A STABLE SECOND-ORDER ACCURATE, FINITE ELEMENT SCHEME FOR THE
ANALYSIS OF TWO-DIMENSIONAL INCOMPRESSIBLE VISCOUS FLOWS

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1. INTRODUCTION

For the last several years (indeed, dating back to the Second Conference on Finite Elements for Flow Problems in 1976), the use of exterior penalty methods and reduced integration as devices for producing very efficient finite element methods for the analysis of incompressible viscous flows has been advocated by numerous investigators. The idea is to use exterior penalty methods to handle the incompressibility constraint, \( \text{div} \, \mathbf{u} = 0 \), and then to use reduced integration of the penalty term to "unlock" the system (which basically frees the dependence of the penalty parameter \( \varepsilon \) on the mesh size \( h \).

In 1980, however, we were able to show both mathematically and experimentally that many of the favorite RIP (reduced-integration-penalty) schemes are numerically unstable. These difficulties were discussed in Oden's lecture at the Bnaff Conference in 1980 and are summarized in the final conference paper by Oden. Complete details of this analysis are given in the papers of Oden, Kikuchi, and Song together with a more complete list of references on this subject.

From the mathematical point of view, the key to the stability of these methods is the so-called LBB-condition, to be given in the next section. The once widely used \( Q_2 \)-elements for velocities (tensor products of quadratures) with \( 2 \times 2 \) Gaussian integration (or, equivalently, with discontinuous bilinear pressures) and the well-known \( Q_1 \)-elements (bilinear) with 1-point integration of the penalty terms (or, equivalently, constant pressures) fail the LBB-condition for most choices of boundary conditions in that they lead to an LBB-parameter \( \alpha_h \) which depends upon the mesh size \( h \). If one uses \( Q_2 \) elements for velocities but only 1-point integration, then the LBB condition is satisfied with \( \alpha_h \) independent of \( h \), but the rate-of-convergence of the method is suboptimal. The question naturally arises as to whether or not it is possible to produce a stable RIP method which exhibits an optimal rate of convergence in the "energy" \( (H^1) \) norm.

At the third symposium at Bnaff, Oden announced two types of elements which were candidates for such optimal RIP methods: the \( Q_2 \)-element for velocities with discontinuous piecewise linear pressures (referred to here as the \( Q_2/P_1 \)-element) and the 8-node isoparametric element with discontinuous piecewise linear pressures (here the \( I8/P_1 \)-element). These were subsequently coded and by July 1980 were producing encouraging numerical results (see Oden). Similar numerical results were obtained by other investigators in the intervening months (see Zienkiewicz and Taylor and Engleman and Sani).
In the present paper, we examine these two elements in greater detail. First, we show that the $Q_2/P_1$-element does, in fact, satisfy the LBB-condition with $\alpha_h$ independent of $h$ whereas the $P_1/P_1$ element apparently does not, except under special circumstances. We also attempt to provide results of numerical experiments which support some of our theoretical findings.

Our main result is that the $Q_2/P_1$-element is stable and exhibits an optimal convergence rate in the $H^1$-norm, while $P_1/P_1$ is "mildly" unstable. We also suggest certain ways to stabilize this element for certain classes of problems.

2. STATEMENT OF THE PROBLEM

It suffices to consider the following two-dimensional Stokes problem:

$$
\begin{align*}
\nu \Delta u + \nabla p &= f \quad \text{in } \Omega \\
\nabla \cdot u &= 0 \quad \text{in } \Omega \\
\quad u &= 0 \quad \text{on } \partial \Omega 
\end{align*}
$$

(1)

Here $\nu$ is the viscosity, $u$ the velocity field, $p$ the hydrostatic pressure, $f$ the body force, and $\Omega$ is an open bounded polygonal domain in $\mathbb{R}^2$. To rewrite (1) in a variational form, we introduce the notation

$$V = (H^1_0(\Omega))^2, \quad H = L^2(\Omega)$$

$(\cdot, \cdot)_0, \| \cdot \|_0$ the inner product and norm on $H$

$$(u, v)_1 = \sum_{1 \leq i, j \leq 2} (u_{i,j}, v_{i,j})_0 \quad \text{inner product on } V$$

$|v|_1 = [(v, v)_1]^{1/2} \quad \text{norm on } V$

$$a(u, v) = \nu (u, v)_1 \quad \text{bilinear form on } V$$

$$f(v) = (f_1, v_1)_0 \quad \text{linear form on } V$$

$$K = \{ v \in V \mid \nabla \cdot v = 0 \} .$$

Then we consider the problem of finding $(u, p) \in V \times H$, $\int_{\Omega} p \, dx = 0$, such that
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\begin{align*}
V &= \left( H_0^1(\Omega) \right)^2, \quad H = L^2(\Omega) \\
(\cdot, \cdot)_0, \| \cdot \|_0 &= \text{the inner produce and norm on } H \\
(u, v)_1 &= \sum_{1 \leq i, j \leq 2} (u_{i,j}, v_{i,j}) = \text{inner product on } V \\
\|v\|_1 &= [(v, v)_1]^{1/2} = \text{norm on } V \\
a(u, v) &= \langle u, v \rangle_1 = \text{bilinear form on } V \\
f(v) &= (f, v)_0 = \text{linear form on } V \\
K &= \{ v \in V \mid \text{div } v = 0 \}.
\end{align*}
\]

