GENERALIZED POTENTIALS IN
FINITE ELASTOPLASTICITY

S. J. KIM and J. T. ODEN
Department of Aerospace Engineering and Engineering Mechanics,
The University of Texas at Austin, TX 78712, U.S.A.

Abstract—This paper deals with the development of a thermodynamic basis for finite elasto-
plasticity which does not require assumptions of convexity and/or differentiability of various free
energy or stress potential functionals. The approach involves an extension of the thermodynamic
arguments of Coleman and Noll to non-differentiable non-convex energies, and thereby overcomes
some of the traditional objections of applying this apparatus to the study of finite plasticity.

1. INTRODUCTION

Classical plasticity theory is, in many ways, a product of inductive thought, growing as it
did from attempts to model observed behavior of metals and soils under loading histories
sufficient to create “permanent” deformation. This approach toward the development of a
mechanical theory is quite different from the deductive methods of modern continuum
mechanics, where the framework for theories of material behavior is derived in a semi-
axiomatic way from a small collection of universal postulates (laws of physics). Over the last
fifteen years, there have been numerous attempts to provide a thermodynamic basis for a
plasticity theory sufficiently general to accommodate finite deformations but, at the same
time, not inconsistent with either continuum thermodynamics or classical plasticity. While
this volume of literature is too large to be adequately referenced here, we mention as
significant examples, the works of Green and Naghdi[1], Perzyna[2], Coleman and
Gurtin[3], Valanis[4, 5], Eringen[6], Lee[7, 8], Halphen and Nguyen[9] and Nemat-
Nasser[10, 11]; for a more complete list of references in this general area, the proceedings
edited by Lee and Mallett[12], Desai and Gallagher[13] and Nemat-Nasser[14] can be
consulted.

While important advances have been made in generalizing the theory of plasticity, many
approaches that have been put forth involve unnecessary assumptions reflecting a bias
toward either classical plasticity (e.g. assumptions of the existence of a yield function,
convexity of stress potentials, etc.) or a bias toward the use of manipulations prevalent in
modern continuum mechanics (e.g. assumptions of differentiability of free energies or stress
potentials, or implied assumptions that strain rates derived from stress potentials are
kinematically well-defined).

Our aim in the present paper is to demonstrate a theory of finite elastoplasticity which
makes use of standard continuum thermodynamics arguments, which does not require
assumptions of convexity or differentiability of various functionals, which addresses issues
of proper decomposition of various “elastic” and “plastic” deformation measures in a
consistent way, and which, under appropriate additional assumptions, reduces to a theory
capable of describing infinitesimal deformations of elastoplastic solids.

The principal mathematical machinery needed for the general techniques we employ is
the theory of non-convex optimization and generalized subdifferentials advanced by
Clarke[15, 16, 17] and Rockafellar[18, 19]. Panagiotopoulos[20] observed that these ideas
had application to certain plasticity problems. We outline the key mathematical concepts in
the section following this introduction and provide a brief review of relevant ideas from
convex analyses in an appendix. These ideas are then used in the formal study of thermo-
dynamic restrictions on the form of constitutive functions characterizing a general class of
materials. We then introduce generalized stress potentials, which may be neither convex nor
differentiable, and which, together with the thermodynamic restrictions provide for a very
general thermomechanical theory which includes classical elastoplasticity as a special case.
We also point out some difficulties which arise when one uses some of the more popular
decompositions of the deformation gradient $F$ proposed in earlier treatments of this subject.
We conclude the work with an example of a material which satisfies the conditions of our
theory and which models a class of elastoplastic materials.

2. MATHEMATICAL PRELIMINARIES AND NON-CONVEX ANALYSIS

A review and summary of many of the concepts of convex analysis are given in the
Appendix. Here we shall record extensions of these concepts to non-convex problems
following the ideas of Clarke [15] and Rockafellar [2]. Unless noted otherwise, $V$ denotes a
topological vector space, $V^*$ the topological dual of $V$ and $\langle \cdot, \cdot \rangle$ duality pairing on $V^* \times V$.

Contingent and tangent cones

Let $K$ be a nonempty subset of a topological vector space $V$. Then the contingent cone
to $K$ at a point $u \in K$ is defined as the set

$$C_K(u) = \limsup_{\delta \to 0^+} \frac{1}{\delta} [K - u].$$

(2.1)

Likewise, the tangent cone to $K$ at $u$ is defined as the set

$$T_K(u) = \liminf_{\delta \to 0^+} \frac{1}{\delta} [K - u].$$

(2.2)

To interpret the notation used in (2.1) and (2.2), we use the concept of a limit superior
(inferior) of a multifunction defined on a topological vector space in the Appendix. Let
$\Gamma$ be a set-valued function from $[0, \infty) \times K$ into $V$ such that

$$\Gamma(\theta, u) \equiv \frac{1}{\theta} (K - u) = \left\{ v \in V | v = \frac{1}{\theta} (w - u), w \in K, \theta \in [0, \infty) \right\}$$

(2.3)

Then

$$C_K(u) = \limsup_{\theta \to 0^+} \Gamma(\theta, u)$$

$$= \bigcup_{\theta \in \mathcal{N}(0)} \bigcap_{\lambda > 0} [\Gamma(\theta, u) + A]$$

(2.4)

and

$$T_K(u) = \liminf_{\theta \to 0^+} \Gamma(\theta, u')$$

$$= \bigcap_{\theta \in \mathcal{N}(0)} \bigcup_{\theta \in \mathcal{N}(0)} [\Gamma(\theta, u') + A]$$

(2.5)

where $\mathcal{N}(0)$ and $\mathcal{N}(u)$ denoted collections of neighborhoods of 0 and $u$, respectively.

To visualize $C_K(u)$ and $T_K(u)$, we note that for $K \subset \mathbb{R}^N$,

$$C_K(u) = \{ v \in \mathbb{R}^N | \exists \theta_k \to 0^+, v_k \to v, \text{ such that } u + \theta_k v_k \in K \}$$

(2.6)

$$T_K(u) = \{ v \in \mathbb{R}^N | \forall \theta_k \to 0^+, u_k \to u, \text{ with } u + \theta_k v_k \in K \}$$

(2.7)
A two-dimensional case is illustrated in Fig. 1. Suppose \( u \) terminates at a cusp in a non-convex set \( K \), as shown. The entire plane can be represented as the union of four cones with vertex \( u \): BOD, DOC, COA, and AOB, with 0 the terminix of \( u \). Clearly, for any point \( v \) inside the cone BOD U DOC U COA, it is always possible to find a sequence of positive numbers \( \{\theta_k\} \) such that \( u + \theta_k v_k \in K \) of any sequence \( v_k \rightarrow v \). Outside of this cone (interior to AOB), it is impossible to find \( \{\theta_k\} \) for which \( u + \theta_k v_k \in K \). Hence,

\[
C_K(u) = BOD \cup DOC \cup AOB
\]

Similarly, pick a sequence \( u_k \rightarrow u \) where \( u_k \) is a sequence of vectors tracing out the arc \( EO \) on \( K \). The legitimate vectors \( v \) with sequences \( v_k \rightarrow v \) such that \( u_k + \theta_k v_k \in K \) as \( \theta_k \rightarrow 0 \) will be those in the half-space BOD U DOC. Similarly, for \( u_k \) approaching \( u \) along FO, we must choose \( v \) in DOC U COA. All other sequences \( u_k \in K, u_k \rightarrow u \) yield acceptable \( v \) in either of these half spaces. Thus, \( T_K(u) \) must represent the intersection:

\[
T_K(u) = DOC.
\]

If \( K \) is convex, then

\[
T_K(u) = C_K(u).
\]

**Normal cone**

For \( K \subset V, K \neq \emptyset \), the normal cone to \( K \) at \( u \) is a subset of the dual space \( V^* \) defined by

\[
N_K(u) = \{u^* \in V^* | \langle u^*, v \rangle \leq 0 \quad \forall v \in T_K(u) \}\]

(2.8)
Fig. 2. Normal and tangent cones at points of a set $K$.

