s_\delta s_\delta^t \end{pmatrix},
\]

(3.10)

where the vectors \( s_\gamma, s_\delta, \) and \( s_\delta \) are given in Appendix A. Then, the element contributions of the coefficients \( a'(H_i, v^h) \) can be computed efficiently.

Various types of load have been tested (Fig. 6) and the comparison between the fully integrated solution and the projection of the underintegrated solution has been made, as the mesh was refined. The results have always been optimal: slopes 3 \((L^2(\Omega)/R\text{-norm})\) and 2
(energy norm) have been obtained for smooth solutions \((u \in H^3(\Omega))\). For composite materials with different densities \((f \in L^2(\Omega), u \in H^2(\Omega))\) the slopes 3 and 2 were still observed. For composite materials with different moduli of elasticity \((u \in H^1(\Omega))\) the slopes 2 and 1 obtained are still optimal.

Unfortunately, even though these results obtained with a global control are optimal, they can only be applied to a very limited number of problems where the spurious modes are exactly known (square elements and only traction boundary conditions). These limitations motivate the use of a similar control applied on each element and next described.

### 3.3. A local control of the spurious modes

We here examine the use of local control; this local procedure consists of eliminating, element-by-element, the components of \(H_x, H_y\) and \(W\), and then averaging the modal values obtained in neighboring elements. For any element we choose to do the following simplifications:
Fig. 7. Displacement of a fixed square: (a) definition of the problem, (b) results.

(i) The modal values of $W$ are taken as if the element was quadrilateral (cf. Remark 3.2).
(ii) $C$ is diagonal.
(iii) $K^{stab}$ is given by (3.10).

These simplifications allow one to obtain a completely geometric control; its easy implementation is described in Appendix A.

The first example is described in Fig. 7(a). Three types of load and boundary conditions $(g \neq 0, p \neq 0, u_0 \neq 0)$ have been applied, all producing a stress singularity which occurs in the neighborhood of the origin. The displacement solutions are shown in Fig. 7(b) and prove that the control efficiently eliminates the oscillations. The shear along the lowest row of integration points is shown in Fig. 8. The results are qualitatively similar for any load. Whereas the shear before control shows oscillations, the results for the filtered solutions are smoother and close to the fully integrated solution.

In the second example, we consider a ring under the action of an external pressure (Fig. 9). Here again the oscillations generated by the underintegration are damped when the control is applied. Only very slight oscillations remain, not exceeding 5%, and these can be easily
interpreted: in the control, the expression taken for the mode $W$ was obtained for quadrilateral elements. For the mesh considered the elements are slightly bent and this difference explains these slight oscillations. The same domain (quarter ring) has also been discretized using straight-sided elements and the control of the underintegrated solution has led to a displacement field without any oscillations and similar to the underintegrated displacements. Calculations of the stress along a radius show behavior identical to the previous example.

The third example involves a concentrated force and illustrates our discussion concerning the excitation of spurious modes and the orthogonalization of the data. A point force is applied at a corner of a fixed-side square discretized with a mesh refining in the neighborhood of the singularity. Strong oscillations appear in this region when underintegration is used, whereas the full-integration solution is smooth (Fig. 10). These oscillations show a pattern similar to the one used to construct the mode $W$ (Fig. 4): amplification of the mode $3H_x t_x$ (respectively $3H_y t_y$) along the $x$- (resp. $y$-) direction. Therefore, according to the previous

---

Fig. 8. Shear along a line AA'.
interpretations of (1.7) and (2.17), a way to prevent oscillations is to consider a system of loads similar to the load concentrated in a point A but orthogonal to $3\mathbf{H} + t$. This is obtained by splitting the force into 3 equal forces applied at A and its two closest nodes. Indeed, the displacements obtained with this system of loads only show slight oscillations. Finally, note that this control produces displacement fields similar to the fields obtained using full integration.

4. Underintegration in nonlinear elasticity

4.1. Introduction

In this section, we show how the local control previously described behaves when applied to a nonlinear incompressible material. Only Mooney–Rivlin materials are considered: they are characterized by a strain-energy function

$$\sigma = C_1(I_1 - 3) + C_2(I_2 - 3).$$ (4.1)
where $I_i, i = 1, \ldots, 3$, are the principal invariants of the Cauchy–Green deformation tensor. The incompressibility constraint is taken into account in a mixed formulation of the equilibrium problem by introducing the hydrostatic pressure $P$. The energy function is then

$$
\sigma = C_1(I_1 - 3) + C_2(I_2 - 3) + P(I_3 - 1),
$$

The solution of this highly nonlinear problem is accomplished using Newton's method. Details of the finite element applied to this particular class of problems are discussed at length in the book of Oden [12]. We noticed in Section 3.3 how the control was only geometric. This is particularly useful for rubberlike materials and makes it very efficient when constitutive relations are numerically expensive to obtain. Also note that the calculation of the projected element solution involves the knowledge of the mode $W$ which is computed using the current
deformed geometry of the element. Also we point out that the control is applied at each load increment. We present two examples where stress singularity is present.