Then we consider the problem of finding \( (u, p) \in V \times H \), \( \int_{\Omega} p \, dx = 0 \), such that
\begin{align*}
\mathbf{a}(u,v) - (p, \text{div } v) = f(v) \quad \forall \, v \in V \\
(q, \text{div } u) = 0 \quad \forall \, q \in H
\end{align*}

(2)

Of course, any smooth solution of (1) satisfies (2) and any solution of (2) is a solution of (1) if interpreted in an appropriate weak sense.

A regularization of (2) can be obtained by introducing a penalty parameter \( \varepsilon > 0 \) and seeking \( u_\varepsilon \in V \) such that

\begin{align*}
\mathbf{a}(u_\varepsilon,v) + \frac{1}{\varepsilon} (\text{div } u_\varepsilon, \text{div } v) = f(v) \quad \forall \, v \in V
\end{align*}

(3)

from which one can calculate an approximate pressure \( p_\varepsilon \) as

\begin{align*}
p_\varepsilon = -\frac{1}{\varepsilon} \text{div } u_\varepsilon.
\end{align*}

(4)

It is easily shown that there exists a unique solution \( u_\varepsilon \) to (3) for any \( \varepsilon > 0 \) and that \( u_\varepsilon \rightarrow u \) in \( V \) and \( p_\varepsilon \rightarrow p \) in \( H \) as \( \varepsilon \rightarrow 0 \), where \( (u,p) \) is a solution of (2) (see, e.g. Oden, Kikuchi, and Song\(^3\)).

For the approximation of (3), we first consider the case in which \( \Omega \) is a rectangle on which a uniform mesh of \( \varepsilon \) rectangular \( Q_2 \) elements \( \Omega_e \) has been constructed. We introduce the discrete spaces

\begin{align*}
V_h = \{ v^h = (v^h_1,v^h_2) \mid v^h_i \in C^0(\Omega) \} , \\
v^h_i|_{\Omega_e} \in Q_2(\Omega_e) \, , \, v^h_i|_{\partial\Omega} = 0 \, , \\
1 \leq e \leq E \, , \, i = 1,2
\end{align*}

(5)

\begin{align*}
H_h = \{ q^h \mid q^h|_{\Omega_e} \in P_1(\Omega_e) \, , \, 1 \leq e \leq E \}
\end{align*}

(6)

We then seek \( (u^h_\varepsilon,p^h_\varepsilon) \in V_h \times H_h \) such that

\begin{align*}
\mathbf{a}(u^h_\varepsilon,v^h) - p^h_\varepsilon \text{div } v^h = f(v^h) \quad \forall \, v^h \in V_h \\
(q^h, \text{div } u^h_\varepsilon)_0 = 0 \quad \forall \, q^h \in H_h
\end{align*}

(7)

As shown by Oden and Kikuchi\(^1\), RIP schemes are numerically stable whenever the following discrete LBB-condition is satisfied for \( \varepsilon_h \) independent
of $h$: There exists $\alpha_h > 0$ such that

$$\alpha_h \|q^h\|_0/\ker B^h \leq \sup_{\psi^h \in V_h} \frac{I(q^h, \text{div} \psi^h)}{\|\psi^h\|_1}$$

$$\forall \psi^h \in H^\prime_h$$

(8)

where $B^h$ is the adjoint of the discrete constraint operator $B_h (= \text{div}_h + \text{boundary conditions})$ and $I(\cdot, \cdot)$ is the numerical quadrature operator approximating $(q^h, \text{div} \psi^h)_0$.

$$B^h : H^\prime_h \rightarrow V^\prime_h$$

3. MAJOR RESULTS

The major results of our study are summarized in this section. Full details and the results of numerical experiments will be given in the expanded version of the conference paper.

Theorem 1. For the $Q_2/P_1$ scheme described earlier, $\ker B^h = \ker B^\prime_h = \mathbb{R}$ and there exists a constant, $\alpha$, independent of $h$, such that $\psi q^h \in H^\prime_h$.

$$\alpha \|q^h\|_0 \leq \sup_{\psi^h \in V_h} \frac{\int_{\Omega} q^h \text{div} \psi^h \text{dx}}{\|\psi^h\|_1}$$

(9)

The pressure of the center node in the $Q_2$-element is essential for the proof of this result. Without it, our results indicate the method is unstable.

Theorem 2. For the $I8/P_1$ scheme for problem (4), $\alpha = O(h)$ and the scheme is generally unstable.