In two dimensions, $N_{k}(u)$ consists of the vectors through $u$ which make obtuse angles with the vectors in $T_{k}(u)$, as shown in Fig. 2.

**Clarke–Rockafellar derivatives**

Let $F$ be any extended real-valued function defined on $V$ and let $F$ be finite at a point $u \in V$. Then various types of subderivatives of $F$ at $u$ can be defined as follows.

- **Upper subderivative.** The upper subderivative of $F$ at $u$ in direction $v$ is defined as

$$D^+F(u; v) = \limsup_{\theta \to 0^+} \frac{1}{\theta} \inf_{\theta \to 0^+} \{ F(u' + \theta v') - \alpha \}. \quad (2.9)$$

The notation $\limsup_{\theta \to 0^+}$ is defined in the Appendix, and by the notation

$$(u; \alpha) \downarrow (u, F(u))$$

we signify the convergence of a sequence $(u', \alpha) \in epi F$ to a point on the graph of $F$:

$$(u', \alpha) \downarrow (u, F(u)) \iff (u', \alpha) \to (u, F(u)), \quad \alpha \geq F(u').$$

If $F$ is l.s.c. (lower semicontinuous) at $u$, then (2.8) reduces to

$$D^+F(u; v) = \limsup_{\theta \to 0^+} \frac{1}{\theta} \inf_{\theta \to 0^+} \{ F(u' + \theta v') - F(u') \}. \quad (2.10)$$

- **Lower subderivative.** The lower subderivative of $F$ at $u$ in direction $v$ is defined by

$$D^\downarrow F(u, v) = \liminf_{\theta \to 0^+} \sup_{(u', \alpha) \in epi F} \frac{1}{\theta} \{ F(u' + \theta v') - \alpha \}. \quad (2.11)$$
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with

\[ (u', \alpha) \uparrow (u, F(u)) \iff (u', \alpha) \to (u, F(u)) \]
\[ \alpha \leq F(u'). \]

If \( F \) is u.s.c. at \( u \),

\[ D \downarrow F(u; v) = \lim \inf \sup_{(u, F(u'))=0, F(u) \to v} \frac{1}{\theta} F(u' + \theta v') - F(u'). \]

The derivatives \( D \downarrow F(u; v), D \uparrow F(u; v) \) are referred to here as Clarke–Rockafellar derivatives (or \( C/R \)-derivatives for compactness).

To help understand the meaning of the \( C/R \)-derivatives, we consider the two examples shown in Figs. 3 and 4.

**Example 1.** Let \( f_1 \) and \( f_2 \) denote the real-valued functions,

\[
 f_1(x) = \begin{cases} 
 x^2 - 4x + 4 & x \leq 1 \\
 -x^2/2 + 2x - 1/2 & x \geq 1 
\end{cases}
\]

and

\[
 f_2(x) = \begin{cases} 
 x^2 - 4x + 1 & x \leq 1 \\
 -x^2/2 + 2x + 3/2 & x \geq 1 
\end{cases}
\]

Fig. 3. The upper (lower) subderivatives of (a) continuous but nondifferentiable and (b) discontinuous at \( x = 1 \).
As we observe in Fig. 3, \( f_1(x) \) is continuous at \( x = 1 \) but nondifferentiable in classical sense and \( f_2(x) \) is discontinuous but lower semicontinuous at \( x = 1 \). Since these functions are defined on \( \mathbb{R} \), the definition of \( C/\mathbb{R} \)-derivatives reduces to

\[
D_{\mathbb{R}} f(x; y) = \lim_{\theta \to 0^+} \sup_{(x', \theta y)} \frac{1}{\theta} |F(x' + \theta y) - F(x')|
\]

and

\[
D_{\mathbb{R}} f(x; y) = \lim_{\theta \to 0^+} \inf_{(x', \theta y)} \frac{1}{\theta} |F(x' + \theta y) - F(x')|
\]

Around \( x = 1 \), we have two subsequences of derivatives with \( y = 1 \) which converges to 1 and \( -2 \) for both \( f_1 \) and \( f_2 \) and we easily conclude that

\[
D_{\mathbb{R}} f_1(1; 1) = 1 \quad (= \sup \{1, -2\})
\]

\[
D_{\mathbb{R}} f_2(1; 1) = 1
\]

\[
D_{\mathbb{R}} f_1(1; 1) = -2
\]

\[
D_{\mathbb{R}} f_2(1; 1) = -2.
\]

We remark that the convergence \((x', y) \downarrow (x, f(x))\) in the definition of \( C/\mathbb{R} \) derivatives (2.8) and the definition of tangent cone (2.2) lead to the fact that the epigraph of the function \( y \to D_{\mathbb{R}} f(x; y) \) is the tangent cone \( T_{\mathbb{R}}(x, f(x)) \) which is shown in Fig. 3(a).

**Example 2.** A better example of \( C/\mathbb{R} \) derivatives can be constructed in \( \mathbb{R}^2 \). Consider the lower semicontinuous function \( f = f(x, y) \) shown in Fig. 4. The numbers indicated in the figure are intended to mean the following:
The slope of line $GH$ at $G$ in the direction $v = (1, 0)$ is $+0.5$.
The slope of line $EF$ at $F$ in the direction $v = (1, 0)$ is $-0.5$.
The slope of line $CD$ at $C$ in the direction $v = (1, 0)$ is $-0.8$.
The slope of line $AB$ at $B$ in the direction $v = (1, 0)$ is $+0.2$.

