ON THE PRINCIPLE OF STATIONARY 
COMPLEMENTARY ENERGY IN FINITE 
ELASTOSTATICS†

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Abstract—In this work, we explore conditions sufficient to guarantee the existence of a principle 
of stationary complementary energy in finite elastostatics. We are then able to derive a principle 
of complementary energy. The results also apply to a large class of incompressible and almost 
incompressible materials.

1. INTRODUCTION

We shall first indicate some existence results for solutions to equilibrium problems in 
finite elastostatics as minimizers of a total potential energy functional. In order to calculate 
the pressure term associated with the incompressibility constraint we shall introduce a 
penalty method which has also the advantage of representing the case of almost 
incompressible materials. After stating the respective existence results associated with the 
penalty method we shall indicate how these results can be used in order to establish a 
principle of the type of stationary complementary energy for “almost” incompressible 
materials. Finally, we shall indicate how some incompressible cases may be recovered from 
the almost incompressible cases.

2. EQUILIBRIUM SOLUTIONS IN FINITE ELASTOSTATICS AS MINIMIZERS OF A 
TOTAL POTENTIAL ENERGY FUNCTIONAL

The problems of interest here are equilibrium problems in which equilibrium states are 
characterized as minimizers of a potential energy functional. Specifically, let \( \Omega \subseteq \mathbb{R}^n \) be an open bounded domain in \( \mathbb{R}^n \) with a Lipschitzian boundary \( \partial \Omega = \partial \Omega_1 \cup \partial \Omega_2 \) with \( \partial \Omega_1 \cap \partial \Omega_2 = \emptyset \) and with meas \( \partial \Omega_1 > 0 \). Let, for example, \( K \) denote the set:

\[
K = \{ v \in (W^{1,p}(\Omega))^n : \det (I + \nabla v) = 1 \text{ a.e. on } \Omega; \quad v = 0 \text{ a.e. on } \partial \Omega_1 \text{ (trace sense)} \} 
\]  

(2.1)

and consider the minimization problem:

\[
\text{Find } u \in K, \text{ such that: } \\
\pi(u, \Omega) = \pi(v, \Omega), \forall v \in K
\]  

(2.2)

where \( \pi(\cdot, \cdot) \) is the total potential energy functional

\[
\pi(v, \Omega) = \int_\Omega W(x, 1 + \nabla v) \, dx - F(v), \forall v \in (W^{1,p}(\Omega))^n. 
\]  

(2.3)

Here \( I \) is the identity second order tensor. \( W(\cdot, \cdot) \) is the stored energy function per unit 
volume in the reference configuration, characterizing the mechanical response of the 
material, and is a positive function of the Carathéodory type (i.e. for almost all \( x \in \Omega \), 
\( W(x, \cdot) \) is continuous and for all \( P \cdot v \), \( W(x, 1 + P \cdot v) \) is measurable on \( \Omega \)). In (2.3) \( -F(v) \) is the potential energy of the applied external forces and is typically of the form

\[
F(v) = \int_\Omega f \cdot v \, dx + \int_{\partial \Omega_2} t \cdot v \, ds, \forall v \in (W^{1,p}(\Omega))^n 
\]  

(2.4)

†Dedicated to Professor Eric Reissner in recognition of his many contributions to variational theory.
with \( f(\{(W^{1,p}(\Omega))\}^r) \) prescribed on \( \Omega \) representing the body force per unit volume of the reference configuration, and \( t(\{(W^{1,(1+p)(p\partial \Omega)}(\Omega))\}) \) prescribed on \( \partial \Omega \), and representing the applied surface tractions per unit area of the reference configuration, \( (1/p) + (1/p') = 1 \).

