This paper describes a finite element approximation scheme for computing finite elastoplastic deformations of a hypothetical class of materials referred to as materials of type N. Kinematical descriptions of motion and strain-rate and rotationally invariant stress-rates measures appropriate for finite elastoplastic deformations are derived and the thermodynamical theory for materials of type N is briefly reviewed. Finite element approximations of the governing nonlinear equations are derived along with numerical algorithms for solving the associated discrete nonlinear system. These new methods are applied to some representative plane-strain problem, including problems with unloading and reloading and the problem of upsetting of a partially constrained rectangular billet.

In earlier papers, we have described a continuum theory of materials characterized by the existence of certain generalized flow potentials which was simultaneously consistent with continuum thermodynamic concepts and sufficiently broad to encompass a wide range of material behavior, including finite elastoplasticity (see [20, 21]). The theory, which in principle need not involve the notion of stress, involves the characterization of possibly non-differentiable and non-convex flow potentials which, together with a free-energy functional, depend upon internal state variables and various natural deformation-rate measures. Because such potentials satisfy an abstract and general normality rule, we refer to such materials as 'materials of type N'. Many thermodynamically sound theories of elastoplasticity can be obtained as special cases of materials of type N under appropriate additional restrictions.

The question remains as to whether or not such a general theory can be specialized in a meaningful way and used as a basis for the numerical analysis of physically relevant problems. The present paper describes a first step toward providing an affirmative answer to this question. Our objective here is to describe finite element approximations of the governing equations of the theory for particular choices of the flow potential and to present algorithms for the numerical solution of specific problems. The theory is not, in a strict sense, a rate-type theory, although it does involve deformation rates, and incremental-type algorithms can be used to solve the discretized problem. The basic approach, however, involves steps and considerations which are quite different from conventional methods used in computational plasticity.

*Research supported by the U.S. Army Research Office under Contract No. DAAG 29-83-K-0036 is gratefully acknowledged.

Following this introduction, we review several key kinematical issues and introduce appropriate deformation-rate measures. Our particular scheme for decomposing deformation rates into elastic and inelastic parts differs from those proposed by other authors, and leads naturally to the rotationally invariant (objective) stress-rate measure of Dienes [13]. In Section 3 we review key features of the theory of materials of type N and suggest examples of forms of the generalized flow potential which are inspired by the flow rule proposed by Bodner and Partom [9, 10]. The finite element approximation of the governing equations is taken up in Section 4 where an incremental, total Lagrangian algorithm is also described. Specific applications are also considered, including the numerical analysis of large-strain uniaxial stretching and compression of a specimen and the crushing (upsetting) of a rectangular billet in a metal-forming simulation. These preliminary results indicate that the theory and methods presented may provide a useful approach to the numerical solution of a significant class of large-deformation plasticity problems.

2. Kinematics and strain-rate measures

While somewhat difficult to define with flawless mathematical precision, the basic idea of a finite ‘plastic’ or ‘inelastic’ deformation of a material is heuristically clear: It is the ‘irrecoverable’ part of the deformation of a material subjected to a loading cycle. One imagines that a material in an ideal reference stress state \( \sigma_0 \) at a particle \( X \) is subjected to motions which carry the stress at this particle through a history which eventually returns the stress to the original state \( \sigma_0 \). If, at the conclusion of this stress cycle, the local state of deformation at \( X \), however one chooses to measure it, differs from what it was before the stress cycle, then a portion was not ‘recovered’ and this is dubbed the ‘plastic’ deformation. It is clear that these ideas are local in character; they may have meaning only in a local neighborhood of a material particle or for bodies in state of homogeneous deformation. Having this in one's mind, we offer in this section an alternative to the multiplicative and the additive decompositions of the deformation gradient proposed by Lee [23, 24] and Nemat-Nasser [33, 34], respectively.

Let us consider the motion of a material body \( B \) relative to a fixed reference configuration \( C_0 \subset \mathbb{R}^N \) \((N \leq 3)\), which is defined by the map \( \kappa_0: B \rightarrow \mathbb{R}^N. \ X = \kappa_0(X) \), where \( X \) is a material particle. The spatial position \( x \) of a particle \( X \) at time \( t \) is then given by a relation of the type

\[
x = \chi(X, t)
\]

with \( X \in \kappa_0(B), t \geq 0 \), and \( \chi \) a continuous invertible map from \( C_0 \) into \( \mathbb{R}^N \). The deformation-gradient tensor \( F \) at \( X \) at time \( t \) is defined by

\[
F = \frac{\partial \chi}{\partial X}.
\]

Let \( \mathcal{N}(X) \) denote a small material neighborhood of particle \( X \). The motion of the body carries \( \mathcal{N}(X) \) from the reference configuration \( C_0 \) to the current configuration \( C_t \). Let the Cauchy stress \( \sigma \) at any particle \( A \in \mathcal{N}(X) \) in \( C_0 \) be denoted \( \sigma(A, 0) \) with

\[
\sigma(A, 0) = \sigma_0(X) + \omega(\Delta X) \quad \forall A \in \mathcal{N}(X).
\]
where $\Delta X = A - X$, and
\[
\lim_{\|\Delta X\| \to 0} \frac{\|\sigma(\Delta X)\|}{\|\Delta X\|} = 0.
\]

$\|\cdot\|$ denotes the Euclidean norm. We shall refer to $\sigma_0(X)$ as the initial stress at particle $X$. For simplicity, we omit other variables (such as temperature, etc.) that could also be listed in defining an ‘initial state’ of the material.

During an interval of time $[0, t]$, $t > 0$, the stress at particles in $\mathcal{N}(X)$ are part of the stress history
\[
H_i(A) = \{\sigma(A, \tau) \mid A \in \mathcal{N}(X), 0 \leq \tau \leq t\}
\]
and the configurations of $\mathcal{N}(X)$ are denoted $x(\mathcal{N}(X), \tau), 0 \leq \tau \leq t$.

In addition to the actual stress history $H_i$, we consider any stress history $H_i^R$, corresponding to a relaxation of the stress at $X$, such that
\[
H_i^R(A) \subset H_i(A), \quad H_i^R \in \mathcal{H}_0.
\]
where $\mathcal{H}_0$ is the family of all stress histories terminating at $\sigma_0$:
\[
H_i^R(A) \in \mathcal{H}_0 \Rightarrow \sigma(A, t) = \sigma_0(A, 0).
\]

For $p$ the values of a continuous map of $\mathcal{N}(X)$ into $\mathbb{R}^N$, we denote by $C_p$ any configuration of $B$ for which the stress history at $A \in \mathcal{N}(X)$ is $H_i^R(A)$. Thus, the introduction of a (possible unattained) configuration $C_p$ provides for the familiar device of comparing the geometries of the body in $C_p$ with $C_0$ to define plastic deformation. The situation is illustrated in Fig. 1.

We shall now review some procedures that have been proposed for the decomposition of total measures of deformation into elastic and plastic parts.

