ALGORITHMS AND NUMERICAL RESULTS FOR FINITE ELEMENT APPROXIMATIONS OF CONTACT PROBLEMS WITH NON-CLASSICAL FRICTION LAWS

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Abstract—This paper represents a continuation of our earlier studies into the numerical analysis of contact problems with non-classical friction laws.

1. INTRODUCTION
This communication is a companion paper to our study published recently in this Journal for certain two-dimensional contact problems in elastostatics [1]. The special feature of the problems of interest here is that a non-standard friction law is assumed to be in force in which (1) the elasticity and elasto-plasticity of the junctions on the contact surface is taken into account and (2) non-local effects due to the deformation of asperities can be modeled. In [2], we discuss physical mechanisms that lead to such non-classical laws and we derive the following non-linear variational inequality characterizing a general variational principle for a non-classical Signorini problem with friction:

The elastic body \( \Omega \) subjected to body forces \( \mathbf{f} \) and surface tractions \( \mathbf{t} \) on \( \Gamma = \partial \Omega \) is in static equilibrium whenever the displacement field \( \mathbf{u} \) is such that \( a(\mathbf{u}, v - \mathbf{u}) + j_{ps}(\mathbf{u}, v - \mathbf{u}) \geq f(v - \mathbf{u}) \) for all admissible virtual displacements \( v \in K \).

In (1.1) we employ again the notations used in [1]. While a complete re-listing of the symbols is not necessary, it is noted that

\[
a(\mathbf{u}, v) = \int_\Omega \sigma(y) : \epsilon(v) \, dx
\]

is the virtual work of stresses \( \sigma \)

\[
= E_{\mu\nu \lambda} : \epsilon_{\mu\nu \lambda}
\]
on strains \( \epsilon_{\mu\nu \lambda} \) =

\[
(v_{ij} + v_{ji})/2. (dx = dx_1, dx_2)
\]

\[
= \int_\Gamma \mathbf{u}_s(\sigma(u)) \phi(\mathbf{v}_s) \, ds
\]

is the virtual work of frictional stresses on the contact surface \( \Gamma_c \).

\[
f(v) = \int_\Omega \mathbf{f} \cdot v \, dx + \int_{\Gamma_c} \mathbf{t} \cdot v \, ds
\]

is virtual work of external forces.

\[
K = \text{unilateral constraint set of admissible displacements}
\]

\[
= \{ v \in V | v \cdot \mathbf{n} \leq g \text{ on } \Gamma_c \}.
\]

\[
V = \{ v = (v_1, v_2) \in H^1(\Omega) \}.
\]

\[
r_i = 0 \text{ on } \Gamma_r \subset \partial \Omega. \quad i = 1, 2.
\]

In the expression for \( f_{ps} \), \( v \) is the coefficient of friction. \( S_p \) is a smoothing operation mollifying the actual contact stress \( \sigma = \sigma_{ps} \) over a disc of radius \( \rho \) and \( \phi \) is a continuous piecewise differentiable function modeling the stiffness of the junctions on the contact surface. Interpretations of the parameters \( \rho \) and \( \epsilon \) are given in Fig. 1.

As a finite element approximation of (1.1), we introduce the discrete problem:

Find \( \mathbf{u}_h \in V_h \) such that

\[
a(\mathbf{u}_h, v_h - \mathbf{u}_h) + j_{ps}(\mathbf{u}_h, v_h - \mathbf{u}_h) = f(v_h - \mathbf{u}_h) + \delta^{-1} \int_{\Gamma_c} (\mathbf{u}_h \cdot \mathbf{n} - g_k) \phi(\mathbf{v}_s) \cdot \mathbf{n} \, ds
\]

\[
\forall \mathbf{v}_h \in V_h
\]

where \( V_h \) is a finite-dimensional subspace of \( V \) obtained using 9-node \((Q_2)\) elements, or 4-node \((Q_1)\) elements,

\[
V_h = \{ v_h = (v_{h1}, v_{h2}) \in H^1(\Omega) \}.
\]

\[
\Omega = \bigcup \Omega_r. \quad 1 \leq r \leq E. \quad i = 1, 2: \quad r = 1 \text{ or } 2
\]

and we have relaxed the unilateral constraint by introducing an exterior penalty: \( \delta^{-1} \int_{\Gamma_c} \mathbf{u}_h \cdot (\mathbf{n} - g_k) \phi(\mathbf{v}_s) \cdot \mathbf{n} \, ds \). Let us finally note that (1.2) can be replaced by the approximate non-linear variational equality:

\[
a(\mathbf{u}_h, v_h) + j_{ps}(\mathbf{u}_h, v_h) \mathbf{n}_{u_h} \cdot \mathbf{v}_h = f(v_h) \quad \forall v_h \in V_h
\]

where \( I[\cdot] \) is a numerical quadrature rule for evaluating the integral \( \int_{\Gamma_c} \mathbf{f} \cdot \mathbf{n} \, ds \) on the approximate contact surface \( \Gamma_c \), and \( \mathbf{n}_{u_h} \) is a unit vector in the direction...
of the tangential displacement $u_{\kappa T}$ of points on the contact surface:

$$n_{\kappa T} = \frac{u_{\kappa T}}{|u_{\kappa T}|} = \sqrt{(u_{\kappa T} - u_{\kappa T})}.$$