The existence of solutions to (2.2) can be guaranteed by the generalized Weierstrass theorem. "If \( \pi : K \rightarrow \mathbb{R} \) is a proper, coercive, weakly lower semicontinuous functional defined on a nonempty sequentially weakly closed subset of a reflexive Banach space, then \( \pi \) is bounded below on \( K \) and attains its minimum on \( K \)." Consequently, there are two main issues concerning the existence of minimizers to (2.2), the sequentially weak closedness of the set of admissible motions \( K \) and the weak lower semicontinuity and coerciveness of the energy functional \( \pi \). The sequential weak closedness of \( K \) follows from the results of Ball ([1], p. 369) and of Reshetnyak ([2], th. 2; [10], th. 4).

In order for \( K \) to have some meaning we must have \( \det (1 + \nabla u) \in L^1(\Omega) \) and it might seem that the restriction on \( p \) is too severe to accommodate certain classes of materials commonly used in nonlinear elastostatics. This is not the case for we can always redefine \( K \) and problem (2.2) in order to account for it (see e.g. LeTallec[3], p. 14).

In order to study the coercivity and the sequential weak lower semicontinuity of \( \pi \) we may again use the analysis of LeTallec[3] or for a more general class of problems use the notion of quasiconvexity and polyconvexity together with the results of Ball[1], from which we conclude that it is thus possible to solve problem (2.2). However, it is desirable not only to characterize the solution to (2.2) as a solution to the equilibrium equation but also to compute the component of the first Piola-Kirchhoff stress tensor associated with the constraint \( \det (1 + \nabla u) = 1 \) a.e. in \( \Omega \). Towards this end and in order to avoid the introduction of additional unknowns and spaces associated with a Lagrange multiplier-type technique we shall use an exterior penalty method.

3. THE PENALTY METHOD

The exterior penalty method for the constrained optimization problem (2.2) involves considering a penalty functional \( P : (A^{1,p}(\Omega))^r \rightarrow \mathbb{R} \) with the following properties:

(i) \( P \) is positive semidefinite on \( K \) in the sense that \( P(v) \geq 0 \) for all \( v \in (A^{1,p}(\Omega))^r \); \( P(v) = 0 \) if \( v \in K \) and \( P(v) > 0 \) if \( v \notin K \).

(ii) \( P \) is sequentially weakly lower semicontinuous.

Where

\[
(A^{1,p}(\Omega))^r = \{ v \in (W^{1,p}(\Omega))^r : v = 0 \ \text{a.e. on} \ \partial \Omega, \ \text{trace sense} \}
\]

we then construct the penalized functional

\[
\pi, : (A^{1,p}(\Omega))^r \rightarrow \mathbb{R}
\]

\[
\pi, (v, \Omega) = \pi(v, \Omega) + \frac{1}{\epsilon} P(v), \forall v \in (A^{1,p}(\Omega))^r
\]

where \( \epsilon \) is a positive real number.

From the above we may conclude that if \( \pi \) verifies the conditions of the generalized Weierstrass theorem so does \( \pi, \). Thus there exists a solution \( u, \in (A^{1,p}(\Omega))^r \) to the minimization problem

\[
\text{Find } u, \in (A^{1,p}(\Omega))^r \text{ such that: } \left\{ \begin{array}{l}
\pi, (u, , \Omega) \leq \pi, (v, \Omega), \forall v \in (A^{1,p}(\Omega))^r
\end{array} \right\}
\]

for every \( \epsilon > 0 \).

The importance of the exterior penalty functional \( \pi, \) can be seen through the following result.
Proposition 3.1

Let $\pi$ and $K$ as defined above satisfy the conditions of the generalized Weierstrass theorem, then there exists at least one solution $u, e(4^1(\Omega))^*$ to the penalized optimization problem (3.3). Moreover, there exists a subsequence of solutions to (3.3) obtained as $\epsilon$ goes to zero which converges weakly in $(4^1(\Omega))^*$ to a solution $u$ of the constrained optimization problem (2.2).