### 2.1. An additive decomposition

We first consider an additive decomposition due to Nemat-Nasser [33, 34] which can be obtained by manipulation of a displacement field as follows:
\[
\begin{align*}
\text{d}x - dX &= (F - I) \text{d}X = \text{d}u, \quad (3) \\
\text{d}p - dX &= (F^p - I) \text{d}X = \text{d}u^p. \quad (4)
\end{align*}
\]
If we demand that a single-valued displacement field be realized in reaching the current configuration, we are led to a definition of an elastic deformation gradient given by
\[
[u(X) + \text{d}u] - [u(X) + \text{d}u^p] = (F^e - I) \text{d}X. \quad (5)
\]
We next use equations (3), (4) and (5) to eliminate $du$ and $du^p$ and obtain

$$F = F^e + F^p - I. \quad (6)$$

In other words,

$$F^e \Delta X = (du - du^p) = \Delta X \quad (7)$$

and

$$F^p \Delta X = dp = (du^p + \Delta X). \quad (8)$$

Since

$$L = \frac{\partial}{\partial x} \frac{\partial u}{\partial t} = \frac{\partial}{\partial t} \frac{\partial x}{\partial X} \frac{\partial X}{\partial x} = \dot{F}F^{-1} \quad (9)$$

and, from equation (6),

$$\dot{F} = \dot{F}^e + \dot{F}^p \quad (10)$$

we have

$$L = \dot{F}^e F^{-1} + \dot{F}^p F^{-1} \overset{\text{def}}{=} L^e + L^p. \quad (11)$$

Finally, the symmetric part of $L$ is given by the sum.

$$D = \frac{1}{2}(L + L') = \frac{1}{2}(L^e + L^e') + \frac{1}{2}(L^p + L^p') = D^e + D^p. \quad (12)$$
if $D^e = \dot{F}^e F^{-1}_{\text{sym}}$, $D^p = \dot{F}^p F^{-1}_{\text{sym}}$, where $I_{\text{sym}}$ denotes the symmetric part of the tensor.

2.2. A multiplicative decomposition

The representation of Lee [23, 24] starts with the definition

$$F = \frac{\partial x}{\partial X} = \frac{\partial x}{\partial p} \frac{\partial P}{\partial X} = \bar{F}^e F^p. \quad (14)$$

Then, by using equations (11) and (14)

$$L = \dot{F} F^{-1} = (\dot{F}^e F^p + \bar{F}^e F^p) F^{-1}_{\text{sym}} F^e_{\text{sym}}$$

$$= \dot{F}^e \bar{F}^e_{\text{sym}} + \bar{F}^e \dot{F}^p_{\text{sym}} F^p_{\text{sym}} F^e_{\text{sym}}$$

$$= \dot{L}^e + \bar{F}^e \dot{L}^p F^e_{\text{sym}}. \quad (15)$$

We recognize from (5) that the rate-of-strain measure $D$ could not be decomposed additively, as Lee noted. In metals, elastic strain rates may be small, and equation (12) can be approximated by

$$L = \bar{L}^e + \bar{L}^p$$

and $D = \bar{D}^e + \bar{D}^p. \quad (16)$

It is desirable to construct a strain-rate measure that is independent of rotation effects and, apparently, neither of the decompositions described above generate fully stretch-only-dependent strain rates for both elastic and plastic deformation measures. For these reasons, we propose an alternative method of decomposition.

2.3. An alternative decomposition method

Consider again a particle $A$ in the neighborhood $\mathcal{N}(X)$ of $X$, in the reference configuration (see Fig. 2). The position vector of point $A$ relative to the origin of the fixed spatial reference frame is denoted

$$\overline{OA}_0 = X + \Delta X. \quad (17)$$

According to the polar decomposition theorem (see e.g. [14]), the deformation gradient $F$ can be represented as the decomposition

$$F = RU = VR. \quad (18)$$

where $R$ is a positive definite orthogonal rotation tensor, $U$ is a right stretch tensor of $F$, and $V$ is a left stretch tensor of $F$.

The location of $A$ in the rotation-free current configuration $C_{\text{rf}}$, as in Fig. 2, is

$$\overline{OA}_{C_{\text{rf}}} = x + R \Delta x = \chi(X, t) + U(X, t) \Delta X + W_1(X, t, \Delta X). \quad (19)$$
where
\[ \lim_{\|\Delta X\| \to 0} \frac{1}{\|\Delta X\|} \|W_1(X, t, \Delta X)\| = 0. \] (20)

Now let us consider an intermediate configuration \( C_{p-} \) which may correspond to a 'rotation-free' state at the initial stress level \( \sigma_0 \). Then,
\[ \overline{OA}_{C_{p-}} = \chi(X, t) + U^p(X, t) \Delta X + W_2(X, t, \Delta X) \] (21)
and \( W_2 \) has the same asymptotic behavior with respect to \( \|\Delta X\| \) as does \( W_1 \).

We next introduce a second-order tensor \( U^e \) which represents the elastic stretch tensor and is defined by
\[ U^e(X, t) \Delta X - \Delta X = \overline{OA}_{C_{p-}} - \overline{OA}_{C_{p-}} \]
\[ = [U(X, t) - U^p(X, t)] \Delta X + W_1(X, t, \Delta X) - W_2(X, t, \Delta X). \] (22)

Thus, in the limit as \( |\Delta X| \to 0 \), we have
\[ U = U^e + U^p - 1 \] (23)
and the relations (18) and (11) give
\[ F = RU = RU^e + RU^p - R, \] (24)
S.J. Kim, J.T. Oden, Finite elastoplastic deformations of type-N materials

\[ L = \dot{FF}^{-1} = \dot{RR}^t + R(\dot{U}^e U^{-1} + \dot{U}^p U^{-1})R^t \]  
and

\[ D = L_{sym} = R(\dot{U}^e U^{-1}_{sym} + \dot{U}^p U^{-1}_{sym})R^t. \]

One can define as the elastic and the plastic strain rates.

\[ \dot{E} = \dot{U}^e U^{-1}_{sym}. \]  
\[ \dot{P} = \dot{U}^p U^{-1}_{sym}. \]

and

\[ R'DR = \dot{E} + \dot{P}. \]

The elastic and the plastic strain tensors can be defined by

\[ E = \int_0^t \dot{U}^e U^{-1}_{sym} dt. \]  
\[ P = \int_0^t \dot{U}^p U^{-1}_{sym} dt. \]

Of course, it may be argued that this decomposition is still imperfect since the definition involves the total stretch \( U \), but \( U \) is a well-defined kinematical property of the motion independent of the choice of \( C_p \) and (28) is a natural consequence of our formulations. It has recently been pointed out to us that our decomposition of the deformation gradient is similar to that proposed by Simo and Marsden [45].