Thus, if $g_{\kappa} \in C^\infty(\Gamma_c^\delta)$,

$$\int_{\Gamma_c^\delta} g_{\kappa} \, dx \approx \int g_{\kappa} \approx \sum_{r=1}^{N} \sum_{i=1}^{N} w_r g_r (\xi_i)$$

where $E'$ is the number of elements in $\Gamma_c^\delta$, $w_r$ = the quadrature weights on element $\Gamma_r \subset \Gamma_c^\delta$, and $\xi_i$ are the quadrature points.

The behavior of the approximation (1.2) depends upon the parameters $\nu$, $\epsilon$, $p$, $\delta$ and $h$. In the present study, we present a new algorithm for solving the variational inequality (1.1) and some preliminary numerical results obtained for various choices of these parameters.

2. AN ALGORITHM FOR NON-CLASSICAL FRICTION PROBLEMS

We outline here an algorithm for solving the non-linear friction problem (1.3) for the case of Signorini's problem with non-classical friction. In subsequent calculations, Simpson's rule is used for the quadrature operator $\iint \ldots$. The algorithm presented here is suggested by the proof of the existence of solutions to the general problem (1.1) given in [3].

Let us assume that standard subroutines are available to generate the finite element mesh, specify material properties, specify boundary conditions, evaluate the stiffness matrix (excluding contact and friction terms), evaluate the load matrix, and impose boundary conditions on $\Gamma_c^\delta$. In addition, let us define, for a given displacement field $u_h$,

$$\tau_s = \frac{\sigma_n (u_h)}{S_p (\sigma_n (u_h))}$$

where $S_p$ is the stress mollifier and $\sigma_n (u_h)$ is the normal stress due to $u_h$ ($\Gamma_r = \text{element in } \Gamma_c^\delta$):
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\begin{align*}
\sigma_{uk}(u_k) & \in \{ \sigma \in C^0(\Gamma_c^k) | \sigma|_c^k \in P_0(\Gamma_c^k) \} \\
\sigma_{uk}(u_k)(\xi_j^k) & = -\delta^{-1}u_k \cdot n - g_k^s(\xi_j^k) \\
S_p(\sigma_{uk}(u_k))(x) & = f(\sigma, (x-y)\left(-\sigma_{uk}(u_k)(y)\right) \\
& \forall x, y \in \Gamma_c^k.
\end{align*}

(2.1)

The algorithm consists of the following steps:

1. Initialize \( \tau^{(0)}_h \): \( \tau^{(0)}_h \) = nodal forces required to maintain equilibrium; \( \tau^{(0)}_h \) = 0.0 when the problem has at least two mutually orthogonal axes of symmetry in the plane; \( \tau^{(0)}_h \) = 0.0 otherwise.

2. Solve the standard Signorini problem with prescribed normal stress \( \tau^{(0)}_h \) on the contact surface for \( u^{(1)}_h \) using the exterior penalty to relax the unilateral contact condition (the case \( \tau^{(0)}_h \) = 0 represents a problem without friction):

\begin{align*}
a(u^{(1)}_h, v_h) + \int [v] \cdot [\phi_j^m] n^{m} \cdot v_{kr} \\
+ \delta^{-1} [v] (u^{(1)}_h - g_h) \cdot v_{kr} & = f(v_h) \forall v_h \in V_h
\end{align*}

or, in general.

3. Compute

\begin{align*}
\sigma_{uk}^{(1)}(\xi_j^k) & = -\delta^{-1}(u^{(1)}_h - g_h)(\xi_j^k).
\end{align*}

(2.3)

4. Compute

\begin{align*}
\sigma_{uk}^{(1)}(\xi_j^k) & = -v \cdot \phi_j^m (u^{(1)}_h) n^{m}.
\end{align*}

(2.4)

5. Set

\begin{align*}
\tau^{(1)}_h & = S_p(\sigma_{uk}(u^{(1)}_h)).
\end{align*}

(2.5)

6. Use \( \tau^{(1)}_h \) as data in problem (2.2) and compute a new corrected displacement field \( u^{(2)}_h \) from (2.2) (with \( t = 2 \)).

7. Let TOL denote a preassigned tolerance that we set for the difference between successive mollified

Fig. 2. Flow chart for non-classical friction algorithm.
When this criteria is satisfied, we terminate the process and plot deformed configuration of the mesh, stress distributions, contact stresses, etc.

A flow chart summarizing this procedure is given in Fig. 2. Note that many different linear/non-linear programming schemes can be used to solve the variational inequality corresponding to (2.2). In the present case, we employ the following successive iteration process to solve the non-linear equation (2.2) for a given iteration $i$:

(i) We set

$$\Gamma_{c_{0}} = \{ \text{all nodal points on } \Gamma_{c} \text{ for which } u_{0}^{(i)} - g > 0 \}$$

and $u_{0}^{(0)} = 0$.