The advantages of considering an exterior penalty formulation to the optimization problem (2.2) reside in the fact that the minimizers of $\pi$ can be sought in the entire space $(4^1(\Omega))^*$ because the constraint set $K$ enters the problem only in the construction of the penalty functional $P$, and also on the fact that minimizers $u, e(4^1(\Omega))^*$ can be chosen so as to approximate an actual minimizer of $\pi$ arbitrarily closely in the weak topology of $(4^1(\Omega))^*$ by taking $\epsilon$ sufficiently small.

Among all the possible penalty functionals we shall use

$$P(v) = \frac{1}{2} \int_{\Omega} [\det (I + P v) - 1]^2 \, dx$$

$$\forall v \in (4^1(\Omega))^*$$

which, as we shall see, also has the advantage of giving an explicit way of calculating the component of the first Piola-Kirchhoff stress tensor associated with the incompressibility constraint $\det (I + P v) = 1$ a.e. in $\Omega$. In the above, exponent $p$ is assumed to be such that $\det (I + P v) \in L^2(\Omega)$.

From its definition it is clear that (3.4) verifies condition (i) of the definition of a penalty term. The sequential weak lower semicontinuity of (3.4) is an immediate consequence of Reshetnyak ([2], th. 2; [10], th. 4) (see also Ball[1], p. 372), and of the following result for a proof of which we refer to Oden and Kikuchi ([4], p. 10).

Theorem 3.2

Let $U$ and $V$ be normed linear spaces and let $H$ be a sequentially weakly closed subset of $U$. Let $F: H \rightarrow \mathbb{R}$ be a functional defined by

$$F = G \circ B$$

$$F(v) = G(B(v)), \forall v \in H$$

where: $B: H \rightarrow V$ is a sequentially weakly continuous map, and $G: H \rightarrow \mathbb{R}$ is a convex Gâteaux differentiable functional defined on a convex set $H \subseteq V$ containing the range of $B$. Then $F$ is sequentially weakly lower semicontinuous on $H$.

If in addition to the condition stated in the generalized Weierstrass theorem, the total potential energy functional to also Gâteaux-differentiable then it is possible to characterize the minimizer $(u, e)$ of (3.3) as a weak solution to the equilibrium equations, that is $\forall \epsilon > 0, u, e$ verifies

$$\int_{\partial \Omega} \frac{\partial W(1 + P u)}{\partial P u} \cdot P v \, dx + \int_{\Omega} s_a \cdot \frac{1}{\epsilon} [\det (1 + P u) - 1] [\text{adj} (1 + P u)]^T \cdot P v \, dx$$

$$= \int_{\Omega} f \cdot v \, dx + \int_{\partial \Omega} t \cdot v \, ds, \forall v \in (4^1(\Omega))^*.$$  

(3.5)

We remark that in the above it is possible to take the limit when $\epsilon$ goes to zero and therefore recover a characterization of a solution to (2.2) if we have a condition of the Ladyshenskaya, Babuska and Brezzi type for the term associated with the penalty functional, together with some regularity (see, e.g. LeTallec[3] or de Campos[5]). However, in this case the results are local and therefore we shall not consider them in this analysis.

We shall now consider a principle of stationary complementary energy.
4. A PRINCIPLE OF COMPLEMENTARY STRAIN ENERGY

The principle of complementary energy for the compressible or slightly compressible materials is concerned with the dual problem associated with the conjugate functional $E^*(\cdot, \Omega) : (A^{1,p}(\Omega))^* \to \mathbb{R}$ defined by

$$E^*(\sigma, \Omega) = \sup_{\nu \in \mathcal{V}(\Omega)} \left\{ \langle \sigma, \nu \rangle - E(\nu, \Omega) \right\}$$

which is called the total complementary energy, where for all $\nu \in (A^{1,p}(\Omega))^*$ and all $\varepsilon > 0$, $E(\nu, \Omega) = E(\varepsilon, \Omega) + \frac{1}{\varepsilon} P(\nu) = \int_a W(1 + \nabla \nu) \, dx + \frac{1}{2\varepsilon} \int_a [\det (1 + \nabla \nu) - 1]^2 \, dx$ represents the total strain energy functional.