Let us calculate $C/R$ derivative at the origin $u = (0, 0)$ with direction $v = (1, 0)$. Recall that

$$D f((0, 0); (1, 0)) = \limsup_{\theta \to 0^+} \inf \left\{ \frac{1}{\theta} \left( f(u' + \theta v) - f(u') \right) \right\}.$$ 

Along the direction of $v = (1, 0)$, we choose two sequences approaching $u = (0, 0)$ from either positive or negative $x$-axis: $\liminf Df(u', v')$ taken from the positive side of $x$-axis will be a sequence of slopes along $EF$ and $\liminf Df(u', v')$ taken from the negative side of $x$-axis will be the slopes along $DC$, where $Df(u', v') = (1/\theta) [f(u' + \theta v) - f(u)]$. Next we take $\limsup$ of the sequences of slopes and then finally the slope at $F$ along the curve $EF$ as $D f((0, 0); (1, 0))$. Similarly, we can get

$$D f((0, 0); (1, 0)) = \text{the slope at } B \text{ along } AB.$$ 

Quantitatively

$$D f((0, 0); (1, 0)) = -0.5$$

and

$$D f((0, 0); (1, 0)) = +0.2.$$ 

**Generalized subdifferentials**

The subdifferential $\partial F(u)$ of a function $F$ at $u$ is a well known concept in convex analysis (see the Appendix for details). By using the subderivatives defined in (2.8) and (2.10), we define the generalized subdifferential of $F: V \to \mathbb{R}$, at a point $u$ where $F(u)$ is finite, as the set

$$\partial F(u) = \{ u* \in V^* | \langle u*, v \rangle \leq D f(u; v) \forall v \in V \}. \quad (2.13)$$

Now we list two useful theorems due to Rockafellar[4].

**Theorem 2.1.** Let $F$ be any extended real-valued function on $V$, and let $u$ be any point at which $F$ is finite. Then $\partial F$ is a weak*-closed convex subset of $V^*$ and

$$\partial F(u) = \{ u* \in V^* | (u*, -1) \in N_{\text{epi}}(u, F(u)) \}. \quad (2.14)$$

If $D f(u; 0) = -\infty$, then $\partial F(u)$ is empty, but otherwise $\partial F(u)$ is nonempty and

$$D f(u; v) = \sup \{ \langle u*, v \rangle | u* \in \partial F(u) \} \quad \text{for all } v \in V. \quad (2.15)$$

**Theorem 2.2.** If $F$ is a convex function on $V$, then $\partial F(u)$ agrees with the subgradient set in the sense of convex analysis:

$$\partial F(u) = \partial F(u) \quad \text{for all } u \in V$$

$$= \{ u* \in V^* | \langle u*, v \rangle \leq F(u) - F(v) \forall v \in V \}. \quad (2.16)$$
Here \( F'(u; v) = \lim_{t \to 0^+} (F(u + tv) - F(u))/t \) is called the one-sided directional derivative which exists for all \( v \) when \( F \) is convex (although it may be infinitive). ∎

**Remark.** If \( F(u) \) is a characteristic function with respect to a set \( K \), i.e. if

\[
F(u) = \psi_K(u) = \begin{cases} 
0 & \text{if } u \in K \\
+\infty & \text{if } u \notin K
\end{cases}
\]

then

\[
\partial F(u) = \{u^* \in V^* | \langle u^*, v \rangle \leq D \psi_K(u; v) \forall v \in V \} 
\]

This fact can be more easily visualized in the case of convex \( F \), i.e.

\[
\partial \psi_K(u) = \{u^* \in V^* | \langle u^*, v - u \rangle \\
\leq \psi_K(v) - \psi_K(u), \forall v \in K \} 
\]

\[
= \{u^* \in V^* | \langle u^*, v - u \rangle \leq 0, \forall v \in K \} 
\]

\[
= N_K(u)
\]

since \( \psi_K(v) = \psi_K(u) = 0 \) when \( u, v \in K \).

### 3. Mechanical Preliminaries

#### 3.1 Classical convex plasticity

Classical plasticity theory rests on the assumption of the existence of a convex function \( F: \mathcal{M} = \mathbb{R}^2 \times \mathbb{R}^3 \to [0, -\infty) \) of the stress tensor \( \sigma \), called the yield function of the material, which has the property that plastic flow at a particle \( X \) of the material is signaled whenever \( F(\sigma(X)) = 0 \); otherwise the deformation at \( X \) is elastic:

\[
F(\sigma(X)) \begin{cases} 
< 0 & \Rightarrow \text{elastic deformation} \\
= 0 & \Rightarrow \text{plastic flow.}
\end{cases}
\]

The only stress states admissible in such theories are those for which \( F(\sigma) \leq 0 \) or, equivalently, those stresses which belong to the convex set

\[
K = \{\sigma \in \mathcal{M} | F(\sigma) \leq 0\}.
\]

The infinitesimal strain tensor \( \varepsilon \) is representable as the sum of an elastic part \( \varepsilon^e \) and a plastic strain \( \varepsilon^p \), and its time rate-of-change is denoted \( \dot{\varepsilon} \):

\[
\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p.
\]

It is meaningful to assume the existence of a plastic stress potential \( \phi: \mathcal{M} \to \mathbb{R} \) which is convex and l.s.c. and which has the property that

\[
\dot{\varepsilon}^p \in \partial \phi(\sigma).
\]

In particular, the indicator function \( \psi_K \) of the set \( K \) may define a specific stress potential:

\[
\psi_K(\sigma) = \begin{cases} 
0 & \text{if } F(\sigma) \leq 0 \\
+\infty & \text{if otherwise.}
\end{cases}
\]

Note that \( \psi_K \) is l.s.c. on \( \mathcal{M} \) and that \( \psi_K \) is convex if \( F \) is convex.
From (3.4) and the definition of the subdifferential,
\[ \dot{\epsilon} \in \partial \psi_X(\delta) \]
for some particular stress \( \delta \) implies that
\[ \langle \dot{\epsilon}, \sigma - \delta \rangle \geq 0 \quad \forall \sigma \in K. \]  
This result, of course, is the classical *normality condition* which establishes that the strain rate is normal to the yield surface or lies in the normal cone of the yield surface at corners. (see Fig. 5).

### 3.2 Continuum thermodynamics of elasto-plastic materials

Our purpose in this subsection is to introduce notation and to review some aspects of continuum thermodynamics that are to be used later.

We begin by considering the motion of a material body \( \mathcal{B} \) relative to a fixed reference configuration \( \mathcal{C}_0 \subset \mathbb{R}^N \) (\( N \) typically 3), which is defined by the map \( \kappa_0: \mathcal{B} \rightarrow \mathbb{R}^N, \kappa_0 = \kappa_0(X) \). The spatial position \( x \) of a particle \( X \) at time \( t \) is then given by a relation of the type
\[ x = \chi(X, t) \]
with \( X \in \kappa_0(\mathcal{B}), t \geq 0 \), and \( \chi \) a continuous invertible map from \( \mathcal{C}_0 \) into \( \mathbb{R}^N \). The deformation gradient tensor \( F \) at \( X \) at time \( t \) is defined by
\[ F = F_{XX} = \frac{\partial \chi}{\partial X} \]

We now focus on the fundamental problem of defining an appropriate decomposition of \( F \) which distinguishes between the current deformed configuration and a "stress-free" state that might correspond to a state of permanent deformation. Let \( \mathcal{N}(X) \) denote an open neighborhood of a material point \( X \) in the reference configuration and let \( A \) denote an arbitrary fixed particle in \( \mathcal{N}(X) \) (see Fig. 6). The position vector of point \( A \) relative to the origin of the fixed spatial reference frame is denoted
\[ \overline{OA}_{C_0} = X + \Delta X \]
whereas the location of \( A \) in the current configuration \( C_t \),
\[ \overline{OA}_{C_t} = x + \Delta x = x(X, t) + F(X, t) \Delta X + \omega_t(X, t, \Delta X) \]
where
\[ \lim_{|\Delta X| \rightarrow 0} \frac{1}{|\Delta X|} |\omega_t(X, t, \Delta X)| = 0. \]
Let us now consider a configuration \( C_p \), intermediate to \( C_0 \) and \( C_\ell \), which may correspond to a "stress-free permanently deformed" state of the body under appropriate restrictions on the motion and material. In this case, we write

\[
\overline{OA_{C_p}} = x(X, t) + F^p(X, t) \Delta X + \omega_1(X, t, \Delta X) + F^2(X, t) \Delta X + \omega_2(X, t, \Delta X).
\]  

(3.12)

and \( \omega_2 \) has the same asymptotic behavior with respect to \( |\Delta X| \) as does \( \omega_1 \).