However, we have also established the following results:

1. In a rectangular mesh of $I8/P_1$-elements, if any corner element is replaced by a $Q_2/P_1$-element, the resulting scheme is stable. (See Fig. 2).

2. As is the case with many RIP methods, the "unstable" $I8/P_1$-schemes can perform well in the case of smooth solutions on a regular mesh on a smooth domain.

The rates of convergence of $Q_2/P_1$-element velocities in the $\|\cdot\|$-norm is $O(h^2)$—which is optimal; likewise, the pressures converge in $L^2(\Omega)$ at a rate of $O(h^2)$. 
PERTURBED

STABLE AND UNSTABLE RIP/ LAGRANGIAN METHODS FOR
STABILITY OF SOME
TWO-DIMENSIONAL VISCOS RJP FINITE ELEMENT METHODS
FLOW PROBLEMS
FOR STOKESIAN FLOWS

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1. INTRODUCTION

In so-called primitive variable formulations of problems of flow of viscous, incompressible, Stokesian fluids, two fields appear as unknowns: the velocity field \( \mathbf{u} \) and the pressure field \( p \), the latter representing a Lagrange multiplier associated with the incompressibility constraint \( \text{div} \mathbf{u} = 0 \). Finite element methods based on such formulations were first introduced over a decade ago [16]. Since the mid-1970s, however, much interest has arisen in a related family of finite element methods developed around the ideas of handling the incompressibility constraint by means of an exterior penalty technique and then under-integrating the penalty terms to free the dependence of the penalty parameter \( \varepsilon \) upon the mesh size \( h \). One great advantage of such RIP (reduced-integration-penalty) finite element methods is that they reduce the number of unknown fields to only one—a regularized velocity \( \mathbf{u}_\varepsilon \); the pressure can be computed a posteriori as a post processing operation.

Such RIP methods were developed and discussed by several authors, and we mention in particular the works of Malkus [12,13], Hughes [10], Malkus...
and Hughes [15], Reddy [21], Bercovier [2], Engleman and Sani [5], and the references therein. In 1980, however, mathematical analyses indicated that some of the more popular RIP methods might be numerically unstable [17,18,19,20]. It was discovered that while certain of these methods perform well in problems with smooth solutions for which regular uniform meshes are employed, serious oscillations in the pressure approximation can occur when the data or the mesh pattern are mildly irregular, and these oscillations increase in amplitude as the mesh is refined.

Oden, Kikuchi, and Song [20] attributed the deficiency of these unstable methods to their failure to satisfy a key stability criterion which they referred to as the "LBB-condition," making reference to the work of Ladyszhenskaya [11] on existence theorems of viscous flow problems and of Babuška [1] and Brezzi [3] on the approximation of elliptic problems with constraints. The discrete LBB-condition of Oden, Kikuchi, and Song is basically the requirement that the discrete approximation $B_h^*$ of the transpose $B^*$ of the constraint operator $B = \text{div}$ be bounded below as a linear operator mapping the space of approximate pressures onto the dual of the space of approximate velocities. For example, one form of this condition is that there exist an $\alpha_h > 0$ such that for all $q_h \in Q_h^*$,

$$\alpha_h \| q_h \|_{L^2(\Omega)/\ker B_h^*} \leq \sup_{v_h} \frac{I(q_h, \text{div } v_h)}{\| v_h \|_1}$$

Related conditions for mixed finite elements were discussed by Fortin [8] and Girault and Raviart [9]. The possibility of unstable pressure approximations of an RIP method is signalled by the existence of a parameter $\alpha_h$.

* Definitions of terms displayed here are given in Section 2.
which depends upon the mesh size $h$. Indeed, the fact that a mesh-dependent $\alpha_h$ corresponds to RIP methods with "spurious pressure modes" is supported by the theoretical and numerical results of Oden et al [2D] and by extensive numerical experiments of Malkus [14]. Equally important, the behavior of $\alpha_h$ as a function of $h$ governs the asymptotic rate of convergence of RIP methods.

An important question that has arisen from these considerations is whether or not stable RIP methods exist which converge at optimal rates in the energy- and $L^2$-norms. The present paper is directed at resolving this question for a restricted class of problems by estimating the stability parameter $\alpha_h$ in the corresponding discrete LBB-condition.

Some results concerning this stability parameter are presented and a series of numerical experiments on RIP methods for the Stokes' problem are here performed.

