Chapter 5

Stable and Unstable Rip/Perturbed Lagrangian Methods for Two-Dimensional Viscous Flow Problems

J. Tinsley Oden and Olivier Jacquotte*

5.1 INTRODUCTION

For the last several years (indeed, dating back to the Second Conference on Finite Elements for Flow Problems in 1976), the use of exterior penalty methods and reduced integration as devices for producing very efficient finite element methods for the analysis of incompressible viscous flows has been advocated by numerous investigators. The idea is to use exterior penalty methods to handle the incompressibility constraint, $\text{div } u = 0$, and then to use reduced integration of the penalty terms to 'unlock' the system (which basically frees the dependence of the penalty parameter $\varepsilon$ on the mesh size $h$.

In 1980, however, we were able to show both mathematically and experimentally that many of the favourite RIP (reduced-integration-penalty) schemes are numerically unstable. These difficulties were discussed in Oden's lecture at the Banff Conference in 1980 and are summarized in the final conference paper by Oden (1). Complete details of this analysis are given in the papers of Oden, Kikuchi, and Song (2,3) together with a more complete list of references on this subject.

From the mathematical point of view, the key to the stability of these methods is the so-called LBB-condition, to be given in the next section. The once widely used $Q_2$-elements for velocities (tensor products of quadratures) with $2 \times 2$ Gaussian integration (or, equivalent, constant pressures) foil the LBB-condition for most choices of boundary conditions in that they lead to an LBB-parameter $\alpha_h$ which depends upon the mesh size $h$. If one uses $Q_2$ elements for velocities but only one-point integration, then the LBB condition is satisfied with $\alpha_h$ independent of $h$, but the rate-of-convergence of the method is suboptimal. The

*Texas Institute for Computational Mechanics, Department of Aerospace Engineering and Engineering Mechanics, The University of Texas.
question naturally arises as to whether or not it is possible to produce a stable RIF method which exhibits an optimal rate-of-convergence in the 'energy' $H^1$ - ) norm.

At the third symposium at Banff, Oden announced two types of elements which were candidates for such optimal RIF methods: the $Q_2$-element for velocities with discontinuous piecewise linear pressures (referred to here as the $Q_2/P_1$-element) and the eight-node isoparametric element with discontinuous piecewise linear pressures (here the $18/P_1$-element). These were subsequently coded and by July 1980 were producing encouraging numerical results (see Oden (4)). Similar numerical results were obtained by other investigators in the intervening months (see Zienkiewicz and Taylor and Engleman and Sani (5)).

In the present paper, we examine these two elements in greater detail. First, we show that the $Q_2/P_1$-element does, in fact, satisfy the LBB-condition with $\alpha_h$ independent of $h$ whereas the $18-P_1$-element apparently does not, except under special circumstances. We also attempt to provide results of numerical experiments which support some of our theoretical findings.

Our main result is that the $Q_2/P_1$-element is stable and exhibits an optimal convergence rate of $H^1$-norm. We also suggest certain ways to stabilize the $18-P_1$ element for certain classes of problems.

### 5.2 STATEMENT OF THE PROBLEM

Let $\Omega$ denote an open bounded region of $\mathbb{R}^2$ with boundary $\partial\Omega$. We consider the two-dimensional Stokes problem an $\Omega$, which involves finding a velocity field $u = (u_1, u_2)$ and a pressure field $p$ such that

$$\begin{cases} - \nu \Delta u + \nabla p = f & \text{in } \Omega \\ \text{div } u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega \end{cases} \tag{5.1}$$

where $\nu$ is the viscosity of the fluid, ($\nu = \text{const.} > 0$), and $f$ is the body force, assumed to be a prescribed vector field with components $f_i \in L^2(\Omega)$.

We recast (5.1) in a weaker variational framework by introducing the spaces

$$V = (H_0^1(\Omega))^2, \quad Q = L^2(\Omega) \tag{5.2}$$

and the forms

$$a(u, v) = \nu(u, v)_1, \quad f(v) = \sum_{i=1}^{2} (f_i, v_i) \tag{5.3}$$

for all $u, v \in V$, where $(\cdot, \cdot)_1$ and $(\cdot, \cdot)$ are inner products on $V$ and $Q$, respectively,
and are given by

\begin{align*}
(v, w)_0 &= \int_{\Omega} vw \, dx, \quad v, w \in Q \\
(u, v)_1 &= \sum_{i,j=1}^{2} \left( \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \right), \quad u, v \in V \tag{5.4}
\end{align*}

The partial derivatives in (5.4) are interpreted in a distributional sense.

We proceed by considering the problem of finding \((u, \rho) \in V \times Q\) such that

\begin{align*}
a(u, v) - (p, \text{div } v) &= f(v) \quad \forall v \in V \\
(q, \text{div } u) &= 0 \quad \forall q \in Q \tag{5.5}
\end{align*}

It is easily verified that any solution of (5.1) satisfies (5.5); any solution of (5.5) satisfies equations of the form (5.1) in a distributional sense. Under the conditions stated, it is also known that (5.5) possesses a solution \((u, \rho)\), with \(u\) uniquely determined by each choice of \(f\) and \(\rho\) unique up to an arbitrary constant.

Problem (5.1) can also be interpreted as the characterization of a saddle point of the functional

\[ L: V \times Q \rightarrow \mathbb{R} \]

\[ L(v, q) = \frac{1}{2} a(v, v) - f(v) - (q, \text{div } v) \tag{5.6} \]

with \(q\) clearly a Lagrange multiplier associated with the constraint, \(\text{div } v = 0\) in \(Q\).