We next introduce a second order tensor \( F' \), defined by

\[
F'(X, t) \Delta X - \Delta X = \overline{OA_{C_i}} - \overline{OA_{C_p}}.
\]  

(3.13)

Thus, in the limit as \( |\Delta X| \to \infty \), we have

\[
F = F' + F'' - I.
\]  

(3.14)

Note that unlike Lee [7, 8], neither \( F' \) nor \( F'' \) need be the gradient of an "elastic" or "plastic" displacement or position vector field by formalizing the idea by Nemat-Nasser [11]. However, if \( u \) is the particle displacement field, we will always have,

\[
F = I + \nabla u.
\]  

(3.15)

When the representation (3.14) is employed, the velocity gradient tensor \( L \) is given by

\[
L = \frac{\partial \dot{x}}{\partial x} = \dot{F}F^{-1} = L' + L^p
\]

where

\[
L' = \dot{F}F^{-1} \quad \text{and} \quad L^p = F''F^{-1}.
\]  

(3.16)

Here superimposed dots (\( \dot{} \)) indicate time-rates.

The symmetric part of \( L \) is the deformation rate tensor \( D \) and is also representable as
the sum of two parts,

\[ D = \frac{1}{2} (L + L^T) = \frac{1}{2} (L' + L'^T + L'' + L''') \]

\[ = D' + D''. \quad (3.17) \]

The thermomechanical behavior of the body is governed by the principles of conservation of mass, energy, balance of linear and angular momentum, and the law of entropy production (the second law). Local forms of these principles are listed as follows:

Conservation of Mass

\[ \rho \det F = \rho_0. \quad (3.18) \]

Balance of Linear and Angular Momentum

\[ \text{div } \sigma + \rho b = \rho \dot{x} \]
\[ \sigma = \sigma^T. \quad (3.19) \]

Conservation of Energy

\[ \rho \dot{\epsilon} = tr(\sigma L) - \text{div } q + \rho r. \quad (3.20) \]

Clausius-Duhem Inequality

\[ \rho \dot{\eta} - \text{div } \frac{q}{\theta} - \frac{r}{\theta} \geq 0. \quad (3.21) \]

Here \( \rho \) is the mass density, \( \rho_0 \) the mass density in the reference configuration, \( \sigma \) is the Cauchy stress tensor, \( b \) to the body force per unit mass, \( \epsilon \) the specific internal energy, \( q \) the heat flux vector, \( r \) the heat supply per unit mass per unit time, \( \eta \) the specific entropy, and \( \theta \) the absolute temperature. The Helmholtz free energy is defined by

\[ \phi = \epsilon - \theta \eta \]

so that (3.22) can also be written in the form,

\[ -\rho \dot{\phi} - \rho \dot{\theta} \eta + tr[\sigma (L' + L'')] - \frac{1}{\theta} q \cdot \theta \geq 0. \quad (3.23) \]

It is well known that the Clausius-Duhem inequality can impose conditions on the forms of constitutive equations for the material of which the body is composed. For example, consider a class of materials characterized by a set of five constitutive equations of the form (see Coleman and Gurtin [3]):

\[ \begin{align*}
\phi &= \phi(F^\epsilon, \theta, g, \alpha) \\
\sigma &= \Sigma(F^\epsilon, \theta, g, \alpha) \\
\eta &= N(F^\epsilon, \theta, g, \alpha) \\
q &= Q(F^\epsilon, \theta, g, \alpha) \\
\dot{\alpha} &= H(F^\epsilon, \theta, g, \alpha) \\
D^\epsilon &= P(F^\epsilon, \theta, g, \alpha)
\end{align*} \quad (3.24) \]
where $g = \text{grad } \theta$, $\alpha$ is a tensor referred to as an internal state variable which is sometimes introduced to model changes in the microstructure of the material or to depict history effects via the evolution equation indicated above.

Thermodynamic restrictions on the forms of the constitutive functionals in (3.25) can be determined using the well-known strategy of Coleman and Noll[22]: we assume that the map

$$(F', \theta, g, \alpha) \rightarrow \Phi (F', \theta, g, \alpha)$$

is $C^1$ in each argument. The rate-of-change of the free energy satisfies

$$\dot{\Phi} = \frac{\partial \Phi}{\partial F'} \dot{F} + \frac{\partial \Phi}{\partial \theta} \dot{\theta} + \frac{\partial \Phi}{\partial g} \dot{g} + \frac{\partial \Phi}{\partial \alpha} \dot{\alpha}$$

wherein

$$\frac{\partial \Phi}{\partial F'} \dot{F} = \frac{\partial \Phi}{\partial F'} F' \quad [A : B \equiv \text{tr}(AB)]$$

$$\frac{\partial \Phi}{\partial g} \dot{g} = \frac{\partial \Phi}{\partial g} \dot{g}, \text{ etc.}$$

and we have used obvious indicial notation and the summation convention. Inequality (3.24) thus yields

$$\left( \sigma F^{-T} - \rho \frac{\partial \Phi}{\partial F'} \right) \dot{F} + L : \sigma + A : \dot{\alpha}$$

$$- \left( \kappa + \frac{\partial \Phi}{\partial \theta} \right) \dot{\theta} - \rho \frac{\partial \Phi}{\partial g} \dot{g} - \frac{1}{\theta} Q \cdot \text{grad } \theta \geq 0$$

(3.26)

where

$$A \equiv - \rho \frac{\partial \Phi}{\partial \alpha}.$$

(3.27)

From standard arguments, it follows that

$$\Sigma F^{-T} = \rho \frac{\partial \Phi}{\partial F'}$$

$$N = - \frac{\partial \Phi}{\partial \theta}$$

(3.28)

and we return the inequality

$$L' : \Sigma + A : H - \frac{1}{\theta} Q \cdot g \geq 0.$$

In these results, it is understood that $\Sigma, A,$ and $N$ now depend upon $(F', \theta, g, \alpha)$ and that $H$ and $Q$ are functions of $(F', \theta, g, \alpha)$.