(ii) We solve the linear problem

$$a(u_{0}^{(i)}, v_{h}) + f(u_{0}^{(i)}, v_{h}) = 0, v_{h} \in V_{h}$$

(iii) With the available function $u_{0}^{(i)}$ we obtain

$$\Gamma_{c_{i+1}}$$

(iv) We continue steps (i) through (iii) until sufficiently small tolerance is reached.

This rather straightforward method has proved to be quite effective for contact problems of this type [4].

3. NUMERICAL EXPERIMENTS

We shall now describe the results of several numerical experiments performed with the algorithm discussed in the preceding section. The first problem considered is that of an elastic block pressed against a rough rigid foundation on which a nonlocal non-linear law of friction is assumed to hold. The elastic block is also being pulled by a tangential force uniformly distributed along one of its lateral sides, as shown in Fig. 3. The geometry and data for the problem are also shown in Fig. 3 as well as the rectangular mesh of 9-node, $Q_{2}$-elements which was used in its finite element approximation. Young's modulus $E$ and Poisson's ratio $\mu$ were taken equal to 1000 per length square and 0.3 respectively: the parameters $\rho$ (appearing in the nonlocal law of friction) and $\delta$ (penalty parameter for the unilateral constraint) were taken equal to 0.01 and $10^{-4}$ respectively.

By varying the two remaining parameters, $\nu$, the coefficient of friction, and $\epsilon$, the measure of the stiffness of the junctions, several experiments were performed, the results of which are shown in Figs. 4-8. In Fig. 4 the computed deformed configuration of the block for the case $\nu = 0.5, \epsilon = 10^{-4}$ is presented. In Figs. 5-8 we show the stress profiles for the cases:

- $\nu = 0.5, \epsilon = 10^{-4}$
- $\nu = 0.5, \epsilon = 0.1$

Fig. 3. An elastic block pressed against a rough rigid foundation and pulled by a tangential force: undeformed configuration.
We notice, besides the observations we made concerning the example we presented in [1], that as $\epsilon$ is increased (for fixed $v$) the values of $|\sigma_T|$ at the nodal points decrease and therefore the region where $|\sigma_T|$ attains its maximum is smaller. This was to be expected since an increase in $\epsilon$ means less stiff junctions which imply that larger tangential displacements have to occur in order to make $|\sigma_T|$ reach its maximum value.

We have also considered the case $v = 0.1$, but then the algorithm did not converge. This implies that equilibrium was not reached, indicating that the applied tangential force is sufficiently large to create sliding at all nodes of the finite element model. In fact, for this case, if $P$ denotes the resultant of the applied normal pressure and $T$ of the applied normal tractions then

$$0.1 P = 0.1 \times 720 = 72 < 120 = T$$

so that sliding is expected to occur.

In the second example, we consider an elastic body indented by a parabolic rigid punch and subjected to applied normal tractions $t$, as shown in Fig. 9. We
Fig. 6. Stresses on contact area.

- □ regularized normal pressures
- ▲ tangential stresses

ν = 0.5
c = 0.1
Fig. 7. Stresses on contact area.

- □ regularized normal pressures
- △ tangential stresses

\[ \nu = 1.0 \]
\[ \varepsilon = 0.0001 \]
Fig. 8. Stresses on contact area.
use, in this case, the 4-node isoparametric element to construct a finite element model for this problem. The trapezoidal rule was employed to evaluate \( f[\cdot] \) in expression (2.2). The finite element mesh, the geometry and the mechanical data are also shown in Fig. 9. Young's modulus \( E \), Poisson's ratio \( \mu \) and the coefficient of friction \( \nu \) were taken equal to 1000 (nondimensional units), 0.25 and 0.6 respectively; the parameters \( \rho, \epsilon \), and \( \delta \) were taken equal to 0.1, \( 10^{-2} \), and \( 10^{-8} \) respectively. Two cases were considered: (i) \( t = 20/\text{unit length} \) and (ii) \( t = 60/\text{unit length} \). The corresponding computed stress profiles are shown in Figs. 11 and 12 and the computed deformed configuration for case (ii) is given in Fig. 10. The normal force necessary to cause an indentation of depth \( d = 0.25 \) is approx. 215 so that in case (i) the applied tangential force is approximately 10 percent of the normal force and in case (ii) it is 30\% of it. If the sliding region on \( \Gamma_c \) is determined by the maximum shear stress, then we see that this region does not increase significantly with a large increase in tangential force. This is due to the rather stiff response of the junctions characterized by a rather large value of \( \epsilon \).

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Fig. 11. Stresses on contact area, Example 2—rigid punch.

Applied normal tractions \( t = 20 \) per unit length

- □ mollified normal pressures
- ▲ tangential stresses
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Fig. 12. Stresses on contact area, Example 2—rigid punch.

REFERENCES