An alternate formulation of problem (5.5) can be obtained using an exterior penalty method. Let \(\varepsilon\) be an arbitrary positive number. By seeking minimizers \(u_\varepsilon \in V\) of the penalized functional,

\[ J_\varepsilon(v) = \frac{1}{2} a(v, v) - f(v) + \frac{1}{2\varepsilon} (\text{div } v, \text{div } v) \tag{5.7} \]

we are led to the variation problem

\[ a(u_\varepsilon, v) + \varepsilon^{-1} (\text{div } u_\varepsilon, \text{div } v) = f(v) \quad \forall v \in V \tag{5.8} \]

This problem, which is uniquely solvable for \(u_\varepsilon\) for any \(\varepsilon > 0\), involves a single unknown field \(u_\varepsilon\). Once \(u_\varepsilon\) is obtained, an approximation \(p_\varepsilon\) of the pressure is obtained by the computation

\[ p_\varepsilon = -\varepsilon^{-1} \text{div } u_\varepsilon \tag{5.9} \]

The forms \(a(\cdot, \cdot)\) and \(f(\cdot)\) are continuous and \(a(\cdot, \cdot)\) is \(V\)-elliptic. In addition, Ladyszhenskaya (6) has shown that a constant \(\alpha > 0\) exists such that

\[ \alpha \| q \|_{L^2(\Omega, \mathbb{R})} \leq \sup_{v \in V} \frac{(q, \text{div } v)}{\| v \|_1} \quad \forall q \in Q \tag{5.10} \]

where \(\| v \| = \sqrt{(v, v)}_1\). Under these conditions, the sequence \(\{(u_\varepsilon, p_\varepsilon)\}_{\varepsilon > 0}\) of
solutions of (5.8) and (5.9) converge strongly in $V \times Q/\mathbb{R}$ to the solution $(u, p)$ of (5.5).

We remark that the penalized problem (5.8) can be interpreted in another way. Returning to the saddle point problem for the functional $L(\cdot, \cdot)$ of (5.6), we introduce the perturbed Lagrangian

$$L_{\varepsilon}(v, q) = L(v, q) - \frac{1}{2\varepsilon}(q, q)$$

(5.11)

for all $q \in Q$, which represents a regularization of $L(\cdot, \cdot)$ with respect to the multipliers $q$. For each $\varepsilon > 0$, saddle points $(u_{\varepsilon}, p_{\varepsilon})$ of $L(\cdot, \cdot)$ are characterized by

$$a(u_{\varepsilon}, v) - (p_{\varepsilon}, \text{div} v) = f(v) \quad \forall v \in V$$

$$\varepsilon p_{\varepsilon} + \text{div} u_{\varepsilon} - q = 0 \quad \forall q \in Q$$

(5.12)

Upon solving the last equation in (5.12) for $p_{\varepsilon}$, we obtain (5.9). Substituting this result into the first equation in (5.12) gives precisely the penalty formulation (5.8).

### 5.3 FINITE ELEMENT APPROXIMATIONS

We shall outline briefly features of certain finite element approximations of (5.8). We confine our attention to cases in which $\Omega$ is rectangular or is the union of rectangles and, for simplicity, to uniform meshes of rectangular elements of maximum length $h$. For a family of such meshes with $E = E(h)$ elements, we introduce the discrete (finite-dimensional) spaces,

$$V^h = \{v_h = (v_{h1}, v_{h2})|v_{hi} \in C^0(\tilde{\Omega})\}$$

$$v_{hi}|_{\tilde{\Omega}e} \in Q_k(\tilde{\Omega}_e); v_{hi} = 0 \text{ on } \partial \Omega, 1 \leq i \leq E, i = 1, 2$$

(5.13)

$$Q^h = \{q_h \in L^2(\Omega)|q_{hi} \in P_r(\Omega_e); 1 \leq i \leq E, r \geq 0\}$$

(5.14)

Here $Q_k(\tilde{\Omega}_e)$ is the space of tensor products of complete polynomials in $x_1$ and $x_2$ of degree $\leq k$ defined on finite element $\tilde{\Omega}_e$ and $P_r(\Omega_e)$ is the space of complete polynomials of degree $\leq r$ defined on $\Omega_e$. Clearly, for every $h$,

$$V^h \subset V \quad \text{and} \quad Q^h \subset Q$$

In addition to the spaces $V^h$, we shall also consider cases in which $V^h$ is constructed using 18 elements:

$$18 = \text{eight-node isoparametric elements}$$

(5.15)

We also consider composite elements which employ both $Q_2$ and 18-subelements.

In any case, the finite-element approximation of the penalty formulation (5.8) consists of seeking $u^h \in V^h$ which satisfies the variational equality (5.8) for all $v \in V^h$. However, the performance of the resulting method depends strongly on the
relationship between $\varepsilon$ and $h$ (see Falk (6)). For fixed mesh size $h$, the use of a small $\varepsilon$ (say $\varepsilon = 10^{-5}$ for most reasonable meshes) leads to a 'locked' solution $u_h^e$ (which approaches zero in $V$ as $\varepsilon$ tends to zero). The use of larger values of $\varepsilon$, on the other hand, unlocks the problem in the sense that a nonzero solution $u_h^e$ can be obtained, but then the incompressibility constraint is not adequately enforced. The solution to this paradox is to use reduced integration for evaluating the penalty terms in (5.8) (7.8, 9 and 10).