4. MATERIALS OF TYPE $N$

Let $\mathcal{S}$ denote an $M$-dimensional linear space ($\mathcal{S} \approx \mathbb{R}^m$). If a functional $\Phi : \mathcal{M} \rightarrow [-\infty, \infty]$ has a nonempty generalized subdifferential $\partial \Phi(T)$ for $T \in \mathcal{S}$, then we are assured that there
exists a vector $a$ in the dual space $\mathcal{S}^*$ of $\mathcal{S}$ such that

$$a \in \partial \phi(T), \quad T \in \mathcal{S}$$

We wish to consider a hypothetical class of materials which can be partially characterized by potentials with the above property.

Let $\Sigma \subset \mathbb{R}^{3 \times 3}$ denote the space of stress values at a point $x \in \Omega$ at time $t$ and let $A \subset \mathbb{R}^{3 \times 3}$ denote the space of values of the thermodynamic force conjugate to the rate of the internal state variable $\dot{\alpha}$. We denote the space of stress-force pairs as

$$\mathcal{W} = \Sigma \times A.$$  

**Materials of type N.** A material is said to be of Type $N$ if and only if it is characterized by constitutive equations of the form (3.25) and there exists a potential function $\psi: \mathcal{W} \to (-\infty, \infty)$ which has a nonempty generalized subdifferential. Then, for any $(\sigma, A) \in \mathcal{W}$, there exists $(D^p, \dot{\alpha}) \in \mathcal{W}^*$ such that

$$(D^p, \dot{\alpha}) \in \partial \psi(\sigma, A). \quad (4.1)$$

From the results in Section 2, the relation (4.1) implies the inequality

$$\langle D^p, \sigma^* \rangle_\Sigma + \langle \dot{\alpha}, A^* \rangle_A \leq D \uparrow \psi((\sigma, A); (\sigma^*, A^*))$$

$$\forall (\sigma^*, A^*) \in \mathcal{W}^{*}. \quad (4.2)$$

($\langle \cdot, \cdot \rangle_\Sigma$ and $\langle \cdot, \cdot \rangle_A$ denote duality pairings on $\Sigma^* \times \Sigma$ and $A^* \times A$, respectively) and, geometrically, we have the normality (hence, $N-$) condition.

$$(D^p, \dot{\alpha}), -1) \in N_{\eta \psi} [(\sigma, A, \psi(\sigma, A)] \quad (4.3)$$

Furthermore, if potential $\psi$ is differentiable at $(\sigma, A)$ then $N_{\eta \psi}$ at $(\sigma, A)$ has a single element. So,

$$(D^p, \dot{\alpha}) = \frac{\partial \psi}{\partial (\sigma, A)}$$

or

$$D^p = \frac{\partial \psi}{\partial \sigma}$$

and

$$\dot{\alpha} = \frac{\partial \psi}{\partial A} \quad (4.4)$$

We remark that if $\psi$ is convex and l.s.c., the materials of type $T$ reduce to the generalized plastic materials introduced by Halphen and Nguyen[9].

**Remark.** At this point, it is important to appreciate possible difficulties inherent in certain decompositions of the deformation gradient proposed by other authors when it is assumed that $F'$ or $F^p$ are gradients of a corresponding displacement field. For example, if

$$F^p = 1 + \nabla u^p \quad (R.1)$$

In addition, suppose that we have the usual system of equations,

$$\text{Div} \sigma + \rho b = \rho \ddot{x}, \quad \sigma = \sigma^T, \quad \varphi = CD^p$$

$$(D^p, \dot{\alpha}) \in \partial \psi(\sigma, A), \quad A = -\rho \partial \phi / \partial \alpha, \quad \rho_0 = \rho \det F \quad (R.2)$$
where, for instance, $C$ is a constant tensor and $\dot{\sigma}$ denotes an appropriate stress rate. If $D'$ is a kinematical quantity [or $D'$ or both], we have, in addition to the above relation, kinematical conditions of the type

$$D' = \dot{g}(u,u'), \quad D = g(u), \quad x = X + u. \quad (R.3)$$

A count of the number of unknowns $(\rho, \sigma, D', D', \alpha, A, u, u, x)$ and equations reveals that we have 46 unknowns and 49 equations, i.e. the system is overdetermined.

The same problem arises if we choose $F = F'FP$ with either $F'$ or $F'$ defined as a gradient of a vector field.

When we follow the decomposition explained in section three, we have the following system of equations

$$u = x - X, \quad F = \frac{du}{dx} + 1, \quad F = F' + F' - 1$$

$$D' = \dot{F}'F^{-1}_{\text{sym}}, \quad D = \dot{F}'F^{-1}_{\text{sym}} \quad (R.4)$$

with eqns (R.2).

Here the number of unknowns $(\sigma, D', D', \alpha, A, u, F, F', x)$ is 70 and the number of equations is 67. The total rotation $R$ can be assigned to either $F'$ or $F'$. If it is assigned to $F'$, $F'$ can be taken to be symmetric (a stretch tensor) thus eliminating three unknowns and equalizing the number of equations and unknowns.

5. EXAMPLE

In this section, we will introduce a potential which is not necessarily convex and examine the properties of the potential.

An elastoplastic potential

We first introduce a function $F: \mathcal{W} \to \mathbb{R}$ and a set $C = \{(\sigma, A) \in \mathcal{W} | F(\sigma, A) \leq 0\}$. A family $\{\psi\}_{\epsilon > 0}$ of elastoplastic potentials are now introduced, where $\psi$ is of the form,

$$\psi (\sigma, A) = \begin{cases} -\frac{1}{F(\sigma, A)} & \text{if } (\sigma, A) \in \mathcal{A} \\ +\infty & \text{if } (\sigma, A) \notin \mathcal{A}. \end{cases} \quad (5.1)$$

Here $\epsilon$ is an arbitrary positive number.

These potentials characterize families of materials of type $N$. Hence

$$\nabla (D', A) \in \partial \psi(\sigma, A) \quad (5.2)$$

$$\Leftrightarrow \langle D', \sigma* \rangle_A + \langle A, \sigma* \rangle_A \leq D^\dagger \psi((\sigma, A), (\sigma*, A)) \quad (5.3)$$

We shall investigate how these properties can be justified mathematically and physically. First we consider some properties of the potential $\psi(\sigma, A)$.

Lemma 5.1 If a scalar valued function $F(x)$ is convex over $\mathbb{R}^n$ and $F(x) < 0$ for all $x \in C$, then the inverse of this function $G(x) = 1/F(x)$ is concave over $C$.