Among specific results we establish here is that the $Q_2/P_1$ (bi-quadratic velocity/linear discontinuous pressure) element is stable with $\alpha_h = O(1)$ while the $I8/P_1$ (eight-node isoparametric for velocity/linear discontinuous pressure) element is unstable with $\alpha_h = O(h)$. We also investigate the effect of averaging the pressures in the unstable elements (pressure filtering). For the $I8/P_1$ element, for example, we find that averaging restores the stability of the method and renders a method which converges at an optimal rate in $L^2$ but which exhibits an accuracy regarded as inferior to the $Q_2/P_1$-element. We also reexamine $\alpha_h$ for the $Q_1/P_0$ (bilinear velocity/one point integration) and find $\alpha_h = O(h)$. This improves the estimate of $O(h^2)$ given by Oden, Kikuchi, and Song [2D] and agrees with the result of Carey and Krishnan [4] obtained independently.
2. STATEMENT OF THE PROBLEM

Let $\Omega$ denote an open bounded region of $\mathbb{R}^2$ with boundary $\partial\Omega$. We consider the two-dimensional Stokes problem in $\Omega$, which involves finding a velocity field $\mathbf{u} = (u_1, u_2)$ and a pressure field $p$ such that

\[
\begin{aligned}
-\nu \Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega \\
div \mathbf{u} &= 0 \quad \text{in } \Omega \\
\mathbf{u} &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]

(2.1)

where $\nu$ is the viscosity of the fluid, $(\nu = \text{const.} > 0)$, and $\mathbf{f}$ is the body force, assumed to be a prescribed vector field with components $f_i \in L^2(\Omega)$. 

...
We recast (2.1) in a weaker variational framework by introducing the spaces

\[ V = (H^1_0(\Omega))^2, \quad Q = L^2(\Omega) \]  

(2.2)

and the forms

\[
\begin{align*}
& a: V \times V \rightarrow \mathbb{R}, \quad f: V \rightarrow \mathbb{R} \\
& a(u,v) = \nu(u,v)_1, \quad f(v) = \sum_{i=1}^{2} (f_i, v_i)
\end{align*}
\]

(2.3)

for all \( u, v \in V \), where \( (\cdot, \cdot)_1 \) and \( (\cdot, \cdot) \) are inner products on \( V \) and \( Q \), respectively, and are given by

\[
\begin{align*}
& (v,w)_0 = \int_\Omega vwdx; \quad v, w \in Q \\
& (u,v)_1 = \sum_{i,j=1}^{2} \left( \frac{\partial u_i}{\partial x_j}, \frac{\partial v_i}{\partial x_j} \right); \quad u, v \in V
\end{align*}
\]

(2.4)

The partial derivatives in (2.4) are interpreted in a distributional sense.

We proceed by considering the problem of finding \( (u,p) \in V \times Q \) such that

\[
\begin{align*}
& a(u,v) - (p, \text{div} v) = f(v) \quad \forall v \in V \\
& (q, \text{div} u) = 0 \quad \forall q \in Q
\end{align*}
\]

(2.5)

It is easily verified that any solution of (2.1) satisfies (2.5); any solution of (2.5) satisfies equations of the form (2.1) in a distributional sense. Under the conditions stated, it is also known that (2.5) possesses a solution \( (u,p) \), with \( u \) uniquely determined by each choice of \( f \) and \( p \) unique up to an arbitrary constant.
Problem (1.1) can also be interpreted as the characterization of a saddle point of the functional

\[ L: V \times Q \rightarrow \mathbb{R} \]

\[ L(v, q) = \frac{1}{2} a(v, v) - f(v) - (q, \text{div } v) \quad (2.6) \]

with \( q \) clearly a Lagrange multiplier associated with the constraint, \( \text{div } v = 0 \) in \( Q \).

An alternate formulation of problem (2.5) can be obtained using an exterior penalty method. Let \( \varepsilon \) be an arbitrary positive number. By seeking minimizers \( u_\varepsilon \in V \) of the penalized functional,

\[ J_\varepsilon (v) = \frac{1}{2} a(v, v) - f(v) + \frac{1}{2\varepsilon} (\text{div } v, \text{div } v) \quad (2.7) \]

we are led to the variational problem

\[ a(u_\varepsilon, v) + \varepsilon^{-1} (\text{div } u_\varepsilon, \text{div } v) = f(v) \quad \forall v \in V \quad (2.8) \]

This problem, which is uniquely solvable for \( u_\varepsilon \) for any \( \varepsilon > 0 \), involves a single unknown field \( u_\varepsilon \). Once \( u_\varepsilon \) is obtained, an approximation \( p_\varepsilon \) of the pressure is obtained by the computation

\[ p_\varepsilon = -\varepsilon^{-1} \text{div } u_\varepsilon \quad (2.9) \]

The forms \( a(\cdot, \cdot) \) and \( f(\cdot) \) are continuous and \( a(\cdot, \cdot) \) is \( V \)-elliptic. In addition, Ladyszhenskaya \[14\] has shown that a constant \( \alpha > 0 \) exists such that
\[ a \| q \|_{L^2(\Omega)/\mathbb{R}} \leq \sup_{v \in V} \frac{(q, \text{div} v)}{||v||_1} \]

\[ \forall q \in Q \quad (2.10) \]

where \( ||v|| = \sqrt{(v,v)}_1 \). Under these conditions, the sequence \( \{(u_{\varepsilon}, p_{\varepsilon})\}_{\varepsilon > 0} \) of solutions of (2.8) and (2.9) converge strongly in \( V \times Q/\mathbb{R} \) to the solution \( (u, p) \) of (2.5).