Let $I(\cdot, \cdot)$ denote a numerical quadrature rule for integrating approximately the product of two element-wise continuous functions $f$ and $g$:

$$I(f, g) = \sum_{e=1}^{E} \sum_{j=1}^{G_e} W_j^e f(\xi_j^e) g(\xi_j^e)$$  \hspace{1cm} (5.16)

Here $G_e$ is the number of quadrature points in element $e$, $W_j^e$ are the quadrature weights, and $\xi_j^e$ are the quadrature points in element $e$. By choosing $G_e$ sufficiently large, it is always possible to obtain the exact value of the integral $\int f g \, dx$ from $I(f, g)$ for polynomial $f$ and $g$.

The RIP-finite element approximation of (5.1) then consists of seeking $u_h^e \in V^h$ such that

$$a(u_h^e, v_h) + \varepsilon^{-1} I(\text{div} \, u_h^e, \text{div} \, v_h) = f(v_h) \hspace{1cm} \forall v_h \in V^h$$  \hspace{1cm} (5.17)

For each choice of $V^h$ and quadrature rule $I(\cdot, \cdot)$ there is defined implicitly in the approximation (5.17) a space $Q^h$ of approximate pressure $q_h$. Indeed, the specification of values of $q_h$ at each integration point within an element in the definition of $I(\cdot, \cdot)$ uniquely defines a piecewise continuous polynomial over each element. For example, if $Q_2$-elements are used for the velocity approximation, a $2 \times 2$ Gaussian quadrature rule for $I(\cdot, \cdot)$ corresponds to a piecewise bilinear pressure approximation, etc. In general, an $m \times m$ Gaussian quadrature rule corresponds to a space $Q^h$ of pressures which are discontinuous at interelement boundaries, but which have restrictions on $Q_m(\Omega_e)$ in each finite element.

Once (5.16) is solved for the velocity field $u_h^e$, the corresponding pressure approximation is computed as

$$p_h \in Q^h : p_h(\xi_j^e) = - \varepsilon^{-1} \text{div} \, u_h(\xi_j^e)$$  \hspace{1cm} (5.18)

where $\xi_j^e$ are, again, the quadrature points in element $\Omega_e$. Note that $Q^h$ is such that (5.18) uniquely determines $p_h^e$.

Under the conditions established thusfar, (5.17) is uniquely solvable for $u_h^e$ for any $\varepsilon > 0$. However, the behavior of $u_h^e$ and $p_h^e$ as $\varepsilon$ or $h$ tend to zero depends upon more delicate features of the approximation.

Let $B_h$ and $B_h^*$ denote the discrete operators,

$$B_h : V^h \rightarrow Q^h ; B_h^* : Q^h \rightarrow V^h$$

$$[q_h, B_h^* v_h] = \langle v_h B_h^* q_h \rangle = I(q_h, \text{div} \, v_h) \hspace{1cm} \forall q_h \in Q^h, \forall v_h \in V^h$$  \hspace{1cm} (5.19)
where \([\cdot , \cdot]\) and \(\langle \cdot , \cdot \rangle\) denote duality pairings on \(Q' \times Q (Q = Q' = L^2(\Omega))\) and \(V' \times V\) respectively (i.e., \(B_h\) and \(B_h^*\) are the discrete approximations of \(\text{div}\) and \(\text{-grad}\) plus boundary conditions defined by \(I(\cdot , \cdot)\)). Then, the discrete LBB-condition for problem (5.17) (or (5.19)) is as follows:

\[
\alpha_h \| q_h \|_{L^2(\Omega)/\ker B_h^*} \leq \sup_{v_h \in V_h} \frac{I(q_h, \text{div} v_h)}{\| v_h \|_1} \quad \text{for all } q_h \in Q_h
\]  

(5.20)

The behaviour of \(\alpha_h\) as \(h\) tends to zero and the structure of \(\ker B_h^*\) governs the stability of RIP methods. In particular, let \(E_h(u, p)\) denote the distance function

\[
E_h(u, p) = \inf_{v_h \in V_h} \| u - v_h \|_1 + \inf_{p_h \in Q_h} \| p - p_h \|_{L^2(\Omega)}
\]  

(5.21)

defined on \(V \times Q, Q = \{ q \in Q | \int_\Omega q \, dx = 0 \}\). Then one can show (see Oden et al. (3)) that if \((u, p)\) is the solution of (5.5) and \((u_h^*, p_h^*)\) are the solutions of (5.17) and (5.18) in \(V_h \times Q_h\),

\[
\| u - u_h^* \|_1 \leq C(1 + \alpha_h^{-1})(E_h(u, p) + \varepsilon)
\]  

\[
\| p - p_h^* \|_{L^2(\Omega)} \leq C(1 + \alpha_h^{-1} + \alpha_h^{-2})(E_h(u, p) + \varepsilon)
\]  

(5.22)

where \(C\) is a generic constant independent of \(u, p, \varepsilon,\) and \(h\).

The remainder of this paper is devoted to the study of (5.20) and estimations of the stability parameter \(\alpha_h\).

### 5.4 LBB CONDITIONS FOR CERTAIN RIP METHODS

We state estimates of the stability parameter \(\alpha_h\) in the discrete LBB condition (5.20) obtained for the following RIP-methods:

1. \(Q_2/P_1\) elements [here a mixed formulation based on the equivalence of (5.8) and (5.12) is used with \(Q_2\) (biquadratic) velocities and linear discontinuous pressures. This formulation was discussed by Oden and Kikuchi (2). The result relative to \(\alpha_h\) is stated in this section and the next section will be devoted to the proof of this estimate. The method of the proof is general in order to get a constant \(\alpha_h\) independent of \(h\).