The classical principle of complementary energy asserts that among all the stress and force fields $\sigma, f, t$ that satisfy the equilibrium equations, those corresponding to the actual states of compatible deformations render the total complementary energy a maximum.

By a "compatible deformation field" one generally means that the strain measures conjugate to the stresses satisfy the strain displacement equations and the displacements satisfy displacement boundary conditions.

The existence of a corresponding principle for the case of finite elastostatics has long been a subject of debate, and we refer to the papers of Koiter [6], Lee and Shield [7, 8], Levinson [9] and particularly Reissner [11] on this subject.

We shall now show that if the properties that guaranteed the existence of solutions to (3.3) still hold, then there exists a complementary strain energy.

**Proposition 4.1 (Existence of a complementary strain energy)**

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be an open bounded domain in $\mathbb{R}^n$ with a Lipschitzian boundary $\partial \Omega = \partial \Omega_1 \cup \partial \Omega_2$ with $\partial \Omega_1 \cap \partial \Omega_2 = \phi$ and with means $\partial \Omega_1 > 0$. Let $(A^{1,p}(\Omega))^*$, $E(\cdot, \Omega)$ be defined as in (3.1) and (4.2). Then if $E(\cdot, \Omega)$ is a proper, coercive (at least $E(\nu, \Omega) \geq C \|\nu\|_{A^{1,p}(\Omega)}^p; \alpha > 1; C$ is a constant), weakly lower semicontinuous functional on $(A^{1,p}(\Omega))^*$ the complementary energy functional (4.1) exists.

**Proof.** Let us rewrite (4.1) in the form

$$E^*(\sigma, \Omega) = \inf_{\nu \in \mathcal{V}(\Omega)} \left\{ E(\nu, \Omega) - \langle \sigma, \nu \rangle \right\}$$

which is called the total complementary energy, where for all $\nu \in (A^{1,p}(\Omega))^*$ and all $\varepsilon > 0$, $E(\nu, \Omega) = E(\varepsilon, \Omega) + \frac{1}{\varepsilon} P(\nu) = \int_a W(1 + \nabla \nu) \, dx + \frac{1}{2\varepsilon} \int_a [\det (1 + \nabla \nu) - 1]^2 \, dx$

from which the result follows.

**Remark 4.2**

We remark that if $E(\cdot, \Omega)$ is also Gâteaux differentiable then the following character-
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\[
\int_\Omega \frac{\partial W(1 + V\mathbf{u}_\epsilon)}{\partial \mathbf{u}_\epsilon} \cdot \mathbf{V} \mathbf{w} \, dx + \frac{1}{\epsilon} \left[ \det (1 + V \mathbf{u}_\epsilon) - 1 \right] \left[ \text{adj} (1 + V \mathbf{u}_\epsilon) \right]^T \cdot \mathbf{V} \mathbf{w} \, dx \right] \\
= \langle \sigma, \mathbf{w} \rangle, \quad \forall \mathbf{w} \in (A^{1,q}(\Omega))^n. 
\]

If now it would be possible to invert (4.5) we would be able to obtain \( \mathbf{u}_\epsilon = g(\sigma, \epsilon) \) which substituted into (4.3) would give \( E^*(\sigma, \Omega) \) explicitly. In general, this is not possible because even if \( \sigma \) represents the stress tensor associated with a solution to a nonlinear elastostatics problem, due to the existence of nonunique solutions in nonlinear elastostatics, the inversion of (4.5) is almost never possible. We shall avoid this problem and at the same time give a principle of stationary complementary energy by working directly with (4.5), and following a technique similar to the one in Lee and Shield\([7]\).

Let \( \mathbf{u}_\epsilon \in (A^{1,q}(\Omega))^n \) be an equilibrium solution for an almost incompressible material in nonlinear elastostatics, characterized by a proper, coercive, weakly lower semicontinuous, Gâteaux differentiable, total potential energy functional defined on the set of admissible displacements \( (A^{1,q}(\Omega))^n \). Then \( \mathbf{u}_\epsilon \) verifies the equilibrium eqns (3.5).