We remark that the penalized problem (2.8) can be interpreted in another way. Returning to the saddle point problem for the functional \( L(\cdot, \cdot) \) of (2.6), we introduce the perturbed Lagrangian

\[ L_{\varepsilon}(v,q) = L(v,q) - \frac{1}{2\varepsilon}(q,q) \quad (2.11) \]

for all \( q \in Q \), which represents a regularization of \( L(\cdot, \cdot) \) with respect to the multipliers \( q \). For each \( \varepsilon > 0 \), saddle points \( (u_{\varepsilon}, p_{\varepsilon}) \) of \( L(\cdot, \cdot) \) are characterized by

\[ a(u_{\varepsilon}, v) - (p_{\varepsilon}, \text{div} v) = f(v) \quad \forall v \in V \]

\[ (\varepsilon p_{\varepsilon} + \text{div} u_{\varepsilon}, q) = 0 \quad \forall q \in Q \quad (2.12) \]

Upon solving the last equation in (2.12) for \( p_{\varepsilon} \), we obtain (2.9). Substituting this result into the first equation in (2.12) gives precisely the penalty formulation (2.8).
3. FINITE ELEMENT APPROXIMATIONS

We shall outline briefly features of certain finite element approximations of (2.8). We confine our attention to cases in which \( \Omega \) is rectangular or is the union of rectangles and, for simplicity, to uniform meshes of rectangular elements of maximum length \( h \). For a family of such meshes with \( E = E(h) \) elements, we introduce the discrete (finite-dimensional) spaces,

\[
V^h = \{ v_h = (v_{h1}, v_{h2}) | v_{hi} \in C^0(\Omega) \},
\]

\[
v_{hi} \big|_{\Omega_e} \in Q_k(\Omega_e); \quad v_{hi} = 0 \text{ on } \partial \Omega,
\]

\[
1 < e < E, \quad i = 1, 2 \}
\]

\[
Q^h = \{ q_h \in L^2(\Omega) | q_{h} \big|_{\Omega_e} \in P_r(\Omega_e); \}
\]

\[
1 < e < E, \quad r > 0 \}
\]

Here \( Q_k(\Omega_e) \) is the space of tensor products of complete polynomials in \( x_1 \) and \( x_2 \) of degree \( \leq k \) defined on finite element \( \Omega_e \) and \( P_r(\Omega_e) \) is the space of complete polynomials of degree \( \leq r \) defined on \( \Omega_e \).

Clearly, for every \( h \),

\[
V^h \subset V \quad \text{and} \quad Q^h \subset Q
\]
In addition to the spaces $V^h$, we shall also consider cases in which $V^h$ is constructed using 18-elements:

$$I8 = \text{eight-node isoparametric elements} \quad (3.3)$$

We also consider composite elements which employ both $Q_2$ and 18-subelements.

In any case, the finite-element approximation of the penalty formulation (2.8) consists of seeking $u^e_h \in V^h$ which satisfies the variational equality (2.8) for all $v \in V^h$. However, the performance of the resulting method depends strongly on the relationship between $\varepsilon$ and $h$ (see Falk [7]). For fixed mesh size $h$, the use of a small $\varepsilon$ (say $\varepsilon = 10^{-5}$ for most reasonable meshes) leads to a "locked" solution $u^e_h$ (which approaches zero in $V$ as $\varepsilon$ tends to zero). The use of larger values of $\varepsilon$, on the other hand, unlocks the problem in the sense that a nonzero solution $u^e_h$ can be obtained, but then the incompressibility constraint is not adequately enforced. The solution to this paradox is to use reduced integration for evaluating the penalty terms in (2.8) ([12,13,14,15]).

Let $I(\cdot, \cdot)$ denote a numerical quadrature rule for integrating approximately the product of two element-wise continuous functions $f$ and $g$:

$$I(f, g) = \sum_{e=1}^E \sum_{j=1}^{G_e} W^e_j f(\xi^e_j) g(\xi^e_j)$$

(3.4)

Here $G_e$ is the number of quadrature points in element $e$, $W^e_j$ are the quadrature weights, and $\xi^e_j$ are the quadrature points in element $e$. By choosing $G_e$ sufficiently large, it is always possible to obtain the exact value of the integral $\int_{\Omega} fg dx$ from $I(f,g)$ for polynomial $f$ and $g$. 
The RIP-finite element approximation of (2.1) then consists of seeking $u^e_h \in V^h$ such that
\[ a(u^e_h, v^e_h) + \varepsilon^{-1} I(\text{div } u^e_h, \text{div } v^e_h) = f(v^e_h) \]
\[ \forall v^e_h \in V^h \] (3.5)

For each choice of $V^h$ and quadrature rule $I(\cdot, \cdot)$ there is defined implicitly in the approximation (3.5) a space $Q^h$ of approximate pressure $q^e_h$. Indeed, the specification of values of $q^e_h$ at each integration point within an element in the definition of $I(\cdot, \cdot)$ uniquely defines a piecewise continuous polynomial over each element. For example, if $Q_2$-elements are used for the velocity approximation, a $2 \times 2$ Gaussian quadrature rule for $I(\cdot, \cdot)$ corresponds to a piecewise bilinear pressure approximation, etc. In general, an $m \times m$ Gaussian quadrature rule corresponds to a space $Q^h$ of pressures which are discontinuous at interelement boundaries, but which have restrictions in $Q_m(\bar{\Omega}_e)$ in each finite element.