3. A theory of materials of type N

3.1. Material of type N

The thermomechanical behavior of the body \( B \) is governed by the principles of conservation of mass, energy, balances of linear and angular momenta, and the law of entropy production. Local forms of these principles can be written as follows:

**Conservation of mass**

\[ \rho \det F = \rho_0. \]  

**Balance of linear and angular momenta**

\[ \text{div } \sigma + \rho b = \rho \ddot{x}. \]  
\[ \sigma = \sigma^t. \]

**Conservation of energy**

\[ \rho \dot{e} = \text{tr}(\sigma L) - \text{div } q + \rho r. \]
S.J. Kim, J.T. Oden. Finite elastoplastic deformations of type-N materials

Clausius-Duhem inequality

\[ \rho \dot{\eta} + \text{div} \frac{\mathbf{q}}{\theta} - \frac{r}{\theta} \geq 0 \]  

(35)

Here \( \rho \) is the mass density, \( \rho_0 \) the mass density in the reference configuration, \( \mathbf{\sigma} \) is the Cauchy stress, \( \mathbf{b} \) the body force per unit mass, \( \varepsilon \) the specific internal energy, \( \mathbf{q} \) the heat-flux vector, \( r \) the heat supply per unit mass per unit time, \( \eta \) the specific entropy, and \( \theta \) the absolute temperature. It is convenient to also introduce the free-energy density

\[ \phi = \varepsilon - \eta \theta \]  

(36)

To characterize a specific class of materials, we must introduce particular constitutive equations. The body \( B \) is said to be composed of a material of type \( N \) if and only if the following hold:

(i) there exist functions \( \Phi, \Sigma, N, \) and \( Q \) of \( (E, \theta, g, \alpha) \) which define the free energy, stress, entropy, and heat flux at each particle \( X \in B \) and at each time \( t \),

\[ \phi = \Phi(E, \theta, g, \alpha), \quad \mathbf{\sigma} = \Sigma(E, \theta, g, \alpha), \quad \eta = N(E, \theta, g, \alpha), \quad q = Q(E, \theta, g, \alpha) \]  

(37)

and

(ii) there exists a potential

\[ \psi : W \to (-\infty, \infty), \]  

(38)

which has a non-empty generalized subdifferential and which is such that

\[ (\mathbf{P}, -\dot{\alpha}) \in \partial \psi(\mathbf{\sigma}, \mathfrak{A}) \quad \forall (\mathbf{\sigma}, \mathfrak{A}) \in W. \]  

(39)

Here the following notations are used:

\( E \) is the deformation (strain) tensor defined in (29),

\( g = \text{grad} \theta \),

\( \alpha \) = an internal state variable introduced to model possible changes in the microstructure of the material or to depict overall history effects through an evolution equation defined by (39),

\( \mathfrak{A} \) = a thermodynamic force conjugate to \( \dot{\alpha} \).

The values of \( \mathbf{\sigma} \) and \( \mathfrak{A} \) at a particle \( X \) at time \( t \) belong to sets \( S \) and \( A \), respectively.

\[ S \subset \mathbb{R}^6, \quad A \subset \mathbb{R}^6 \]  

(40)

and we denote the stress-thermodynamic force pairs as elements of a set \( W \).

\[ W = S \times A. \]  

(41)

The notion of a generalized subdifferential of a non-convex, non-differentiable potential is
due to Clarke [12] and Rocke1ellar [43] and is discussed in some detail in Kim and Oden [20].

The generalized subdifferential can be reduced to the subdifferential in the case of convex functional and to the usual gradient in the case of a differentiable functional, i.e.

\[(\dot{\mathbf{r}} - \dot{\mathbf{a}}) \in \partial \psi(\mathbf{r}, \mathbf{a})\],

\[(\dot{\mathbf{r}} - \dot{\mathbf{a}}) \in \partial \psi(\mathbf{r} - \mathbf{a}, \mathbf{A})\], when \(\psi\) is convex.

\[(\dot{\mathbf{r}} - \dot{\mathbf{a}}) = \frac{\partial \psi(\mathbf{r} - \mathbf{a}, \mathbf{A})}{\partial (\mathbf{r} - \mathbf{a}, \mathbf{A})}\], when \(\psi\) is differentiable. (42)

We regard (39) as a definition of \(\dot{\mathbf{r}}\) for a given potential \(\psi\): a kinematical interpretation of \(\dot{\mathbf{r}}\) might be obtained from the relations in the preceding section.

Eliminating \(r\) between (34) and (35) and taking into account (36), we obtain

\[\mathbf{r} : \mathbf{L} - \rho \mathbf{\phi} - \rho \dot{\mathbf{\eta}} - \frac{1}{\theta} \mathbf{q} \cdot \nabla \theta \geq 0.\] (43)

Assume that the map: \((\mathbf{E}, \theta, \mathbf{g}, \mathbf{a}) \rightarrow \mathbf{\phi}(\cdot, \cdot, \cdot, \cdot)\) is \(C^1\) in each argument. Then the rate of change of the free energy is given by

\[\dot{\mathbf{\phi}} = \frac{\partial \mathbf{\phi}}{\partial \mathbf{E}} : \dot{\mathbf{E}} + \frac{\partial \mathbf{\phi}}{\partial \theta} \dot{\theta} + \frac{\partial \mathbf{\phi}}{\partial \mathbf{g}} \cdot \dot{\mathbf{g}} + \frac{\partial \mathbf{\phi}}{\partial \mathbf{a}} : \dot{\mathbf{a}}.\] (44)

Here \(\mathbf{\phi}\) denotes the contraction of two tensors, e.g. \(\mathbf{A} : \mathbf{B} = \text{tr}(\mathbf{A}^T \mathbf{B})\).

Introducing (28) and (44) into (43) gives

\[(\mathbf{S} - \rho \frac{\partial \mathbf{\phi}}{\partial \mathbf{E}}) : \dot{\mathbf{E}} + \mathbf{S} : \dot{\mathbf{r}} - \rho \left(\frac{\partial \mathbf{\phi}}{\partial \theta} + \mathbf{\eta}\right) \dot{\theta} - \rho \frac{\partial \mathbf{\phi}}{\partial \mathbf{g}} \cdot \dot{\mathbf{g}} - \rho \frac{\partial \mathbf{\phi}}{\partial \mathbf{a}} : \dot{\mathbf{a}} - \frac{1}{\theta} \mathbf{q} \cdot \nabla \theta \geq 0.\] (45)

Here the stress measure \(\mathbf{S}\) has the following relation with the Cauchy stress

\[\mathbf{S} = \mathbf{R}' \mathbf{\sigma} \mathbf{R}'.\] (46)

We shall refer to \(\mathbf{S}\) as the Dienes stress since the quantity \(\mathbf{R}' \mathbf{\sigma} \mathbf{R}\) appeared in the work of Dienes [13] in an attempt to provide another definition of an objective stress rate. Kinematically, Dienes stress \(\mathbf{S}\) has a special meaning. Since \(\mathbf{S}\) is defined as a reverse rotation of the Cauchy stress \(\mathbf{\sigma}\), it is the stress in the configuration \(\mathbf{C}_0\) in Fig. 2. The configuration \(\mathbf{C}_0\) is designed to be rotation-free. Therefore, variables \(\mathbf{U}, \mathbf{S}\), etc., referred to \(\mathbf{C}_0\) are rotation-free. Discussions of difficulties surrounding the use of appropriate objective stress rates were contributed by e.g., Dienes [13], Nagtegaal and DeJong [30], Atluri [6, 7] and Lee and Wertheimer [26] among others.