Proof. $F(x)$ is convex, i.e.

$$F(\lambda x + (1 - \lambda)y) \leq \lambda F(x) + (1 - \lambda)F(y) \quad \forall x, y \in C \quad \text{and} \quad \forall \lambda \in [0, 1]. \quad (5.4)$$
Now,

\[ R(x, y, \lambda) = G[\lambda x + (1 - \lambda)y] - \lambda G(x) - (1 - \lambda)G(y) \]

\[ = \frac{1}{D} \left\{ F(x)F(y) - \lambda F(y) P(x, y, \lambda) - (1 - \lambda)x P(x, y, \lambda) \right\} \]

where

\[ P(x, y, \lambda) = F[\lambda x + (1 - \lambda)y] \quad \text{and} \]

\[ D = P(x, y, \lambda)x F(x)F(y) < 0. \] (5.5)

Since \(-\lambda F(y)/D \leq 0\) and \(-(1 - \lambda)x F(x)/D \leq 0\), from (5.4), we have

\[ R(x, y, \lambda) \geq \frac{1}{D} \left\{ F(x)F(y) - \lambda^3 F(x)F(y) + (1 - 2\lambda + \lambda^2)F(x)F(y) \right\} + (\lambda - \lambda^2)(F^2(x) + F^2(y)) \]

\[ = -\frac{(\lambda - \lambda^2)}{D} \left\{ F(x) - F(y) \right\}^2 \]

\[ \geq 0 \quad \text{since} \quad -(\lambda - \lambda^2)/D \geq 0. \]

\[ \square \]

**Theorem 5.1** If the set \( C \) is convex, then the potential \( \psi_c(\sigma, \mathcal{A}) \) is convex. Otherwise, \( \psi_c \) is nonconvex.

**Proof**

(i) The set \( C \) is convex

\( \rightarrow F(\sigma, \mathcal{A}) \) is a convex function

\( \rightarrow 1/F \) is a concave function (from lemma 5.1)

\( \rightarrow -1/F \) is convex.

(ii) Take a point \( M \) along the line \( LN \) in the Fig. 7 with nonconvex set \( C \). \( F \) is infinity at \( M \) but values of \( f \) at \( L \) and \( N \) are finite. So it is not convex.

Next, we will establish that the proposed potential converges to the characteristic function \( \psi_c \) in an appropriate sense as \( \epsilon \to 0 \). To do this, we introduce the notion of infimal convergence of Wijsman's [28, 24].

**Definition 5.1**

(i) For any function \( g \) defined on \( \mathbb{R}^N \) and for any \( r < 0 \), define the function \( r^g \) on \( \mathbb{R}^N \) by

\[ r^g(x) = \inf_{y \in B_r(x)} g(y) \] (5.6)

where \( B_r(x) \) is the closed ball of radius \( r \) with center \( x \).
(ii) Let \( \{g_k; k = 1, 2, \ldots\} \) be a sequence of functions on \( \mathbb{R}^N \). The sequence is said to converge \textit{infimally} to a function \( g \) if, for every \( x \in \mathbb{R}^N \),

\[
\lim_{r \to 0} \liminf_{k \to \infty} g_k(x) = \lim_{r \to 0} \limsup_{k \to \infty} g_k(x) = g(x)
\]

and we write

\[ g_k \rightarrow_{\inf} g. \]

**Theorem 5.2** When \( \varepsilon \to 0 \), then \( \psi_r(v) \rightarrow \psi_r(v) \), where \( v \equiv \{\sigma, \mathcal{A}\} \), for \( C \neq \phi \).

**Proof.** To prove this theorem, it suffices to show that

\[
\lim_{r \to 0} \liminf_{\varepsilon \to 0} \psi_r(v) \geq 0 \quad \text{if} \quad v \in C \tag{5.7}
\]

\[
\lim_{r \to 0} \limsup_{\varepsilon \to 0} \psi_r(v) \leq 0 \quad \text{if} \quad v \in C \tag{5.8}
\]

and

\[
\lim_{r \to 0} \limsup_{\varepsilon \to 0} \psi_r(v) = +\infty \quad \text{if} \quad v \notin C. \tag{5.9}
\]

It is easy to prove (5.7) since

\[ \psi_r(v) \geq 0, \quad \forall v \in \mathcal{W} \]

from definition (5.1). So \( \psi_r(v) \geq 0, \forall v \in C \) which leads

\[
\lim_{r \to 0} \liminf_{\varepsilon \to 0} \psi_r(v) \geq 0.
\]

Next, for any \( r > 0 \), there is

\[ w \in C \cap B_r(v), \quad \forall v \in C. \]

By the definition (5.6),

\[ \psi_r(v) \leq \psi_r(w) \quad \forall \varepsilon. \]

Thus

\[
\lim_{\varepsilon \to 0} \sup_{r \to 0} \psi_r(v) \leq \lim_{\varepsilon \to 0} \psi_r(w)
\]

\[
= \lim_{\varepsilon \to 0} \varepsilon \cdot \frac{1}{-F(w)}
\]

\[
\leq \lim_{\varepsilon \to 0} \varepsilon M = 0
\]

where \( m \) is finite positive real number. Consequently,

\[
\lim_{r \to 0} \liminf_{\varepsilon \to 0} \psi_r(v) \leq 0.
\]

Finally, we assume the contrary to (5.9) i.e.

\[
\lim_{r \to 0} \liminf_{\varepsilon \to 0} \psi_r(v) < +\infty, \forall v \in C.
\]
but the openness of the set $\mathcal{C} = \{ v \in \mathcal{V} | v \in \mathcal{V} \}$ guarantees the existence of infinitely many $r$ which satisfy $0 < r < r_i$, where $r_i = \text{distance} (C, v)$ for $v \in \mathcal{C}$. In the limit as $r \to 0$, $r$ has to reach the region $0 < r < r_i$, where $\psi_i(v) = +\infty$. So none of the sequence can assume a finite value. Therefore

$$\lim_{r \to 0} \liminf_{r \to 0} \psi_i(v) = +\infty, \quad v \in \mathcal{C}.$$

**Remark.** In fact, the potential (5.1) is an interior penalty function for the yield constraint $F(\sigma, \mathcal{A}) \leq 0$ (see Oden and Kim [25, 26]). For details on convex interior penalty functions see Mine and Fukushima [27].

If $\psi_i$ is convex (i.e. $\mathcal{C}$ is convex), then from (2.15) we have the inequality

$$\langle \sigma^* - \sigma, D^p \rangle + \langle \mathcal{A}^* - \mathcal{A}, \dot{\alpha} \rangle \leq \psi_i(\sigma^*, \mathcal{A}^*) - \psi_i(\sigma, \mathcal{A})$$

\forall (\sigma^*, \mathcal{A}^*) \in \mathcal{V}.

Obviously, $(\sigma, \mathcal{A})$ should belong to $\mathcal{C}$. Then the inequality (5.10) need be true only for all $(\sigma^*, \mathcal{A}^*) \in C$, i.e.

$$\langle \sigma^* - \sigma, D^p \rangle + \langle \mathcal{A}^* - \mathcal{A}, \dot{\alpha} \rangle \leq \psi_i(\sigma^*, \mathcal{A}^*) - \psi_i(\sigma, \mathcal{A})$$

\forall (\sigma^*, \mathcal{A}^*) \in \mathcal{C}.