2. \(I8/P_1\) elements [eight-node isoparametric elements for velocities, linear pressures].

3. \(Q_1/P_0\) elements [bilinear velocities, piecewise constant (one-point integration) pressures].

4. Composite elements [elements consisting of two or more of the above elements as subelements].

Again, we note that in all of the cases we study, we shall assume that

\[
\Omega = \Omega_h \text{ is a rectangle (or a union of rectangles) discretized by a uniform mesh of rectangular finite elements:}
\]
The principal results concerning \( \ker B^*_h \) and the LBB-constant \( \alpha_h \) are stated in the following theorems.

**Theorem I**
Let conditions (5.23) hold and let the discrete spaces \( V^h \) and \( Q^h \) be constructed using \( Q_2/P_1 \)-elements. Then \( \ker B^*_h = \ker B^* \) and the stability parameter \( \alpha_h \) in the discrete LBB-condition (5.20) is a positive constant independent of \( h (\alpha_h = O(1)) \).

**Theorem II**
Let conditions (5.23) hold and suppose that \( V^h \) and \( Q^h \) are defined by \( I8/P_1 \)-elements. Then \( \dim \ker B^*_h = 3 \) and the stability parameter \( \alpha_h \) in the discrete LBB-condition (5.20) depends linearly on \( h \):

\[
\alpha_h = O(h)
\]

**Theorem III**
Under the assumptions of Theorem II, if \( V^h \) and \( Q^h \) are defined using \( Q_1/P_0 \)-elements (i.e., bilinear velocities and a one-point integration for the penalty), then \( \dim \ker B^*_h = 2 \) and \( \alpha_h = O(h) \).

**Theorem IV**
Let conditions (5.23) hold and suppose that \( V^h \) and \( Q^h \) are defined using composite \( I8/P_1 - Q_2/P_1 \)-elements of the type shown in Figure 5.1. *This

* It suffices to define \( q_h \) only to within an arbitrary constant or to demand that all \( q_h \) be such that \( (1, q_h) = 0 \).
condition does not hold for all RIP-methods; for instance, it is not true for the $Q_2/P_0$-elements studied in (3). Then $\dim \ker B^*_h = 1$ and the stability parameter $\alpha_h$ appearing in the discrete LBB-condition is a positive constant independent of $h$; ($\alpha_h = O(1)$).

5.5 THE LBB-CONDITION FOR $Q_2/P_1$-ELEMENTS

In this section, we will describe a general method for establishing the LBB-Condition when $\ker B^*_h$ and $\ker B^*$ coincide. This procedure will then be used for $Q_2/P_1$-elements. The method is embodied in the following four steps.

I. Let $q_h$ be an arbitrary element in $Q^h$. Construct a vector $u_h \in V^h$ such that

$$(q_h, \text{div } u_h) = \| q_h \|_0^2 \quad \| u_h \|_1 \leq C \| q_h \|_0$$

(5.23)

where $\| \cdot \|_0 = \| \cdot \|_{L^2(\Omega)}$ and $C$ is a constant. Then

$$\sup_{v_h \in V^h} \frac{(q_h, \text{div } v_h)}{\| v_h \|_1} \geq \frac{1}{C} \| q_h \|_0$$

so that $\alpha = 1/C$.

To construct such a $u_h$, we continue as follows.

II. For each $q_h \in Q^h \subset Q$, $q_h \neq$ constant, it can be shown (Ladyszhenskaya (11)) that a $v_q \in V$ can be found such that

$$\text{div } v_q = q_h \text{ in } \Omega \text{ and } \| v_q \|_1 \leq C_1 \| q_h \|_0$$

(5.24)

Let $w_h$ denote the $V$-orthogonal projection of $v_q$ onto $V^h$:

$$(w_h - v_q, v_q)_1 = 0 \quad \forall v_q \in V^h$$

(5.25)

Then

$$\| w_h \|_1 \leq \| v_q \|_1 \leq C_1 \| q_h \|_0$$

III. Set

$$e = v_q - w_h$$

(5.26)

We attempt to construct a $u_h$ with the desired property (5.23) by demanding that

$$(q_h, \text{div } (e - e_h)) = \sum_{e=1}^E \int_{\Omega_e} q_h \text{div } (e - e_h) \, dx = 0$$

(5.27)

where

$$e_h = u_h - w_h$$

(5.28)

Then it is clear that

$$(q_h, \text{div } v_q) = \| q_h \|_0^2 = (q_h, \text{div } u_h)$$

which is (5.23), and it remains only to verify that (5.23)_2 holds. Assuming that this is possible, we see that the original problem reduces to one of constructing a $u_h$ such that (5.27) holds.
IV. To satisfy (5.27) it is sufficient to require that

\[ \int_{\Omega_e} q_h \text{div}(e - e_h) \, dx = -\int_{\Omega_e} V q_h \cdot (e - e_h) \, dx + \int_{\partial \Omega_e} q_h n \cdot (e - e_h) \, dx = 0 \]

for each finite element \( \Omega_e \), \( n \) being a unit outward normal to \( \partial \Omega_e \). In many finite element meshes, each \( \Omega_e \) is the image of a fixed master element \( \tilde{\Omega} \) under an invertible affine map \( F_e \).

\[ F_e : \tilde{\Omega} \to \Omega_e \quad F_e \hat{x} = X = T_e \hat{x} + b_e \]  

(5.29)

\( T_e \) being a \( 2 \times 2 \) matrix and \( b_e \) a translation vector. Then it is sufficient to construct \( u_h \) such that

\[ \int_{\tilde{\Omega}} \hat{q} \cdot (\hat{e} - \hat{e}_h) \, d\hat{x} - \int_{\partial \tilde{\Omega}} q_{h1} \cdot (\hat{e} - \hat{e}_h) \, d\hat{x} = 0 \]  

(5.30)

where \( \hat{q} = q_h \circ F_e^{-1} \), \( \hat{e} = e \circ F_e^{-1} \), etc.