Once (3.4) is solved for the velocity field $u^e_h$, the corresponding pressure approximation is computed as
\[ p^e_h \in Q^h : p^e_h(\xi^e_j) = -\varepsilon^{-1} \text{div } u^e_h(\xi^e_j) \] (3.6)

where $\xi^e_j$ are, again, the quadrature points in element $\Omega_e$. Note that $Q^h$ is such that (3.6) uniquely determines $p^e_h$.

Under the conditions established thusfar, (3.5) is uniquely solvable for $u^e_h$ for any $\varepsilon > 0$. However, the behavior of $u^e_h$ and $p^e_h$ as $\varepsilon$ or $h$ tend to zero depends upon more delicate features of the approximation.

Let $B^*_h$ and $B^*_h$ denote the discrete operators,
where \([\cdot, \cdot]\) and \(<\cdot, \cdot\rangle\) denote duality pairings on \(Q' \times Q\) (\(Q = Q' = L^2(\Omega)\)) and \(V' \times V\) respectively (i.e., \(B_h^*\) and \(B_h^*\) are the discrete approximations of \(\text{div}\) and \(-\text{grad}\) plus boundary conditions defined by \(I(\cdot, \cdot)\)). Then, the discrete LBB-condition for problem (3.5) (or (3.7)) is as follows:

There exists a number \(\alpha_h > 0\) such that

\[
\alpha_h \|q_h\|_{L^2(\Omega)}/\ker B_h^* \leq \sup_{v_h \in V_h} \frac{I(q_h, \text{div} v_h)}{\|v_h\|_1}
\]

for all \(q_h \in Q^h\)

The behavior of \(\alpha_h\) as \(h\) tends to zero and the structure of \(\ker B_h^*\) governs the stability of RIP methods. In particular, let \(E_h(u, p)\) denote the distance function

\[
E_h(u, p) \equiv \inf_{v_h \in V_h} \|u - v_h\|_1 + \inf_{q_h \in Q^h} \|p - q_h\|_{L^2(\Omega)}
\]

defined on \(V \times Q\), \(Q = \{q \in Q \mid \int_{\Omega} q dx = 0\}\). Then one can show (see Oden and Kikuchi [20]) that if \((u, p)\) is the solution of (2.5) and \((u_h^e, p_h^e)\) are the solutions of (3.5), (3.6) in \(V^h \times Q^h\).
\[ \| u - u_h^\varepsilon \|_1 \leq C (1 + \alpha_h^{-1}) (E_h(u,p) + \varepsilon) \]  
\[ \| p - p_h^\varepsilon \|_{L^2(\Omega)} \leq C (1 + \alpha_h^{-1} + \alpha_h^{-2}) (E_h(u,p) + \varepsilon) \]  

where $C$ is a generic constant independent of $u$, $p$, $\varepsilon$, and $h$.

The remainder of this paper is devoted to the study of (3.8) and estimations of the stability parameter $\alpha_h$.

4. LBB CONDITIONS FOR CERTAIN RIP METHODS

We state estimates of the stability parameter $\alpha_h$ in the discrete LBB condition (3.11) obtained for the following RIP-methods:

1) $Q_2/P_0$ elements [here a mixed formulation based on the equivalence of (2.8) and (2.12) is used with $Q_2$ (biquadratic) velocities and linear discontinuous pressures. This formulation was discussed by Oden and Kikuchi [19].]

2) $P_1$ elements [eight-node isoparametric elements for velocities, linear pressures]

3) $Q_1/P_0$ elements [bilinear velocities, piecewise constant (one-point integration) pressures]

4) Composite elements [elements consisting of two or more of the above elements as subelements]

In all of these cases, we have*

\[ I(q_h, \text{div} v_h) = (q_h, \text{div} v_h) \quad \forall q_h \in Q_h, v_h \in V_h \quad (4.1) \]
even though $I(\text{div} v_h, \text{div} u_h) \neq (\text{div} v_h, \text{div} u_h)$.

* This condition does not hold for all RIP-methods; for instance, it is not true for the $Q_2/P_0$-elements studied in [20].
The principal results concerning \( \text{ker } B_h^* \) and the LBB constant \( \alpha_h \) are stated in the following theorems.