From (46), we have

\[\dot{\mathbf{S}} = \dot{\mathbf{R}}' \mathbf{\sigma} \mathbf{R} + \mathbf{R}' \dot{\mathbf{\sigma}} \mathbf{R} + \mathbf{R}' \mathbf{\sigma} \dot{\mathbf{R}} = \mathbf{R}'(\dot{\mathbf{\sigma}} - \omega \mathbf{\sigma} + \mathbf{\omega} \mathbf{\sigma}) \mathbf{R}.\] (47)
where
\[ \mathbf{\omega} = \mathbf{\dot{R}R^i} = \mathbf{R\dot{R}^i}. \]  

Then an objective stress rate can be defined as
\[ \mathbf{\sigma = R\dot{S}R^i} = \mathbf{\dot{\sigma}} - \mathbf{\omega \sigma} + \mathbf{\sigma \omega}. \]  

Notice that \( \mathbf{\sigma} \) is a stress rate in the configuration \( C_i \) and \( \mathbf{S} \) is a stress rate in \( C_C \).

We note that the set of constitutive variables characterizing materials of type N do not include \( \dot{\mathbf{\theta}} \) and \( \dot{\mathbf{g}} \). For this reason, the variables \( \dot{\theta}, \dot{g} \) can be changed without changing other arguments. This fact leads to the following equalities:
\[ S = \rho \frac{\partial \phi}{\partial E} \]  
(50)
\[ \eta = - \frac{\partial \phi}{\partial \theta} \]  
(51)
\[ \frac{\partial \phi}{\partial g} = 0 \]  
(52)

and the ensuing inequality
\[ S : \dot{\mathbf{P}} + \mathbf{\mathcal{A}} : \mathbf{\alpha} - (1/\theta)q \cdot \nabla \theta \geq 0. \]  
(53)

where \( \mathbf{\mathcal{A}} = - \frac{\partial \phi}{\partial \alpha} \).

It remains to be shown how this theory can be applied to concrete cases. We consider specific examples in the next section.

3.2. An example: a modified Bodner and Partom theory

Bodner and Partom [9, 10] proposed a flow rule which was drawn from the classical Prandtl-Reuss flow rule and improved by phenomenological observations, but which does not involve specification of a yield surface. This theory attempts to characterize isotropic hardening of certain materials.

Since non-isotropic hardening may not be too significant in some large-deformation elastoplasticity problems such as those encountered in metal-forming, a modification of this theory so that it fits into the setting of our theory may be worthwhile.

First, we introduce a pair of potentials:

Free-energy functional
\[ \phi = \frac{1}{2} \left[ \frac{\lambda}{\rho_0} (\text{tr} \mathbf{E})^2 + \frac{2\mu}{\rho_0} (\text{tr} \mathbf{E}^2) \right] - h_1 A - \frac{1}{m} (h_1 - h_0) \exp(-mA). \]  
(54)

Flow potential functional
\[ \psi = D_0 \sum_{i=0}^{\infty} (-1)^{i+1} \frac{B^i}{i!(2ni-1)} \frac{h_1^{2ni-1}}{J_2^{ni-1/2}}. \]  
(55)
Here
\[ J_2 = \frac{1}{2} \text{tr}(S'^2), \quad S' = S - \frac{1}{2} \text{tr}(S), \]
\[ B = \frac{1}{3^n} \left( \frac{n+1}{n} \right). \]

and \( \Lambda \) is an internal state variable (since these potentials will characterize an isotropic-hardening effect, here the internal state variable is scalar-valued). \( h \) is a hardness variable which is conjugate to the internal variable. \( \lambda \) and \( \mu \) are Lamé constants and \( D_0, h_0, h_1, m \) and \( n \) are material constants.

Applying the axiom (39) and results (50)–(53), we obtain the following constitutive relations:

\[ S = \rho \frac{\partial \phi}{\partial E} = \frac{\rho \lambda}{\rho_0} (\text{tr} \ E) I + \frac{\rho}{\rho_0} 2\mu E. \]  
(56)

\[ h = -\rho \frac{\partial \phi}{\partial \Lambda} = h_1 + (h_0 - h_1) \exp(-m\Lambda). \]  
(57)

\[ \dot{\mathbf{P}} = \frac{\partial \psi}{\partial S} = D_0 \frac{1}{h \sqrt{J_2}} \exp\left(-\frac{Bh^{2n}}{J_2^2}\right) S'. \]  
(58)

\[ \dot{\mathbf{A}} = -\frac{\partial \psi}{\partial h} = \frac{1}{h} S' : \mathbf{P}. \]  
(59)

**REMARKS 3.1.** Several remarks are in order:

(i) The exponential term in (58) becomes very sensitive to the magnitude of the argument \( h^2/J_2 \) as \( n \to \infty \). Indeed, these equations can almost model non-work-hardening materials (i.e., elastic-perfectly plastic materials) by choosing very large \( n \) and \( D_0 \).

(ii) Let us check whether or not the constitutive equations (56)–(59) satisfy the principle of material frame indifference, i.e.

\[ T^* = T(C^*). \]  
(60)

where \( T \) is a response function and \( * \) denotes value at the rotated frame. Set

\[ \dot{\mathbf{E}} = \dot{\mathbf{U}}^* \mathbf{U}^{-1} |_{\text{sym}}, \quad \dot{\mathbf{P}} = \dot{\mathbf{U}}^p \mathbf{U}^{-1} |_{\text{sym}}. \]

\[ \alpha = \Lambda 1, \quad \mathbf{A} = h 1 \quad \text{and} \quad g = 0. \]

Since a rotation of either a stretch tensor or the identity tensor leaves both invariant and the stress \( S \) is rotation-free, the relations (56)–(59) are completely invariant under rotation. In other words, the requirement (60) is satisfied since \( T^* = T \) and \( C^* = C \) for all four equations.

(iii) From thermodynamic restrictions, we obtained the inequality (53). We must verify that the given constitutive relations satisfy this inequality. Direct substitution gives
\[ S : \dot{P} = S : \left[ D_0 \frac{1}{h \sqrt{J_2}} \exp\left( -\frac{Bh^{2n}}{J_2^2} \right)(S - \frac{1}{3} \text{tr}(S)1) \right] \]

and

\[ -\frac{\partial \phi}{\partial \alpha} : \dot{\alpha} = -\frac{\partial \phi}{\partial \lambda} : \dot{\Lambda} \]

\[ = H \cdot \Lambda = S' : \dot{P} = \text{[positive]} S' : S' \geq 0. \]

Thus, \((1/\theta)q \cdot \nabla \theta \leq 0\), as generally required.