4.2. Thick cracked rubber bar

This example models a thick rubber bar with a nonpropagating crack, submitted to a prescribed stretching displacement at its extremities. The initial configuration is shown in Fig. 11(a). The solution obtained with underintegration (Fig. 11(b)) shows severe oscillations; they disappear when the control has been applied and the controlled solution (Fig. 11(c)) and the fully integrated solution (Fig. 11(d)) behave similarly.

4.3. Compression of a fixed rubber material domain

We consider a rectangular domain, with a fixed lower side and we apply a compressive displacement to the top side. The displacement increments are 5, 10, 15 and 17.5% of compression with respect to the original shape. The solution of this problem is complicated by a stress singularity that occurs in the neighborhood of the corner A. We consider two discretizations of the domain with 25 and 49 elements.

For the crude mesh, oscillations appear very soon when underintegration is performed, but
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Fig. 12. Compression (15, 17.5%) of a rubber domain (25 element mesh): (a) underintegration, (b) controlled underintegration, (c) full integration.

The control easily corrects the solution and a displacement field close to the fully integrated field is obtained (Fig. 12(a)). But when the mesh is refined, the oscillations become more important and deform the element to such a degree that the control is not able to restore the shape of the element corner (Fig. 13(b)). We interpret this lack of performance to the fact that the control has been exactly obtained for quadrilateral elements. In this case, the element sides curve and the element is too deformed. Also this lack of performance is observed when looking at the number of iterations required for the convergence of the Newton algorithm: they are 5, 5, 5 and 5 (respectively 5, 5, 6, 7) for the fully (respectively under-) integration in the 25-element mesh and 6, 6, and 6 (respectively 5, 6, and 13) in the 49-element mesh.
5. Conclusion

In this paper, we have proposed several ways to a posteriori eliminate hourglass-type instabilities from solutions obtained from underintegrated stiffness matrices. Several previous theoretical works have shown that this elimination was possible and accurate when the spurious modes are exactly known on the entire domain. In the present work, we have observed that the knowledge of the element stiffness matrix kernel is sufficient and that an element-by-element control was also accurate. Moreover, this control is efficient and can be applied independently of the material. Several questions have however arisen:

(1) When an element is too distorted, the control cannot restore a reasonable shape. The accuracy of the control relies on the approximation made on the mode $W$. Does an exact calculation of $W$ for very distorted elements give a better answer?

(2) Do the results generalize to three-dimensional elasticity?
(3) How does the method behave in various other nonlinear problems (stability bifurcation, viscoelasticity, plasticity)?

The answer to these questions may provide a substantial gain in computation time.

Appendix A.

The notations are the following:

\[ h = (1, 1, 1, 1, -1, -1, -1, -1, 0)^T, \]
\[ s_7 = (-1, 1, 1, -1, 0, -2, 0, 2, 0)^T, \]
\[ s_8 = (-1, -1, 1, 1, 2, 0, -2, 0, 0)^T, \]
\[ s_9 = (1, 1, 1, -2, -2, -2, -2, -2, 4)^T. \]

The underintegrated solution \( U \), the controlled solution \( \hat{U} \), and the spurious modes are denoted

\[ U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \hat{U} = \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \end{pmatrix}, \quad H_x = \begin{pmatrix} h \\ 0 \end{pmatrix}, \quad H_y = \begin{pmatrix} 0 \\ h \end{pmatrix}, \quad W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}. \]

A member of \( W \) can be chosen such that \( s_9 \cdot w_i^* = 0 \):

\[ w_1 = (3y_1 - y_2 - y_3 - y_4, y_1 - 3y_2 + y_3 + y_4, -y_1 - y_2 - 3y_3 - y_4, y_1 + y_2 + y_3 - 3y_4, \]
\[ y_3 - y_4, y_1 - y_4, y_1 - y_2, y_3 - y_2, 0)^T. \]

and \( w_2 \) is obtained changing \( y_1 \) into \(-x_i\).