Remark

This procedure is next used for the \( Q_2/P_1 \)-element. But in the case where \( \Omega \) is partitioned into \( 18/P_1 \) elements, except one \( Q_2/P_1 \) element, we can show that

\[ \ker B_h^* = \ker B^* \]

For this mesh, the construction II, III, IV can theoretically be made, but the essential estimate of (5.23) cannot be obtained. This remark suggests the introduction of the composite element described in Theorem IV.

For the \( Q_2/P_1 \) element, the two discrete finite-dimensional spaces \( V^h \) and \( Q^h \) are defined as

\[ V^h = \{ v_h = (v_{h1}, v_{h2}) | v_{hi} \in C^0(\tilde{\Omega}) \quad v_{hi}(e) \in Q_2(\tilde{\Omega}_e) : v_{hi} = 0 \text{ on } \partial \Omega, \quad |i = e \leq E, i = 1, 2 \} \]

\[ Q^h = \{ q_h \in L^2(\Omega) | q_h|_{\Omega_e} \in P_1(\Omega_e) | e \leq E \} \]

and then using the definition

\[ \ker B_h^* = \{ q_h \in Q^h \text{ such that } \int_{\Omega} q_h \text{div} v_h \, dx = 0 \text{ for all } v_h \in V^h \} \]  

(5.31)

a simple calculation reveals that

\[ \ker B_h^* = \ker B^* = \mathbb{R} \]  

(5.32)

Then it suffices to construct a \( u_h \) such that (5.30) holds for the master element \( \tilde{\Omega} \).
shown in Figure 5.2 and to then show that \( u_h \) satisfies (5.23). We use the notation indicated in the figure; the integral appearing on the left side of (5.33) is denoted \( \mathcal{I} \), and we seek \( \hat{u} \) with \( \hat{u}_i \), \( Q_2(\hat{\Omega}) \). Observe that the shape functions associated with the indicated nodes are of the form

\[
\begin{align*}
\hat{\psi}_5 &= (1 - \xi^2)(1 - \eta^2), \\
\hat{\psi}_{12} &= \frac{1}{2}(\xi^2 - 1)\hat{\eta}(1 - \hat{\eta}) \\
2\hat{\psi}_{23} &= \frac{1}{2}\xi(1 + \xi)(1 - \hat{\eta}^2), \\
1\hat{\psi}_{34} &= \frac{1}{2}(1 - \xi^2)\hat{\eta}(1 + \hat{\eta}) \\
2\hat{\psi}_{41} &= \frac{1}{2}\xi(\hat{\xi} - 1)(1 - \hat{\eta}^2),
\end{align*}
\]

and that each \( \hat{q} \in P_1(\hat{\Omega}) \) is of the form

\[
\hat{q} = q_0 + q_1 \hat{x} + q_2 \hat{y}
\]

where \( q_x, x = 0, 1, 2 \) are real numbers. A simple calculation reveals that

\[
\mathcal{I} = q_0 \oint_{\partial \hat{\Omega}} (\hat{\epsilon}_h - \hat{\epsilon}) \cdot \hat{n} \, \hat{d}s
\]

\[
+ q_1 \left[ - \int_{\hat{\Omega}} (\hat{\epsilon}_{h1} - \hat{\epsilon}_1) \, \hat{x} \, \hat{d}y + \oint_{\partial \hat{\Omega}} \hat{x}(\hat{\epsilon}_h - \hat{\epsilon}) \cdot \hat{n} \, \hat{d}s \right] + q_2 \left[ - \int_{\hat{\Omega}} (\hat{\epsilon}_{h2} - \hat{\epsilon}_2) \, \hat{d}x \, \hat{d}y + \oint_{\partial \hat{\Omega}} \hat{y}(\hat{\epsilon}_h - \hat{\epsilon}) \cdot \hat{n} \, \hat{d}s \right] \tag{5.33}
\]
It is clear that we can make \( \tilde{I} = 0 \) by choosing \( \hat{e}_h \) (equivalently, choosing a \( u_h \)) such that the following five conditions hold:

(i) \[ \hat{e}_h(\hat{a}^i) = 0, \quad 1 \leq i \leq 4 \]

(ii) \[ \hat{e}_h(a^{ij}) \hat{e}^{ij} = 0, \quad 1 \leq i < j \leq 4 \]

where \( \hat{e} \) is the unit vector tangent to \( \partial \Omega \)

(iii) \[ \int_{\partial \Omega} \hat{e}_h \cdot \hat{n} \, ds = \int_{\partial \Omega} \hat{e} \cdot \hat{n} \, ds, \quad 1 \leq i < j \leq 4 \]

(iv) \[ -\int_{\partial \Omega} \hat{e}_h \hat{r} \cdot d\hat{r} + \int_{\partial \Omega} \hat{e}_h \\cdot \hat{n} \, ds = -\int_{\partial \Omega} \hat{e}_1 \hat{r} \cdot d\hat{r} + \int_{\partial \Omega} \hat{e}_1 \\cdot \hat{n} \, ds \]

(v) \[ -\int_{\partial \Omega} \hat{e}_h \hat{r} \cdot d\hat{r} + \int_{\partial \Omega} \hat{e}_h \\cdot \hat{n} \, ds = -\int_{\partial \Omega} \hat{e}_2 \hat{r} \cdot d\hat{r} + \int_{\partial \Omega} \hat{e}_2 \\cdot \hat{n} \, ds \]

This set of conditions must determine the 18 independent components of \( \hat{e}_h(\hat{e}_h, \hat{e}_h, \hat{e}_h) \).