For all \( \mathbf{y} \in (W^{1,p}(\Omega))^n \) let us define the following functional \( Q(., \Omega):(W^{1,p}(\Omega))^n \rightarrow \mathbb{R} \) by

\[
Q(\mathbf{y}, \Omega) = \int_\Omega \frac{\partial W(1 + V\mathbf{y})}{\partial \mathbf{y}} \cdot \mathbf{V} \mathbf{y} \, dx \\
+ \int_\Omega \frac{1}{\epsilon} \left[ \det (1 + V \mathbf{y}) - 1 \right] \left[ \text{adj} (1 + V \mathbf{y}) \right]^T \cdot \mathbf{V} \mathbf{y} \, dx \\
- \int_\Omega W(1 + V \mathbf{y}) \, dx - \int_\Omega \frac{1}{2\epsilon} \left[ \det (1 + V \mathbf{y}) - 1 \right]^2 \, dx. 
\]

Introducing now multipliers \( \mu \in (W^{1,p}(\Omega))^n \) we define the new functional \( \tilde{Q}(., \mu, \Omega):(W^{1,p}(\Omega))^n \times (W^{1,p}(\Omega))^n \rightarrow \mathbb{R} \) by

\[
\tilde{Q}(\mathbf{y}, \mu, \Omega) = Q(\mathbf{y}) + \int_\Omega \frac{\partial W(1 + V\mathbf{y})}{\partial \mathbf{y}} \cdot \mathbf{V} \mu \, dx + \int_\Omega \frac{1}{\epsilon} \left[ \det (1 + V \mathbf{y}) - 1 \right] \left[ \text{adj} (1 + V \mathbf{y}) \right]^T \times \mathbf{V} \mu \, dx \\
- \int_\Omega f \cdot \mathbf{V} \mu \, dx - \int_{\partial \Omega} t \cdot \mathbf{u} \, ds - \int_{\partial \Omega} t \cdot \mu \, ds. 
\]

**Proposition 9.2**

If \( Q(.) \) as defined by (4.6) is Gâteaux differentiable then it is stationary with respect to all solutions to the equilibrium equations (3.5).

**Proof.** If for fixed \( \mu \), \( \tilde{Q}(., \mu, \Omega) \) is stationary for some \( \mathbf{y} \in (W^{1,p}(\Omega))^n \) we must have, for all \( \mathbf{w} \in (W^{1,p}(\Omega))^n \)

\[
\int_\Omega \frac{\partial^2 W(1 + V\mathbf{y})}{\partial \mathbf{y} \partial \mathbf{w}} \cdot \mathbf{V} \mathbf{y} \cdot \mathbf{V} \mathbf{w} \, dx + \int_\Omega \frac{\partial W(1 + V\mathbf{y})}{\partial \mathbf{y}} \cdot \mathbf{V} \mathbf{w} \, dx \\
+ \int_\Omega \frac{1}{\epsilon} \left[ \text{adj} (1 + V \mathbf{y}) \right]^T \mathbf{V} \mathbf{w} \left[ \text{adj} (1 + V \mathbf{y}) \right]^T \mathbf{V} \mathbf{y} \, dx \\
+ \int_\Omega \frac{1}{\epsilon} \left[ \det (1 + V \mathbf{y}) - 1 \right] \left[ \text{adj} (1 + V \mathbf{y}) \right]^T \mathbf{V} \mathbf{w} \, dx \\
+ \int_\Omega \frac{1}{\epsilon} \left[ \det (1 + V \mathbf{y}) - 1 \right] \mathbf{\varepsilon}_{\text{max}} \mathbf{\varepsilon}_{\text{max}}^T (\delta_{i} + y_{j}) u_{i}^m \cdot u_{j}^m \, dx 
\]
- \int_\Omega \frac{\partial W(1 + \nabla y)}{\partial \nabla y} \cdot \nabla w \, dx - \int_\Omega \frac{1}{\epsilon} \left[ \det (1 + \nabla y) - 1 \right] \left[ \text{adj}(1 + \nabla y) \right]^T \cdot \nabla w \, dx