The following is a summary of the principal equations governing the behavior of a body composed of a material characterized by (54) and (55).

**Kinematics**

\[ F = \frac{\partial x}{\partial X}, \quad F = RU, \quad U = U^e + U^p - 1. \]

\[ E = \int_0^t \dot{U}^e U^{-1}_\text{sym} dt, \quad P = \int_0^t \dot{U}^p U^{-1}_\text{sym} dt. \] (61)

**Thermomechanical laws**

\[ \rho \text{ det } F = \rho_0, \]
\[ \text{div } \sigma + \rho b = \rho \ddot{x} \quad (\sigma = \sigma^i). \]
\[ \rho \ddot{\varepsilon} = \sigma : L - \text{div } q + \rho \dot{r} \quad (L = \partial \ddot{x}/\partial X). \]
\[ -\frac{1}{\theta} q \cdot \nabla \theta \geq 0. \]
\[ \phi = \varepsilon - \eta \theta. \]

** Constitutive relations**

\[ S = \frac{\rho \lambda}{\rho_0} (\text{tr } E)1 + \frac{\rho}{\rho_0} 2\mu E \quad (S = R^i \sigma R). \]
\[ h = h_1 + (h_0 - h_1) \exp(-m \Lambda). \]
\[ \dot{P} = \frac{D_0}{h \sqrt{J_2}} \exp\left( -\frac{Bh^{2n}}{J_2^2} \right) S'. \]
\[ \dot{\Lambda} = S' : \dot{P}/h. \quad q = Q(E, \theta, g, \alpha). \]
\[ \eta = N(E, \theta, \alpha) \quad \text{and} \quad (54). \] (63)
It is important to check to determinacy of the system of equations (61)–(63). We, therefore, construct Table 1.

(i) In the equations (61), if \( x \) and either one of \( U^e \) or \( U^p \) are known, the remaining variables are determined. So 3 unknowns for \( x \) and 6 for \( U^e \) make a total of 9 more unknowns than equations.

(ii) In the equations (62), 14 new unknowns (\( \rho, \sigma, \varepsilon, q, \phi, \theta, \eta \)) and 6 new equations give 8 more unknowns. Thus there is a total of 17 more unknowns.

(iii) In the equations (63), 8 new unknowns (\( s, h, A \)) and 25 equations are introduced and 17 more equations are obtained. These provide a total number of equations equal to the total number of undetermined variables.

In subsequent applications, we limit ourselves to quasi-static problems involving no body forces and no temperature dependence. That means we do not solve the energy equation directly and that the equation of momentum balance in the reference configuration will be of the form

\[ \nabla_X \cdot \mathbf{T} = 0 \text{ in } C_0, \quad (64) \]

where \( \mathbf{T} \) is the first Piola–Kirchhoff stress and \( \nabla_X = \partial/\partial X \).

4. Finite element approximation

Early accounts of finite element models of elastoplasticity were reported by Oden and Kubitza [42], Marcal and King [27], Argyris [1], and Zienkiewicz et al. [48], and finite

<table>
<thead>
<tr>
<th>Number of unknowns</th>
<th>Number of equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F )</td>
<td>9</td>
</tr>
<tr>
<td>( R )</td>
<td>3</td>
</tr>
<tr>
<td>( U )</td>
<td>6</td>
</tr>
<tr>
<td>( U^e )</td>
<td>6</td>
</tr>
<tr>
<td>( U^p )</td>
<td>6</td>
</tr>
<tr>
<td>( E )</td>
<td>6</td>
</tr>
<tr>
<td>( P )</td>
<td>6</td>
</tr>
<tr>
<td>( x )</td>
<td>3</td>
</tr>
<tr>
<td>( \rho )</td>
<td>1</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>1</td>
</tr>
<tr>
<td>( \varepsilon )</td>
<td>6</td>
</tr>
<tr>
<td>( q )</td>
<td>3</td>
</tr>
<tr>
<td>( S )</td>
<td>6</td>
</tr>
<tr>
<td>( h )</td>
<td>1</td>
</tr>
<tr>
<td>( \Lambda )</td>
<td>1</td>
</tr>
<tr>
<td>( \theta )</td>
<td>1</td>
</tr>
<tr>
<td>( \eta )</td>
<td>1</td>
</tr>
<tr>
<td>( \phi )</td>
<td>1</td>
</tr>
</tbody>
</table>

| 67 | 67 |
deformations were treated by Oden [36–38] and others. A variety of different formulations of
the large-deformation problem have been explored, and we mention the incremental Lagrangian
formulation of Hibbit, Marcal and Rice [15], the updated Lagrangian scheme of, e.g.,
McMeeking and Rice [28], and various related schemes proposed by Needleman [32], Larsen
and Popov [22], Bathe, Ramm and Wilson [8], Argyris et al. [4, 5], Cesscotto, Frey and Fonder
problems can be found in the literature and we mention as examples those in the papers of
Lee, Mallet and Yang [25], Nagtegaal and DeJong [30], Key, Krieg and Bathe [16], Argyris
and Doltsinis [2, 3], Taylor and Becker [46], and Kikuchi and Cheng [18]. A good survey of
current theories and numerical methods for finite-deformation plasticity can be found in the
volume edited by Nemet-Nasser [35] and in the proceedings edited by Willam [47]; see also
the recent work of Simo and Marsden [45]. Simo and Ortiz [44] and the references therein.

4.1. Formulations

Here we adopt to use the incremental total Lagrangian formulation where the reference
configuration is always the initial configuration. Let \( q \) denote a test function which belongs to
a set of \( V \) admissible displacement increments. Then a weak form of equation (64) can be
written

\[
\int_{\Omega_0} \mathbf{T} : \nabla q \, dx = \int_{\partial\Omega_0} \mathbf{T} \cdot \mathbf{n} \cdot q \, dS \quad \forall \, q \in V. \quad (65)
\]

where \( \mathbf{n} \) is the outward normal unit vector in the reference configuration.