Conditions (i) and (ii) make 12 of the 18 degrees of freedom of \( \hat{e}_h \) zero. We are left with six coefficients:

\[
\hat{e}_{h1} = \hat{e}_{h1}(a^{5})\hat{v}_5 + \hat{e}_{h1}(a^{23})\hat{v}_{23} + \hat{e}_{h1}(a^{14})\hat{v}_{14} \\
\hat{e}_{h2} = \hat{e}_{h2}(a^{5})\hat{v}_5 + \hat{e}_{h2}(a^{12})\hat{v}_{12} + \hat{e}_{h2}(a^{34})\hat{v}_{34}
\]

But four of these coefficients are immediately determined from (iii) by a direct integration:

\[
\hat{e}_{h2}(a^{12}) = \frac{3}{4} \int_{a_1}^{a_2} \hat{e}_2 \, d\hat{s}; \quad \hat{e}_{h1}(a^{23}) = \frac{3}{4} \int_{a_1}^{a_2} \hat{e}_1 \, d\hat{s} \\
\hat{e}_{h2}(a^{34}) = -\frac{3}{4} \int_{a_3}^{a_4} \hat{e}_2 \, d\hat{s}; \quad \hat{e}_{h1}(a^{14}) = -\frac{3}{4} \int_{a_3}^{a_4} \hat{e}_1 \, d\hat{s}
\]

Thus, it remains only to determine \( e_{h_i}(a^{5}) \), \( i = 1, 2 \), using the last two conditions (iv) and (v). But a direct calculation leads to the pair of equalities

\[
\frac{16}{9} \hat{e}_{h1}(a^{5}) = \int_{\Omega} \hat{e}_1 \hat{r} \cdot d\hat{r} + \int_{a_1}^{a_2} \hat{e}_2 \cdot d\hat{s} - \frac{1}{3} \int_{a_1}^{a_3} \hat{e}_1 \, d\hat{s} \\
- \int_{a_3}^{a_4} \hat{e}_2 \cdot d\hat{s} - \frac{1}{3} \int_{\Omega} \hat{e}_1 \, d\hat{s} \\
\frac{16}{9} \hat{e}_{h2}(a^{5}) = \int_{\Omega} \hat{e}_2 \hat{r} \cdot d\hat{r} - \frac{1}{3} \int_{a_1}^{a_2} \hat{e}_2 \cdot d\hat{s} - \int_{a_3}^{a_4} \hat{e}_1 \, d\hat{s} \\
+ \frac{1}{3} \int_{a_3}^{a_4} \hat{e}_2 \cdot d\hat{s} - \int_{a_1}^{a_3} \hat{e}_1 \, d\hat{s}
\]
Hence, conditions (i)–(v) determine a vector $\hat{e}_h$ for which $\hat{I} = 0$, as required. We easily verify that $e_h \in V^h$.

It remains to be proven that the vector $u_h = e_h + w_h$ satisfies (5.23)$_2$. We note that it is sufficient to prove that

$$\| e_h \|_1 \leq C_1 \| \hat{e} \|_1$$

because

$$\| u_h \|_1 - \| w_h \|_1 \leq \| u_h - w_h \|_1 \leq C_1 \| v_q - w_h \|_1 \leq C_1 (\| v_q \|_1 + \| w_h \|_1)$$

so, since

$$\| w_h \|_1 \leq \| v_q \|_1, \quad \| u_h \|_1 \leq (1 + 2C_1) \| v_q \|_1 \leq C \| q_h \|_0$$

To establish (5.34), we note that for the master element*,

$$\hat{e}_h = \sum_{i=1}^{9} \hat{e}_h(b_i) \hat{y}_i, \quad \| \hat{e}_h \|_{1,\Omega}^2 \leq C \sum_{i=1}^{9} \| \hat{e}_h(b_i) \|^2$$

(5.35)

where $\{b_i\}$ are the nine nodes of the element and $\| \cdot \|$ denotes the euclidean norm in $\mathbb{R}^2$. Using the fact that

$$\left| \int_{a}^{b} \hat{e} \cdot \hat{n} \, ds \right| \leq C \| \hat{e} \|_{0,\partial\Omega} \leq C \{ \| \hat{e} \|_{0,\hat{\Omega}}^2 + \| \hat{e} \|_{1,\hat{\Omega}}^2 \}^{1/2}$$

and the previously computed nodal values of $\hat{e}_h$ obtained via steps (i)–(v) above, we can verify that $\| \hat{e}_h(b_i) \|^2 \leq C \| \hat{e} \|_{0,\hat{\Omega}}^2 + \| \hat{e} \|_{1,\hat{\Omega}}^2$. Thus, a constant $C$ exists such that

$$\| \hat{e}_h \|_{1,\hat{\Omega}} \leq C \{ \| \hat{e} \|_{0,\hat{\Omega}}^2 + \| \hat{e} \|_{1,\hat{\Omega}}^2 \}^{1/2}$$

(5.36)