Note that

\[ s_7 \cdot h^* = s_8 \cdot h^* = 0, \quad s_9 \cdot h^* = 12. \]

According to the simplification in Section 3.3 we have

\[ K_{\text{stab}} = \frac{\Omega}{135} \begin{pmatrix} 3(s_7 \cdot s_7^* + s_8 \cdot s_8^*) + s_9 \cdot s_9^* & 0 \\ 0 & 3(s_7 \cdot s_7^* + s_8 \cdot s_8^*) + 4s_9 \cdot s_9^* \end{pmatrix}. \]

Then the control is

\[ \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \lambda_1 \begin{pmatrix} h \\ 0 \end{pmatrix} - \lambda_2 \begin{pmatrix} 0 \\ h \end{pmatrix} - \lambda_3 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \]

with

\[ \lambda_i = \frac{1}{135} s_i \cdot u_i^*, \quad i = 1, 2. \]
This control is easily implemented in a code by a call to the following subroutine.

```fortran
SUBROUTINE PROJ(U,XY)
C
! THIS SUBROUTINE PROJECTS THE ELEMENT SOLUTION
! ORTHOGONALLY W.R.T. HX, HY AND W
! INPUT : U SOLUTION
! XY NODAL COORDINATES (CURRENT CONFIGURATION)
! OUTPUT : U PROJECTED SOLUTION
!
DIMENSION U(2,9), XY(2,9), W(2,9), S7(9), S8(9), S9(9), H(9)
DIMENSION S7U(2), S8U(2), S9U(2), S7W(2), S8W(2)
INTEGER SIGN
DATA S7U, S8U, S9U, S7W, S8W /10*0./
DATA S7 /-1., 1., 1., 1., -1., 1., -1., 1., 0./
DATA S8 /-1., 1., 1., 1., 2., 0., -2., 0., 0./
DATA H/ 1., 1., 1., -1., -1., -1., -1., 1., 1., 1., 1., 1., -1., -1../
SIGN = -1
DO 1 K = 1, 2
K1 = 3 - K
IF(K.EQ.2) SIGN = 1
W(K,1) = SIGN*(+3.*XY(K1,1) - XY(K1,2) - XY(K1,3) - XY(K1,4))
W(K,2) = SIGN*(+XY(K1,1) - 3.*XY(K1,2) + XY(K1,3) + XY(K1,4))
W(K,3) = SIGN*(-XY(K1,1) - XY(K1,2) + 3.*XY(K1,3) - XY(K1,4))
W(K,4) = SIGN*(+XY(K1,1) + XY(K1,2) + XY(K1,3) - 3.*XY(K1,4))
W(K,5) = SIGN*(XY(K1,3) - XY(K1,4))
W(K,6) = SIGN*(XY(K1,1) - XY(K1,4))
W(K,7) = SIGN*(XY(K1,1) - XY(K1,2))
W(K,8) = SIGN*(XY(K1,3) - XY(K1,2))
1 W(K,9) = 0.
DO 2 K = 1, 9
DO 2 J = 1, 2
S7U(J) = S7U(J) + S7(K)*U(J,K)
S8U(J) = S8U(J) + S8(K)*U(J,K)
S9U(J) = S9U(J) + S9(K)*U(J,K)
S7W(J) = S7W(J) + S7(K)*W(J,K)
2 S8W(J) = S8W(J) + S8(K)*W(J,K)
S9U(1) = S9U(1)/12.
S9U(2) = S9U(2)/12.
W1 = (S7W(1)*S7U(1) + S8W(2)*S8U(2) + S7W(2)*S7U(2) + S8W(1)*S8U(1))/
     (S7W(1)*S7W(1) + S8W(2)*S8W(2) + S7W(2)*S7W(2) + S8W(1)*S8W(1))
DO 3 K = 1, 9
DO 3 J = 1, 2
3 U(J,K) = U(J,K) - S9U(J)*W1*W1/J + 1./3.) - W1*W(J,K)
RETURN
END
```

\[
\lambda_3 = \frac{(s_7 \cdot w_1')(s_7 \cdot u_1') + (s_8 \cdot w_1')(s_8 \cdot u_2') + (s_7 \cdot w_2')(s_7 \cdot u_2') + (s_8 \cdot w_2')(s_8 \cdot u_2')}{(s_7 \cdot w_1')^2 + (s_8 \cdot w_1')^2 + (s_7 \cdot w_2')^2 + (s_8 \cdot w_2')^2}.
\]
References