(5.11)

Furthermore, if we let $\epsilon \to 0$, then $\psi_i \to \psi_C$ and $\psi_C(\sigma^*, \mathcal{A}^*) = 0 = \psi(\sigma^*, \mathcal{A}^*)$. So, we recover the result of Halphen and Nguyen [8], i.e.

$$\langle \sigma^* - \sigma, D^p \rangle + \langle \mathcal{A}^* - \mathcal{A}, \dot{\alpha} \rangle \leq 0, \quad \forall (\sigma^*, \mathcal{A}^*) \in \mathcal{C}.$$

(5.12)

By using the potential in the form of interior penalty for the set $\mathcal{C}$, we can easily visualize the result (2.16) from (2.13).

The indicator function $\psi_C$ is nothing but the half cylinder, the cross-section of which is the set $\mathcal{C}$ shown in Fig. 8.

From the definition (2.13), $((D^p, \dot{\alpha}), -1)$ is in $\mathcal{N}_{\psi}(\sigma, \mathcal{A}), \psi_i(\sigma, \mathcal{A}))$. Upon allowing $\epsilon \to 0$, this element of the normal cone for $\psi_i$ will coincide with an element of the normal cone for $\psi_C$ and the projection of this vector to the space including set $\mathcal{C}$ (i.e. $\mathcal{V}$) gives

![Fig. 8. The potential functions $\psi_i$ and $\psi_C$.](image)
the result (2.16), i.e.
\[ \bar{\psi}_e(\sigma, \mathcal{A}) = N_e(\sigma, \mathcal{A}) = \left\{ (D, \theta) \in W^* \mid \langle D, \sigma \rangle \geq 0 \right\} \]
\[ + \left\langle \theta, \mathcal{A} \right\rangle \leq D \int \psi((\sigma, \mathcal{A}); (\sigma^*, \mathcal{A}^*)) \quad (\sigma^*, \mathcal{A}^*) \in W \} \]  
(5.13)

We will call the relation (5.13) as the generalized normality rule for the materials of Type T.

When we have a plastic potential of the form (5.1), there exists plastic deformation for any finite value of \( \epsilon \), no matter how small, since we have nonzero elements \((D, \theta)\) even though \( F(\sigma, \mathcal{A}) < 0 \).

But this property may, in fact, be physically reasonable for many engineering materials. Generally, in single crystalline models, we have four distinct stages in the stress-strain curve as seen in Fig. 9. After an elastic stage, one often observes an “easy-glide” stage (Stage 1 in the figure) during which all the free (mobile) dislocations move and a large amount of plastic strain is realized. Stage 2 is called the work-hardening stage, and unit dislocations are generated from, say, Frank–Read sources and interactions between dislocations making barriers like Lomer–Cottrell locks. In Stage 3, breakdowns of these barriers may occur, giving very low hardening with some plastic strain. For more detailed descriptions of these behavior, see Dieter[28] or Wilkov[24].

In polycrystalline materials, the easy glide stage is not observed as frequently as in the single crystal since the movements of dislocations are stopped more readily by a larger density of built-in obstacles due to the nature of the grain boundaries. Therefore, the usual stress strain curve, i.e. a linear elastic region and a strain hardening region, is obtained. In fact, the easy glide stage is included in the elastic region since the flow of dislocations and plastic strain are ignorable. But, even though dislocations move very short distances, if the irrecoverable dislocation movement is interpreted as plastic deformation, some changes in plastic strain may actually occur in the “elastic” region, specifically in the unloading process. This point is discussed in some papers, e.g. Hutchinson[20] and Nemat-Nasser[11].

For materials with a significant elastic response regions characterized by such potentials, it may be convenient to regard \( \epsilon \) as an interior penalty parameter associated with the constraint set \( C \) and to choose \( \epsilon \) small so that negligible plastic strain rates and rate of internal state variable, \((D, \theta)\), occur for elastic deformations. As indicated in the reduction of our theory to the cases covered by (5.10) to (5.12), taking \( \epsilon \to 0 \) provides for the concept of a convex yield function and other features of classical plasticity within the framework of our theory. However, if this theory were used to develop computational methods for the solution of classical plasticity problems, one may still wish to choose a small value of \( \epsilon > 0 \) to model the yielding constraint by an interior penalty method.

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APPENDIX: ELEMENTS OF CONVEX ANALYSIS

We shall provide here a brief summary of some of the concepts of convex optimization theory which are prerequisite to the ideas discussed in the body of the paper. For more detailed accounts, the books of Ekeland and Temam[31] or Rockafellar[18, 19] or the recent text of Oden[25] can be consulted.

We begin by introducing the following notations:

\( \mathbb{R} \) = the extended real numbers; if \( \mathbb{R} \) is the real number system, \( \mathbb{R} = \mathbb{R} \cup \{ \pm \infty \} \)

\( U, V \) = topological vector spaces

\( U^*, V^* \) = topological dual spaces of \( E \) and \( S \) respectively

\( \langle \cdot, \cdot \rangle \) = duality pairing on \( V^* \times V \) and \( U^* \times U \), respectively; i.e. if \( v \in V^* \) and \( v \in V \), then \( v^*(v) = \langle v^*, v \rangle \), etc.

It is worthwhile to recall the definition of the limit-superior (lim sup) and the limit-inferior (lim inf) of sequences of real numbers, extended real-valued functions of sequences, and sequences of sets in a topological vector space \( V \).

\[ \lim \sup \left\{ a_n \right\} = \lim \left\{ \sup \{ a_n : n \geq N \} \right\} \]

\[ \lim \inf \left\{ a_n \right\} = \lim \left\{ \inf \{ a_n : n \geq N \} \right\} \]

\[ \lim \sup_{x \to+} f(x') = \lim_{x \to+} \sup_{x' \in [a, b]} f(x') \]

\[ \lim \inf_{x \to+} f(x') = \lim_{x \to+} \inf_{x' \in [a, b]} f(x') \]
Fig. A1. Limit superior, limit inferior of discontinuous function $f: \mathbb{R} \to \mathbb{R}$ at $x_0$.

**For** $\{A_n\}$ **a sequence of subsets of the underlying set of topological space** $V$,

$$
\limsup_{n \to \infty} A_n = \bigcup_{n=1}^{\infty} \left( \bigcap_{n=1}^{\infty} A_n \right),
$$

$$
\liminf_{n \to \infty} A_n = \bigcap_{n=1}^{\infty} \left( \bigcup_{n=1}^{\infty} A_n \right).
$$

(A3)

For example, if $\{x_n\}$ **is a sequence of real numbers which converges to** $x$ **and** $f: \mathbb{R} \to \mathbb{R}$, **lim sup $f(x_n)$** **is the supremum of all cluster points of** $f$ **at** $x$, **as indicated in Fig. A1 (with an analogous interpretation for** lim inf). The concept can also be applied to multifunctions from one topological vector space to another. Indeed, if $r: U \to V$ **(with** $r(u)$ **a subset of** $V$ **for each vector** $u \in U$), then

$$
\limsup_{u \to u_0} r(u') = \bigcup_{r(u') + A \in 0} \bigcap_{u \in B} r(u) + A
$$

(A4)

and

$$
\liminf_{u \to u_0} r(u') = \bigcap_{r(u') + A \in 0} \bigcup_{u \in B} r(u) + A
$$

(A5)

where $\mathcal{N}(0)$ and $\mathcal{N}(u)$ **are collections of neighborhoods of** $0$, $u$, respectively.