We next transform this result so that it applies to a typical element $\Omega_e$ of the mesh and sum over all elements to obtain

$$\| e_h \|_{1,\Omega} \leq C \{ \| \hat{e} \|_{0,\hat{\Omega}}^2 + \| \hat{e} \|_{1,\hat{\Omega}}^2 \}^{1/2}$$

(5.37)

In this last calculation, we used the affine map $F_e$ of (5.29) the fact that $\| T_\epsilon \| \leq C_1 h$, $\| T^{-1}_\epsilon \| \leq C_2 h^{-1}$, and standard relations between $\| \hat{e} \|_{1,\hat{\Omega}}$ and $\| e \|_{1,\Omega}$. We shall next verify that

$$\| e \|_0 \leq C h \| e \|_1$$

(5.38)

We will then arrive at (5.34) via (5.37) and thereby complete the proof of the theorem.

* Here and elsewhere in this paper, $C$ denotes a generic constant independent of $h$ and does not necessarily have the same value throughout.
To prove (5.38), we employ a duality argument of Girault and Raviart (12). Note that
\[
\|e_i\|_{0,\Omega} = \sup_{v \in L^2(\Omega)} \frac{(e_i, v)}{\|v\|_{0,\Omega}}, \quad i = 1, 2 \tag{5.39}
\]

Let \(g\) be in \(L^2(\Omega)\) and \(\phi_g\) be the solution to the Dirichlet problem
\[
\begin{cases}
-\Delta \phi_g = g \\
\phi_g|_{\partial \Omega} = 0
\end{cases}
\tag{5.40}
\]
Then
\[
\phi_g \in H^2(\Omega) \cap H^1_0(\Omega) \quad \text{and} \quad \|\phi_g\|_{2,\Omega} \leq C \|g\|_{0,\Omega} \tag{5.41}
\]

The variational formulation for the problem (5.40) is:
\[
(\phi_g, v)_{1,\Omega} = (g, v)_{0,\Omega} \quad \forall v \in H^1_0(\Omega)
\]

It is permissible to take \(v = e_i\) so that \((\phi_g, e_i)_{1,\Omega} = (g, e_i)_{0,\Omega}\). But \(e_i\) is orthogonal to \(V^h\); hence, \((v_h, e_i)_{1,\Omega} = 0, \forall v_h \in V^h\).

It follows that \((e_i, g)_{0,\Omega} = (e_i, \phi_g - v_h)_{1,\Omega} \forall v_h \in V^h\) and \(\|(g, e_i)_{0,\Omega}\| \leq \|e_i\|_{1,\Omega}\) \(\phi_g - v_h\|_{1,\Omega} \forall v_h \in V^h\).

Choosing \(v_h = \phi^h\) to be the interpolant in \(V^h\) of \(\phi_g\), we have
\[
\|\phi_g - \phi^h\|_{1,\Omega} \leq C_h \|\phi\|_{2,\Omega} \leq C_h \|g\|_{0,\Omega}
\]
Hence,
\[
\frac{|(g, e_i)|}{\|g\|_{0,\Omega}} \leq C h \|e_i\|_{1,\Omega} \quad \text{and by (5.7),} \quad \|e\|_{0,\Omega} \leq C h \|e\|_{1,\Omega}
\]
This completes the proof of the theorem. \(\blacksquare\)

### 5.6 NUMERICAL EXAMPLES

The results of several numerical experiments are described which are designed to verify the theoretical results with regard to the \(Q_2/P_1\), \(18/P_1\), and the composite elements described earlier. We consider an L-shaped domain \(\Omega\) partitioned into 64 square subdomains, as shown in Figure 5.3. The fluid is subjected to a constant body force \(f = (0, -100)\). We take \(v = 333\), and the penalty parameter \(\varepsilon = 10^{-5}\). We will be interested in the computed hydrostatic pressure across the Section \(AA'\) defined by: \(y = 0.80\). Each sub-domain corresponds to a finite element; the velocity on each element is interpolated at eight and nine nodes and the pressure by its value at three points. Thus, various choices of how to handle the ninth node lead to meshes with \(18/P_1\), \(Q_2/P_1\) or Composite/P_1 elements. We will be interested in three cases involving these elements:

- **Mesh 1**: All the elements are \(Q_2/P_1\) elements.
- **Mesh 2**: All the elements are \(18/P_1\) elements.
Figure 5.3 A mesh of 64 elements on an L-shaped domain

Figure 5.4 Mesh with composite elements
Mesh 3: Adding 16 centroid nodes, we obtain 16 composite elements as shown in Figure 5.4.

The results reported here were obtained using the FIDAP code for problems of incompressible viscous flow (13).

Figures 5.5 and 5.6 show the comparison between the results obtained with the $Q_2/P_1$ element (Mesh 1) and those obtained with Meshes 2 and 3. Figure 5.5 illustrates the major difference between the $Q_2/P_1$ and the $18/P_1$ element: the former involves a pressure which seems to be smoothly distributed along the section AA' while the latter yields a pressure with severe oscillations. We note, however, that the values of the pressure obtained at the centroid of each element are close to the values obtained with the $Q_2/P_1$ element, which suggest that this unstable solution can be stabilized by a filtering operation which effectively uses these averaged values of pressure.

It is also remarked that the oscillations seem to come from the spurious

![Figure 5.5 Computed pressure profile along Section AA'](image)
modes in $\ker B_h^*$. The smoothing devise may be equivalent to an a posteriori elimination of these spurious modes.