We assume that the equilibrium state is achieved at \((n-1)\)th incremental step, i.e.,

\[
\int_{\Omega_0} \mathbf{T}_{n-1} : \nabla q \, dx = \int_{\partial\Omega_0} \mathbf{T}_{n-1} \cdot \mathbf{n} \cdot q \, dS \quad \forall \, q \in V. \quad (66)
\]

Then at the \(n\)th step,

\[
\mathbf{T}_n = \sum_{i=1}^{n} \Delta \mathbf{T}_i = \mathbf{T}_{n-1} + \Delta \mathbf{T}_n,
\]

\[
t_n = T_n \mathbf{n} = t_{n-1} + \Delta t_n
\]

and

\[
\int_{\Omega_0} \Delta \mathbf{T}_n : \nabla q \, dx = \int_{\partial\Omega_0} \Delta t_n \cdot \mathbf{n} \cdot q \, dS \quad \forall \, q \in V. \quad (67)
\]

From the definition of the first Piola–Kirchhoff stress and equations (28) and (46),

\[
\Delta \mathbf{T} = \Delta (JRSU^{-1}) = \Delta (\lambda (\text{tr} \, \mathbf{E}) F^{-1} + 2\mu \mathbf{R} \mathbf{E}^{-1})
\]

\[
= \lambda (\text{tr} \, \Delta \mathbf{E}) F^{-1} + 2\mu \mathbf{R} \Delta \mathbf{E} \mathbf{U}^{-1} + \lambda (\text{tr} \, \mathbf{E}) \Delta F^{-1} + 2\mu \Delta \mathbf{R} \mathbf{E} \mathbf{U}^{-1} + 2\mu \mathbf{R} \Delta \mathbf{U}^{-1}. \quad (68)
\]
By applying the decomposition relation (28) and the form (68), equation (67) becomes

\[
\int_{\Omega_0} \left[ \lambda \text{tr}(\Delta t D_n)F_n^{-1} + 2\mu \Delta t D_n F_n^{-1} \right] : \nabla q \, dx \\
- \int_{\Omega_0} \left[ \lambda (\text{tr } \Delta P) F_n^{-1} + 2\mu (R \Delta P U^{-1})_n \right] : \nabla q \, dx \\
+ \int_{\Omega_0} \left[ \lambda (\text{tr } E) \Delta F_n^{-1} + 2\mu (\Delta REU^{-1} + RE \Delta U^{-1}) \right]_n : \nabla q \, dx \\
= \int_{\partial \Omega_0} \Delta t_n : q \, ds \quad \forall \, q \in \mathcal{V}.
\]

(69)

Now we apply the tangent stiffness scheme with the idea of successive approximation during iterations in an incremental step. We arrive at the following linearized form at the \(i\)th iteration in the \(n\)th increment:

\[
\int_{\Omega_0} \left[ \lambda \text{tr}(\Delta t D_n^i) + 2\mu \Delta t D_n^i (F_n^{-1})_{i-1} \right] : \nabla q \, dx = \\
\int_{\Omega_0} \left[ \lambda (\text{tr } \Delta P) F_n^{-1} + 2\mu (R \Delta P U^{-1})_{i-1} \right] : \nabla q \, dx \\
- \int_{\Omega_0} \left[ \lambda (\text{tr } E) \Delta F_n^{-1} + 2\mu (\Delta REU^{-1} + RE \Delta U^{-1}) \right]_{i-1} : \nabla q \, dx \\
+ \int_{\partial \Omega_0} \Delta t_n^i : q \, ds - \int_{\partial \Omega_0} \Delta T_{i-1}^n : \nabla q \, ds \quad \forall \, q \in \mathcal{V}.
\]

(70)

Here the last integrals on the right-hand side of (70) is the left-hand side of (69) at the \(i\)th iteration and the first two integrals of (70) are obtained by taking \(\Delta P, \Delta R\) and \(\Delta U^{-1}\) as functions of \(U\) and transferring these to the right-hand side. i.e.,

\[
P_{n-1}^i = \frac{\partial \Delta P}{\partial \Delta U} \bigg|_{n}^{i-1} \Delta U_{n}^i, \quad \Delta R_{n-1}^i = \frac{\partial \Delta R}{\partial \Delta U} \bigg|_{n}^{i-1} \Delta U_{n}^i, \quad \text{etc.}
\]

Note that \(\Delta P_{n-1}^i, \text{ etc.}\), go to zero when the iteration converges since \(P_n, R_n, \text{ etc.}\) are defined in the following fashion:

\[
P = P_{n-1}^i + \sum_{i=1}^{M} \Delta P_n^i, \quad \text{etc.}
\]
Adding (66) to (70) gives

\[
\int_{\Omega_0} \left[ \lambda \text{tr}(\Delta t D_n^i) + 2\mu \Delta t D_n^i \right] (F^{-1})_{n-1} : \nabla q \, dx = \\
= \int_{\Omega_0} \left[ \lambda (\text{tr}(P)) F^{-1} + 2\mu (R \Delta P U^{-1}) \right]_{n-1} : \nabla q \, dx \\
- \int_{\Omega_0} \left[ \lambda (\text{tr}(E)) \Delta F^{-1} + 2\mu (\Delta R E U^{-1} + RE \Delta U^{-1}) \right]_{n-1} : \nabla q \, dx \\
+ \int_{\partial \Omega_0} t_n^i : q \, ds - \int_{\Omega_0} T_{n-1}^i : \nabla q \, dx \quad \forall \ q \in V.
\]

(71)

4.2. Finite element approximation

First we interpolate the functions \( \Delta u_n^i \) and \( q \) as follows:

\[
\Delta u_n^i = u_n \phi_n(x), \quad u_n = \begin{bmatrix} u_n^1 \\ u_n^2 \end{bmatrix}, \quad q = q_n \phi_n(x), \quad q_n = \begin{bmatrix} q_n^1 \\ q_n^2 \end{bmatrix}.
\]

(72)

where \( \phi_n(x) \) is a shape function, we have used the summation convention, and \( u_n = u_n - \sum_i^M \Delta u_n^i \), \( M \) being the iteration index. Then

\[
\Delta t D_n^i = \frac{1}{2} \left[ \frac{\partial \Delta u_n^i}{\partial x} (F^{-1})_{n-1} + (F^{-1})_{n-1} \left( \frac{\partial \Delta u_n^i}{\partial x} \right)^{-1} \right] \\
= \frac{1}{2} \left[ (F_1 \phi_{\beta,1} + F_2 \phi_{\beta,2}) u_{\beta}^1 + (F_3 \phi_{\beta,1} + F_4 \phi_{\beta,2}) u_{\beta}^2 \right].
\]

(73)

where

\[
[F^{-1}]_{n-1} = \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix} \quad \text{and} \quad \nabla \phi_{\beta} = \begin{bmatrix} \phi_{\beta,1} \\ \phi_{\beta,2} \end{bmatrix}.
\]

(74)

Setting

\[
F_1 \phi_{\beta,1} + F_2 \phi_{\beta,2} = \beta A, \quad F_3 \phi_{\beta,1} + F_4 \phi_{\beta,2} = \beta B
\]
gives

\[
\Delta t D_n^i = \begin{bmatrix} \beta Au_{\beta}^1 & \frac{1}{2} (\beta Bu_{\beta}^1 + \beta Au_{\beta}^2) \\ \frac{1}{2} (\beta Bu_{\beta}^1 + \beta Au_{\beta}^2) & \beta Bu_{\beta}^2 \end{bmatrix}.
\]

(75)