**Theorem I.** Let conditions (4.12) hold and let the discrete spaces \( V^h \) and \( Q^h \) be constructed using \( Q_2/P_1 \)-element. Then \( \text{ker } B_h^* = \text{ker } B^* \) and the stability parameter \( \alpha_h \) in the discrete LBB-condition (3.11) is a positive constant independent of \( h \) \( (\alpha_h = O(1)) \) \( \square \)

**Theorem II.** Let conditions (4.12) hold and suppose that \( V^h \) and \( Q^h \) are defined by \( I8/P_1 \)-elements. Then \( \dim \ker B_h^* = 3 \) and the stability parameter \( \alpha_h \) in the discrete LBB-condition (3.11) depends linearly on \( h \):

\[
\alpha_h = O(h) \quad \square
\]

**Theorem III.** Under the assumptions of Theorem II, if \( V^h \) and \( Q^h \) are defined using \( Q_1/P_0 \)-elements (i.e., bilinear velocities and a one-point integration for the penalty), then \( \dim \ker B_h^* = 2 \) and \( \alpha_h = O(h) \) \( \square \)

**Theorem IV.** Let conditions (4.12) hold and suppose that \( V^h \) and \( Q^h \) are defined using composite \( I8/P_1 - Q_2/P_1 \)-elements of the type shown in Fig. 1. Then \( \dim \ker B_h^* = 1 \) and the stability parameter \( \alpha_h \) appearing in the discrete LBB-condition is a positive constant independent of \( h \); \( (\alpha_h = O(1)) \) \( \square \)
5. NUMERICAL EXAMPLES

The results of several numerical experiments are described which are designed to verify the theoretical results with regard to the $Q_2/P_1$, $I_8/P_1$, and the composite elements described earlier. We consider an L-shaped domain $\Omega$ partitioned into 64 square subdomains, as shown in Fig. 2. The fluid is subjected to a constant body force $f = (0, -100)$. We take $\nu = 333$, and the penalty parameter $\varepsilon = 10^{-5}$. We will be interested in the computed hydrostatic pressure across the section AA' defined by: $y = 0.80$. Each subdomain corresponds to a finite element; the velocity on each element is interpolated at 8 or 9 nodes and the pressure by its value at 3 points. Thus, various choices of how to handle the ninth node lead to meshes with $I_8/P_1$, $Q_2/P_1$ or Composite/$P_1$ elements. We will be interested in three cases involving these elements:

Mesh 1: All the elements are $Q_2/P_1$ elements

Mesh 2: All the elements are $I_8/P_1$ elements

Mesh 3: Adding 16 centroid nodes, we obtain 16 composite elements as shown in Fig. 3.

The results reported here were obtained using the FIDAP code for problems of incompressible viscous flow [6].
Figures 4 and 5 show the comparison between the results obtained with the $Q_2/P_1$ element (Mesh 1) and those obtained with meshes 2 and 3. Fig. 4 illustrates the major difference between the $Q_2/P_1$ and the $B^h/P_1$ element: the former involves a pressure which seems to be smoothly distributed along the section AA' while the latter yields a pressure with severe oscillations. We note, however, that the values of the pressure obtained at the centroid of each element are close to the values obtained with the $Q_2/P_1$ element, which suggest that this unstable solution can be stabilized by a filtering operation which effectively uses these averaged values of pressure.

It is also remarked that the oscillations seem to come from the spurious modes in $\ker B^h$. The smoothing device may be equivalent to an a posteriori elimination of these spurious modes.

Finally, the composite elements lead to a quite smooth solution as indicated in Fig. 5, which is close to the solution obtained with 9-node elements, except that for this element $h^2 = 0.25$, while for the $Q_2/P_1$ element $h^2$ was equal to 0.0625.
We also note that when the body force \( \mathbf{f} \) derives from a potential:

\[
\mathbf{f} = -\nabla \mathbf{v}
\]

then the unique solution for the Stokes Problem is

\[
\begin{align*}
\mathbf{u} &= \mathbf{z} \\
p &= -\mathbf{v}
\end{align*}
\]

In this example, \( \mathbf{f} = (0,-100) \) and \( \mathbf{v} = 100\mathbf{y} = -\mathbf{p} \).

The numerical results obtained by different methods are summarized in the Table I.

| Method 1: \( \mathbb{Q}_2/\mathbb{P}_1 \) elements |
| Method 2: composite elements |
| Method 3: \( \mathbb{Q}_0/\mathbb{P}_1 \) elements and filtering the pressures by using only the centroidal value |

**Exact Solution:**

\[
\| \mathbf{p} \|_{L^2(\Omega)} / 100 = 100 \sqrt{\frac{37}{12}} = 175.5942 ; \quad h^2 = 0.0625
\]

<table>
<thead>
<tr>
<th>Method 1</th>
<th>167.1254</th>
<th>20.0310</th>
<th>0.1141</th>
</tr>
</thead>
<tbody>
<tr>
<td>Method 2</td>
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<td>36.3181</td>
<td>0.2068</td>
</tr>
<tr>
<td>Method 3</td>
<td>171.5845</td>
<td>26.6219</td>
<td>0.1516</td>
</tr>
</tbody>
</table>
As a second example, we consider a Dirichlet Stokes Problem, which is designed for numerical verification of the convergence theory for three schemes considered:

a) Solution obtained with $Q_2/P_1$ elements.
b) Solution obtained with $I_8/P_1$ elements.
c) Solution obtained by averaging pressures of the $I_8/P_1$-scheme.