+ \int_\Omega \frac{\partial^2 W(1 + \nabla y)}{\partial \nabla y \partial \nabla y} \cdot \nabla \mu \cdot \nabla w \, dx

+ \int_\Omega \frac{1}{\epsilon} \left[ \text{adj}(1 + \nabla y) \right]^T \cdot \nabla w \left[ \text{adj}(1 + \nabla y) \right]^T \cdot \nabla \mu \, dx

+ \int_\Omega \frac{1}{\epsilon} \left[ \det (1 + \nabla y) - 1 \right] \text{tr} \epsilon \epsilon^m (\delta^k_j + \gamma^k_j) w^m_j \mu^k \, dx = 0.

If now we are able to find \( \mu \) such that (4.8) holds for all \( u \), satisfying the equilibrium equations (3.5) then we shall also have \( Q(\cdot, \Omega) \) stationary for the same \( u \), because for \( u \), satisfying (3.5) \( Q(\cdot, \Omega) \) and \( Q(u, \mu, \Omega) \) are the same. It is easily seen that this case can be accomplished if we take \( \mu = \mu_c \), from which the result follows.

The next result identifies \( Q(\cdot, \cdot) \) with the complementary strain energy functional \( E^*(\cdot, \cdot) \).

**Proposition 4.3**

For any solution \( u_c \) to the equilibrium equations (3.5), \( Q(u_c, \Omega) \) as defined by (4.6) is identical to \( E^*(\sigma, \Omega) \) where \( \sigma \) is the stress tensor associated with the solution \( u_c \).

**Proof.** Let \( u_c \) be a solution to the equilibrium equations (3.5). Then

\[
\langle \sigma, v \rangle = \int_\Omega f \cdot v \, dx + \int_{\partial \Omega} t \cdot v \, ds,
\]

where by definition \( \langle \sigma, v \rangle \) is given by the l.h.s. of (3.5). Substituting back into (4.3) we obtain

\[
E^*(\sigma, \Omega) = -\inf_{\text{ad} A^*(\Omega)^n} \left\{ \int_\Omega W(1 + \nabla v) \, dx + \frac{1}{2\epsilon} \int_\Omega \left[ \det (1 + \nabla v) - 1 \right]^2 \, dx

- \int_\Omega f \cdot v \, dx - \int_{\partial \Omega} t \cdot v \, ds \right\}

\]

and from the definition of \( u_c \), the above infimum is attained for \( v = u_c \) and using again the definition of \( \sigma \) and (4.5) we obtain

\[
E^*(\sigma, \Omega) = -\int_\Omega W(1 + \nabla u_c) \, dx - \frac{1}{2\epsilon} \int_\Omega \left[ \det (1 + \nabla u_c) - 1 \right]^2 \, dx

+ \int_\Omega \frac{\partial W(1 + \nabla u_c)}{\partial \nabla u_c} \cdot \nabla u_c \, dx + \int_\Omega \frac{1}{\epsilon} \left[ \det (1 + \nabla u_c) - 1 \right] \left[ \text{adj}(1 + \nabla u_c) \right]^T \cdot \nabla \mu \, dx

\]

which is exactly \( Q(u_c, \Omega) \).

**Remark 4.4**

In order to be able to establish an analogous result for the incompressible case it is necessary to take the limit when \( \epsilon \) goes to zero in the above formulation. This is possible to do if a stability condition of the Ladyshenskaya, Babuska and Brezzi type holds for the penalized part of (3.5). This in turn is true if the solution \( u_c \) is sufficiently smooth which has only been established for local results (see e.g. Valent [12, 13] or de Campos [5]). Consequently we conclude that it is possible to establish an analogous result for the incompressible case if such conditions for local existence and regularity in finite elastostatics hold.
REFERENCES