To reduce bookkeeping, we note that the terms \( \Delta F_{n-1}^{i-1} \) and \( (\Delta U^{-1})_{n-1}^{i-1} \) should tend to zero when convergence in each iteration procedure is achieved. This would allow us to delete the
second integral on the right-hand side of (71). Then the discretized equations of (71) at each
node in \( N \)th element would assume the form

\[
\{q^1, q^2\} = \int_{\partial \Omega} \left\{ \lambda \mathbf{TRD} \left[ \begin{array}{c} \alpha A \\ \alpha B \end{array} \right] + 2\mu \left[ \begin{array}{c} (F1\phi_{a,1}B + \frac{1}{2}(F3\phi_{a,1}B + F1\phi_{a,2}B))u_0^1 \\ (F2\phi_{a,1}B + \frac{1}{2}(F4\phi_{a,1}B + F2\phi_{a,2}B))u_0^1 \\
\frac{1}{2}(F3\phi_{a,1}B + F1\phi_{a,2}B)u_0^2 \\
\frac{1}{2}(F4\phi_{a,1}B + F2\phi_{a,2}B)u_0^2 \end{array} \right] \right\} \, dx
\]

\[
+ \int_{\partial \Omega} \left\{ (t_1 + \Delta t_1) \right\} dS - \int_{\partial \Omega} \left\{ \begin{array}{c} (t_2 + \Delta t_2) \end{array} \right\} dS \right\} \right\} \forall \left\{ \begin{array}{c} q^1_n \\ q^2_n \end{array} \right\}.
\]

(76)

where \( \Omega^N_0 \) is an \( N \)th element and \( \partial \Omega_0 \) is the side of the element on which the traction is
prescribed. Summations on \( \alpha \) and \( \beta \) are implied. Also,

\[
[R]_{n-1} = \begin{bmatrix} R1 & R3 \\ -R3 & R2 \end{bmatrix}, \quad [\Delta P]_{n-1} = \begin{bmatrix} P1 & P3 \\ P3 & P2 \end{bmatrix}, \quad [U]_{n-1} = \begin{bmatrix} U1 & U3 \\ U3 & U2 \end{bmatrix}.
\]

\[
[C] = \frac{2\mu}{\text{Det } U} \begin{bmatrix} RP1 \cdot U2 - RP2 \cdot U3 & -RP1 \cdot U3 + RP2 \cdot U1 \\ RP3 \cdot U2 - RP4 \cdot U3 & -RP3 \cdot U3 + RP4 \cdot U1 \end{bmatrix}, \quad [\text{RP}] = [R][\Delta P].
\]

\[
[S]_{n-1} = \begin{bmatrix} S1 & S3 \\ S3 & S2 \end{bmatrix}, \quad [\text{RSU}] = \begin{bmatrix} RSU1 & RSU2 \\ RSU3 & RSU4 \end{bmatrix} = [R][S][U^{-1}]
\]

(77)

and

\[
\alpha A = F1\phi_{a,1} + F3\phi_{a,2}, \quad \alpha B = F2\phi_{a,1} + F4\phi_{a,2}.
\]

Again, by setting

\[
\alpha C = F3\phi_{a,1} + F1\phi_{a,2}, \quad \alpha D = F4\phi_{a,1} + F2\phi_{a,2}.
\]

equation (76) can be reduced to the following set of simultaneous equations for each element:

\[
\sum_{\beta_1, \beta_2, \ldots, \beta_{NN}} \alpha \beta \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} u_0^1 \\ u_0^2 \end{bmatrix} = \begin{bmatrix} F_1^a \\ F_2^a \end{bmatrix}
\]

for every \( \alpha = \alpha_1, \alpha_2, \ldots, \alpha_{NN} \).

(78)

Here the \( \alpha_i \) and \( \beta_i \) are global node numbers in an element and \( NN \) is a number of nodes in
each element.

In global form,

\[
\sum_{k=1}^N \sum_{\beta_1, \beta_2, \ldots, \beta_{NN}} \alpha \beta \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} u_0^1 \\ u_0^2 \end{bmatrix} = \begin{bmatrix} F_1^a \\ F_2^a \end{bmatrix}
\]

for every \( \alpha \).

(79)
where \( N \) is a number of elements, \( k \) is the element number and

\[
\begin{align*}
\alpha^B K_{11} &= \int_{\Omega_N^k} \{ \lambda \cdot \alpha A \cdot \beta A + 2\mu \cdot F_1 \cdot \phi_{\alpha,1} \beta A + \mu \cdot \alpha C \cdot \beta B \} \, dx, \\
\alpha^B K_{12} &= \int_{\Omega_N^k} \{ \lambda \cdot \alpha A \cdot \beta B + 2\mu \cdot F_3 \cdot \phi_{\alpha,2} \beta B + \mu \cdot \alpha C \cdot \beta A \} \, dx, \\
\alpha^B K_{21} &= \int_{\Omega_N^k} \{ \lambda \cdot \alpha B \cdot \beta A + 2\mu \cdot F_2 \cdot \phi_{\alpha,1} \beta A + \mu \cdot \alpha D \cdot \beta B \} \, dx, \\
\alpha^B K_{22} &= \int_{\Omega_N^k} \{ \lambda \cdot \alpha B \cdot \beta B + 2\mu \cdot F_4 \cdot \phi_{\alpha,2} \beta B + \mu \cdot \alpha D \cdot \beta A \} \, dx.
\end{align*}
\]

\[
\begin{align*}
F^1_\alpha &= \int_{\Omega_N^k} \{ (\lambda (P_1 + P_2) F_1 + C_{11} - RSU1) \phi_{\alpha,1} + (\lambda (P_1 + P_2) F_3 + C_{21} - RSU3) \phi_{\alpha,2} \} \, dx \\
&\quad + \int_{\partial \Omega_0} t^1_\alpha \phi_\alpha \, ds, \\
F^2_\alpha &= \int_{\Omega_N^k} \{ (\lambda (P_1 + P_2) F_2 + C_{12} - RSU2) \phi_{\alpha,1} + (\lambda (P_1 + P_2) F_4 + C_{22} - RSU4) \phi_{\alpha,2} \} \, dx \\
&\quad + \int_{\partial \Omega_0} t^2_\alpha \phi_\alpha \, ds.
\end{align*}
\]

**REMARK 4.1.**

(i) In the first iteration at each incremental step, we impose the incremental essential boundary conditions; but after the first iteration, we must impose zero values.