We consider the unit square domain partitioned in square subdomain, and the following body forces $f = (f_1, f_2)$ are applied:

$$f_1 = -4y + 12x^2 + 24xy + 12y^2 - 24x^3 - 48x^2y - 72xy^2$$
$$-8y^3 + 12x^4 + 48x^3y + 72x^2y^2 + 48xy^3 - 24x^4y - 48x^2y^3$$
$$-2(x - x_0) + \alpha(x)$$

$$f_2 = 4x - 12x^2 - 24xy - 12y^2 + 8x^3 + 72x^2y + 48xy^2 + 24y^3$$
$$-12y^4 - 48xy^3 - 72x^2y^2 - 48x^3y + 24xy^4 + 48x^3y^2$$

where $\alpha(x) = -1$ if $0 \leq x \leq x_0$, $\alpha(x) = 1$ if $x_0 < x \leq 1$. Then $f(u,p)$ is defined by

$$u = (u_1, u_2) ; \begin{cases} u_1 = x^2(1 - x)^2 (2y - 6y^2 + 4x^3) \\ u_2 = (-2x + 6x^2 - 4x^3) y^2 (1 - y)^2 \end{cases}$$

and

$$\begin{cases} p = x - x - (x - x_0)^2 & \text{if } 0 \leq x \leq x_0 \\ p = x - x_0 - (x - x_0)^2 & \text{if } x_0 < x \leq 1 \end{cases}$$
(u, p) satisfies:

\[
\begin{align*}
\frac{\partial u}{\partial n} &= 0 \\
\text{div } u &= 0 \quad \text{in } \Omega \\
-\Delta u + p &= f \quad \text{in } \Omega
\end{align*}
\]

As before, we are interested by the plot of the pressure across a section of the domain. Figure 6 shows the results obtained by partitioning the domain \( \Omega \) in 64 square subdomains. For this mesh, \( h \) is equal to 1/8. The computations are made with \( Q_2 \)-on I8-elements. Whereas the \( Q_2 \)-solution seems to be stable, clearly the I8-solution shows oscillations around the exact solution. However, it is noted that both solutions coincide at the centroid of the elements and this again suggests that the "smoothed I8-solution," obtained using only the pressure at the centroid, is stable, and may converge at a rate of \( O(h^2) \).

Finally, Fig. 7: confirms this suspicion showing the computed rate of convergence is precisely \( O(h^2) \) for the pressure for the \( Q_2 \)-element, and for the smoothed I8-element. However, it is also observed that the \( Q_2/P_1 \)-pressures are considerably more accurate than the filtered I8/P1-pressures for all mesh sizes considered.

With the results from these examples we can conclude that

- The \( Q_2/P_1 \) element is stable and the optimal \( L^2 \)-rate of convergence of the pressures of \( O(h^2) \) is attained.
- The I8/P1 element yields unstable pressure approximations, but these can apparently be stabilized considering only the values at the centroids.
- Spurious oscillations (checkerboarding) can also appear when \( \ker B_h^* = \mathbb{R} \).
- Filtering the pressures in the \( 18/P_1 \)-element by using only the centroidal value leads to a pressure approximation which may converge in \( L^2 \) at a rate of \( O(h^2) \); however, the accuracy of the filtered scheme is quite inferior to that of the \( Q_2/P_1 \)-elements.

These computed results underline once again the critical role played by the LBB-condition in studying the stability of finite element schemes by reduced integration. These and other results we have computed also indicate that the estimates obtained in Section 4 for the discrete LBB-constant \( \alpha_h \) are sharp. Indeed, the theoretical result that the use of a composite element of the type employed here leads to a stable pressure field, while not of great practical value, is fully confirmed by the numerical results. This suggests again that these calculated estimates of \( \alpha_h \) are a good indication of the actual numerical performance of these methods.
REFERENCES


Figure 4. Examples of stable composite elements.
Figure 2. A mesh of 64 elements on an L-shaped domain.
Figure 1. Mesh with composite-elements.
Figure 4. Computed pressure profiles along Section AA'.

CROSS SECTION AA'

- Q2 Element
- --- 18 Element
Figure 5. Computed pressure profiles along Section AA'.

CROSS SECTION AA'

- Q2 Element
- Composite EL
Figure 6. Pressure profiles for second example.
Figure 7. Computed rate of convergence of the $I8/P_1$ - pressures in $L^2$ - with and without filtering.