Finally, the composite elements lead to a quite smooth solution as indicated in Figure 5.5, which is close to the solution obtained with nine-node elements, except that for this element $h^2 = 0.25$, while for the $Q_2/P_1$ element $h^2$ was equal to 0.0625.

We also note that when the body force $f$ derives from a potential: $f = -\nabla v$, then the unique solution for the Stokes Problem is

$$\begin{cases} 
  u = 0 \\
  p = -v 
\end{cases}$$

In this example, $f = (0, -100)$ and $v = 100y = -p$.

The numerical results obtained by different methods are summarized in the Table 5.1.
Table 5.1: Norm-evaluation obtained by:

<table>
<thead>
<tr>
<th>Method</th>
<th>Method 1: $Q_2/P$, elements.</th>
<th>Method 2: composite elements.</th>
<th>Method 3: $18/P_1$ elements and filtering the pressures by using only the centroidal value.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$|p|_{L^2(\Omega)} = 100 \sqrt{\frac{12 \pi}{13}} \approx 175.5942; h^2 = 0.0625$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Method 1</td>
<td>167.1254</td>
<td>20.0310</td>
<td>0.1141</td>
</tr>
<tr>
<td>Method 2</td>
<td>171.5448</td>
<td>36.3181</td>
<td>0.2068</td>
</tr>
<tr>
<td>Method 3</td>
<td>171.5845</td>
<td>26.6219</td>
<td>0.1516</td>
</tr>
</tbody>
</table>

As a second example, we consider a Dirichlet Stokes Problem, which is designed for numerical verification of the convergence theory for three schemes considered:

(a) Solution obtained with $Q_2/P_1$ elements.
(b) Solution obtained with $18/P_1$ elements.
(c) Solution obtained by averaging pressures of the $18/P_1$-scheme.

We consider the unit square domain partitioned in square subdomain, and the following body forces $f = (f_1, f_2)$ are applied:

$$f_1 = -4y + 12x^2 + 24xy + 12y^2 - 24x^3 - 48x^2y - 72xy^2$$
$$-8y^3 + 12x^4 + 48x^3y + 72x^2y^2 + 48xy^3 - 24x^4y - 48x^2y^3$$
$$-2(x - x_0) + \alpha(x)$$

$$f_2 = 4x - 12x^2 - 24xy - 12y^2 + 8x^3 + 72x^2y + 48xy^2 + 24y^3$$
$$-12y^4 - 48xy^3 - 72x^2y^2 - 48x^3y + 24xy^4 + 48x^3y^2$$

where $\alpha(x) = -1$ if $0 \leq x \leq x_0$, $\alpha(x) = 1$ if $x_0 < x \leq 1$. Then $f(u, p)$ is defined by

$$u = (u_1, u_2); \begin{cases} u_1 = x^2(1-x)^2(2y-6y^2+4y^3) \\ u_2 = (2x + 6x^2 - 4x^3)y^2(1 - y)^2 \end{cases}$$

and

$$\begin{cases} p = x_0 - x - (x - x_0)^2 & \text{if } 0 \leq x \leq x_0 \\ p = x - x_0 - (x - x_0)^2 & \text{if } x_0 < x \leq 1 \end{cases}$$

$(u, p)$ satisfies:

$$\begin{cases} \text{null} & \text{in } \Gamma \\ \text{div } u = p & \text{in } \Omega \\ -\Delta u + p = f & \text{in } \Omega \end{cases}$$

As before, we are interested by the plot of the pressure across a section of the domain. Figure 5.7 shows the results obtained by partitioning the domain $\Omega$ in 64
square subdomains. For this mesh, $h$ is equal to one-eighth. The computations are made with $Q_2$-on 18-elements. Whereas the $Q_2$-solution seems to be stable, clearly the 18-solution shows oscillations around the exact solution. However, it is noted that both solutions coincide at the centroid of the elements and this again suggests that the 'smoothed 18-solution', obtained using only the pressure at the centroid, is stable, and may converge at a rate of $O(h^2)$.

Finally, Figure 5.8 confirms this suspicion showing the computed rate of convergence is precisely $O(h^2)$ for the pressure for the $Q_2$-element, and for the smoothed 18-element. However, it is also observed that the $Q_2/P_1$ pressures are considerably more accurate than the filtered 18/$P_1$-pressures for all mesh sizes considered.
With the results from these examples we can conclude that

- The $Q_2/P_1$ element is stable and the optimal $L^2$-rate of convergence of the pressures of $O(h^2)$ is attained.
- The $18/P_1$ element yields unstable pressure approximations, but these can apparently be stabilized considering only the values at the centroids.
- Spurious oscillations (checkerboarding) can also appear when $\ker B^*_h = \mathbb{R}$.
- Filtering the pressures in the $18/P_1$-element by using only the centroidal value leads to a pressure approximation which may converge in $L^2$ at a rate of $O(h^2)$; however, the accuracy of the filtered scheme is quite inferior to that of the $Q_2/P_1$-elements.

These computed results underline once again the critical role played by the LBB-condition in studying the stability of finite element schemes by reduced integration. These and other results we have computed also indicate that the estimates obtained in Section 4 for the discrete LBB-constant $\alpha_h$ are sharp. Indeed, the theoretical result that the use of a composite element of the type employed here leads to a stable pressure field, while not of great practical value, is fully confirmed by the numerical results. This suggests again that these calculated estimates of $\alpha_h$ are a good indication of the actual numerical performances of these methods.

REFERENCES

10. D. S. Malkus, and T. J. R. Hughes. ‘Mixed finite element methods — Reduced and