**Lim sup inf** $\lim inf sup$

In addition to the notion of limit superior and limit inferior, it is convenient to introduce the concepts of lim sup inf and lim sup introduced by Rockafellar [21].

Let $F$ **be an extended real-valued function from** $U \times V$ **into** $\mathbb{R}$, **let** $u' \to u$ **in** $U$ **and** $v' \to v$ **in** $V$. **Then we define**

$$
\limsup_{u \to u_0} F(u', v') \triangleq \lim sup F(u', v').
$$

(A6)

Likewise,

$$
\liminf_{u \to u_0} F(u', v') \triangleq \lim inf F(u', v').
$$

(A7)

Similarly, **Lim sup sup** and **Lim inf inf** can be defined in an analogous way.

The meaning of these operations can be more easily understood in the case of a real-valued function $F$ defined on $\mathbb{R}^2$, such as the discontinuous at the origin shown in Fig. A2. **To compute** Lim inf inf $F(x', y')$, **for example,**

we compute lim inf $F(x', y')$ **for a fixed** $y'$. **This gives the a function of** $y'$ **which has as its graph the curve** $\overline{AC} \cup EH$. **The lim inf of this function is the point** $E$, **denoted** $F_1$ **in the figure. Similarly lim sup $F(x', y')$ **for fixed** $y'$, **is the curve** $\overline{HD} \cup BA$ **and lim inf of this curve is the point** $D$, **denoted** $F_2$ **in the figure. In summary, for this example,**

$$
\liminf_{x' \to x_0} F(x', y') = F_1
$$

$$
\limsup_{x' \to x_0} F(x', y') = F_2
$$

$$
\liminf_{y' \to y_0} F(x', y') = F_3
$$

$$
\limsup_{y' \to y_0} F(x', y') = F_4
$$
Convex Set. A set $C \subseteq V$ is convex if and only if
\[ \theta u + (1 - \theta)v \in C \quad \forall u, v \in C \quad \text{and} \quad \forall \theta \in [0, 1] \]
(A8)
i.e. $C$ is convex iff the line segment connecting any two points $u$ and $v$ of $C$ lies totally in $C$.

Epigraph. The epigraph of an extended real-valued function $F: V \to \mathbb{R}$ is defined as the set
\[ \text{epi } F = \{(v, \lambda) \in V \times \mathbb{R} | \lambda > F(v)\} \]
(A9)

Convex Function. $F: V \to \mathbb{R}$ is convex iff $\text{epi } F$ is a convex set (see Fig. A3). This is equivalent to $F$ satisfying the condition,
\[ F(\theta u + (1 - \theta)v) \leq \theta F(u) + (1 - \theta)F(v) \quad \forall u, v \in V, \quad \forall \theta \in [0, 1] \]
(A10)

Lipschitzian function. A function $F: V \to \mathbb{R}$ is Lipschitzian around $v \in V$ iff there exists a neighborhood $\mathcal{N}(v)$ of $v$ on which $F$ is finite and such that
\[ |F(u) - F(v)| \leq C P(u - v) \quad \forall u, v \in \mathcal{N}(v) \]
(A11)
where $C$ is a positive constant and $P$ is a continuous seminorm on $V$.

Lower semicontinuity. A function $F: V \to \mathbb{R}$ is lower semicontinuous (l.s.c.) at a point $u \in V$ iff
\[ \liminf_{u' \to u} F(u') \geq F(u). \]
(A12)
Thus, $F$ is l.s.c. at $u$ if $F(u)$ is less than or equal to the values of all cluster points of $F$ at $u$. For example, the function shown in Fig. A4(a) is l.s.c. at $x_0 \in \mathbb{R}$ whereas that in Fig. A4(b) is not. It is easily shown that $F$ is l.s.c. on all of $V$ iff $\text{epi } F$ is closed (in the topology of $V \times \mathbb{R}$). The concept of upper semicontinuity (u.s.c.) is defined in an analogous manner.

Differentiable function. A function $F: V \to \mathbb{R}$ is differentiable (or Gâteaux differentiable) at a point $u \in V$ iff a unique continuous linear functional $DF(u) \in V^*$ exists such that

$$
limit_{\theta \to 0} \frac{1}{\theta} (F(u + \theta v) - F(u)) = \langle DF(u), v \rangle \forall v \in V.
$$

(A13)

If $F: V \to \mathbb{R}$ is differentiable on $V$, then it is easily shown that $F$ is convex iff

$$
F(v) - F(u) \geq \langle DF(u), v - u \rangle \forall u, v \in V.
$$

(A14)

Subdifferentials and subgradients. Let $F: V \to \mathbb{R}$ be a proper function. The subdifferential of $F$ at $u \in V$ is the (possibly empty) subset

$$
\partial F(u) \subset V^*
$$

defined by

$$
\partial F(u) = \{ u^* \in U^* | F(v) - F(u) \geq \langle u^*, v - u \rangle \forall v \in V \}.
$$

(A15)

The elements $u^* \in \partial F(u)$ are called subgradients of $F$ at $u$. Subdifferentials and subgradients have the following properties:

If $F$ is differentiable at $u$, then its subdifferential consists of only the gradient of $F$,

$$
\partial F(u) = \{DF(u)\}
$$

$u^* \in \partial F(u)$ iff

$$
\inf_{v \in V} \{ F(v) - \langle u^*, v \rangle \} = F(u) - \langle u^*, u \rangle.
$$

$$
F^*: V^* \to \mathbb{R}
$$

$F^*(u^*) = \sup_{v \in V} \{ u^*, v \} - F(v)\}

(A16)

Fig. A5. Subdifferential of a non-differentiable function $F$ at a point $u$. 

*Fig. A4. Semicontinuity (and absence of it) of a discontinuous function.*
Thus, \( u^* \in \partial F(u) \) whenever
\[
F^*(u^*) + F(u) = \langle u^*, u \rangle_V.
\] (A17)

Any convex function \( F \) continuous at \( u \) has a non-empty compact subdifferential \( \partial F(u) \).

Geometrically, the subdifferential of a continuous convex function \( F \), not differentiable at a point \( u \), can be regarded as the set of slopes contained in the cone defined by tangents to \( F \) at \( u \), as illustrated in Fig. A5.

**Indicator function.** Let \( K \) be a non-empty subset of \( V \). Then the indicator function \( \psi_K : V \to \mathbb{R} \) of the set \( K \) is defined by
\[
\psi_K(u) = \begin{cases} 
0 & \text{if } u \in K \\
+\infty & \text{if } u \notin K.
\end{cases}
\] (A18)

If \( K \) is convex, \( \psi_K \) is a convex function. In this case,
\[
\partial \psi_K(u) = \{ u^* \in V | \psi_K(v) - \psi_K(u) \geq \langle u^*, v - u \rangle \}
\] (A19)
or
\[
\partial \psi_K(u) = \{ u^* \in V^* | u^* \cdot v - u \leq 0 \ \forall v \in K \}.
\] (A20)