(ii) It is interesting to note that (79) reduces to the equation for stiffnesses in linear infinitesimal elasticity upon appropriate specialization. In the case of infinitesimal deformation,

\[
F = 1, \quad R = 1, \quad U^n = 1.
\]

Then

\[
\begin{align*}
\alpha A &= \phi_{\alpha,1}, \quad \alpha B = \phi_{\alpha,2}, \quad \alpha C = \phi_{\alpha,2}, \quad \alpha D = \phi_{\alpha,1}, \\
\beta A &= \phi_{\beta,1}, \quad \beta B = \phi_{\beta,2}.
\end{align*}
\]

Therefore, as expected.

\[
\begin{align*}
\alpha^B K_{11} &= \int_{\Omega_N^k} [(\lambda + 2\mu) \phi_{\alpha,1} \phi_{\beta,1} + \mu \phi_{\alpha,2} \phi_{\beta,2}] \, dx, \\
\alpha^B K_{12} &= \int_{\Omega_N^k} [\lambda \phi_{\alpha,1} \phi_{\beta,1} + \mu \phi_{\alpha,1} \phi_{\beta,2}] \, dx, \\
\alpha^B K_{21} &= \int_{\Omega_N^k} [\lambda \phi_{\alpha,2} \phi_{\beta,1} + \mu \phi_{\alpha,1} \phi_{\beta,2}] \, dx, \\
\alpha^B K_{22} &= \int_{\Omega_N^k} [(\lambda + 2\beta) \phi_{\alpha,2} \phi_{\beta,2} + \mu \phi_{\alpha,1} \phi_{\beta,1}] \, dx.
\end{align*}
\]
Note that the assembled matrix then becomes symmetric.

In each iteration, the following constitutive routine has to be solved:

\[ h_n^t = h_t + (h_0 - h_t)\exp(-mA_n^{-1}) \]
\[ \Delta P_n^t = \Delta t_n \hat{P}(h_n^t, \sigma_n^t) \]
\[ \Delta E_n^t = \Delta U U^{-1}|_{\text{sym}}^t - \Delta P_n^t \]
\[ S_n^t = \frac{1}{J_n} [\lambda (\text{tr} \Delta E_n^t) I + 2\mu \Delta E_n^t], \quad \Delta A_n^t = \frac{1}{h_n^t} S_n^t : \Delta P_n^t. \]

In the actual computation, we subdivide the incremented solution \( \Delta U U^{-1}|_{\text{sym}} \) by a prescribed number and proceed by using the previous forward Euler method.

Since the previous constitutive equations assume incompressibility in the plastic deformation (see Nagtegaal, Parks and Rice [31] and the detailed stability analysis of Oden et al. [39-40]), here the rectangular element which consists of four 3-node triangles: so-called, four constant strain triangles (4CST element) is used. A mathematical analysis of this 4CST element can be found in Oden, Kikuchi and Song [41] and Kikuchi [17].

4.3. Numerical examples

In this section several example problems are solved to verify the algorithm described above.

The proposed set of constitutive equations (equations (56), (57), (58) and (59)) contains 7 constants to be determined from experimental data. The two elastic constants can be determined by standard procedures, but the plastic constants require at least two uni-axial tests at different strain rates. A detailed discussion of how to determine these constants from test data can be found in the thesis of Kim [19].

Following Bodner and Partom, we consider estimated constants on titanium for our constitutive equations, which are similar to theirs. The material parameters are as follows:

(i) Elasticity constants

\[ \lambda = 93667 \text{ N/mm}^2 \quad (1 \text{ N/mm}^2 = 10^6 \text{ Pa}). \]
\[ \mu = 44000 \text{ N/mm}^2. \]

(ii) Plasticity constants

\[ n = 1, \quad m = 50, \]
\[ D_0 = 1.35 \times 10^7 \text{ sec}^{-1}. \]
\[ h_0 = 1150 \text{ N/mm}^2, \quad h_1 = 1450 \text{ N/mm}^2. \]

As a first computational example the homogeneous plane-strain elongation of a test block is computed. Results are shown in Fig. 3. The computed results reflect up to 20 percent engineering strain and show strong sensitivity to strain rates. Note that slower loading results in lower yield stress.
A result of loading, unloading and reloading is shown in Fig. 4. with a strain rate of $1.5 \times 10^{-3}$ per second. The so-called ratchet effect at yield point in the reloading process is not apparent because of a large amount of total deformation.

In the third example, the variation of the internal state variable with strain is shown in Fig. 5. In this case, the internal state variable represents hardness of the material. It is seen to vary with strain in a way which is qualitatively the same as the stress-strain relation.

Before solving a complicated plane-strain problem, we next check the algorithm's ability to simulate rotational rigid-body motions. This is done by fixing a corner of the stressed element and prescribing the essential boundary conditions at each incremental step as in [46]. Suppose that a block element, such as that in Fig. 6, is subjected to an elastoplastic deformation according to the following program: We prescribe 0.5 percent engineering strain with a $0.5 \times 10^{-3}$ per second rate. Next the block is rotated with prescribed incremental rotation angles while maintaining the preloaded deformation. The results are listed in Table 2 with increments 9°, 10°, 15°, 30°, and 45° to make 90° rotation. If the computed stress rates are appropriately objective, the stress should not change during these rotations. We observe errors in the stress of only about 0.3 percent and note that there is little in error between a large step size (45°) and a small size (9°), as expected from the way we defined deformation measures and rotationally-invariant stresses.

The final example is a head-forming problem in plane strain. A 4 by 5 unit rectangular billet, which is confined at the lower boundary is loaded on the upper part without friction. Incremental displacements are prescribed at the five nodes at the top of the billet. Computed, progressive deformed shapes, and $J_2$-stress contours are shown in Figs. 8 and 9. Figs. 7 and 10
Fig. 4. Stress-strain with loading-unloading-reloading at a strain rate $= 1.5 \times 10^{-3}$.

Fig. 5. Variation of the internal state variable vs. strain.
Fig. 6. Calculated rigid-body motion with 30° increments. (a) loading before rotation. (b) progressive configurations.

Table 2
Dienes stress versus incremental rotation angle

<table>
<thead>
<tr>
<th>Increment</th>
<th>$S_x$</th>
<th>$S_y$</th>
<th>$S_{xy}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>9°</td>
<td>323.4</td>
<td>0.6</td>
<td>0.2</td>
</tr>
<tr>
<td>10°</td>
<td>323.3</td>
<td>0.9</td>
<td>-0.1</td>
</tr>
<tr>
<td>15°</td>
<td>323.6</td>
<td>1.9</td>
<td>0.1</td>
</tr>
<tr>
<td>30°</td>
<td>323.7</td>
<td>0.4</td>
<td>0.1</td>
</tr>
<tr>
<td>45°</td>
<td>323.5</td>
<td>0.1</td>
<td>-0.2</td>
</tr>
</tbody>
</table>
Fig. 7. Finite element model of a 4 x 5 billet.

Fig. 8. Deformed shapes at progressive stages of head-forming.
show the undeformed and the deformed Lagrangian finite element mesh. The residual $J_2$-stress contour is shown in the left-hand part of the deformed configuration.

Throughout the finite element computations, the convergence at each incremental step was checked by calculating the maximum relative error of successive incremented displacements. The relative error is computed as the ratio of the correction between iterations to the first solutions (incremental displacements) of the incremental step. A range of tolerances was set as 0.01–1 percent, depending on step size. Generally, the convergence was achieved in two iterations except when severe changes in the deformation from elastic to plastic states are experienced.
References


302  

S.J. Kim, J.T. Oden. Finite elastoplastic deformations of type-N materials


