REDUCED INTEGRATION AND EXTERIOR PENALTY METHODS FOR FINITE ELEMENT APPROXIMATIONS OF CONTACT PROBLEMS IN INCOMPRESSIBLE ELASTICITY

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TICOM Report 80-2
March 1980

Supported by
Air Force Office of Scientific Research
Contract F-49620-78-C-0083

THE TEXAS INSTITUTE for COMPUTATIONAL MECHANICS
THE UNIVERSITY OF TEXAS AT AUSTIN
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Abstract. Finite element approximations of penalty formulations of the problem of unilateral contact of an incompressible linearly elastic body are considered. Such problems are characterized by a pair of constraints on the displacement vector $\mathbf{u}$ of the type $\text{div} \, \mathbf{u} = 0$ in $\Omega$, $\mathbf{u} \cdot \mathbf{n} - s < 0$ on $\Gamma_C$, where $\Omega \subset \mathbb{R}^N$, $\mathbf{n}$ is the unit normal to the contact boundary $\Gamma_C$, and $s$ is given. The key to the success of such methods is the use of "reduced integration" by which is meant the use of low-order (inexact) numerical quadrature rules to integrate the penalty terms in the formulation. Use of reduced integration makes it possible to satisfy a crucial stability condition, referred to here as the LBB-condition, which establishes a basic compatibility condition between finite-dimensional approximation spaces. Our theory is quite general and extends and generalizes previous results on penalty and mixed methods for linear equality and inequality constraints.
1. INTRODUCTION

This study is concerned with the numerical analysis of the Signorini-type problem of equilibrium of an isotropic, incompressible, linearly elastic body in unilateral contact with a rigid frictionless foundation. A variational principle governing this class of problems is stated as follows:

Find \((u,p,\sigma) \in V \times Q \times M\) such that

\[
\begin{align*}
B(u,v) - (p, \text{div } v) - \langle \sigma, \gamma_n(v) \rangle &= f(v) \quad \forall v \in V \\
(q, \text{div } u) &= 0 \quad \forall q \in Q \\
\langle \tau - \sigma, \gamma_n(u) - s \rangle &\geq 0 \quad \forall \tau \in M
\end{align*}
\]

In (1.1), the following notations and conventions are used:

- \(u = \) the displacement vector, with cartesian components \(u_i = u_i(x), \quad x = (x_1,x_2,...,x_N)\) being a point in an open bounded region \(\Omega \subset \mathbb{R}^N\).
- \(p = p(x) = \) the hydrostatic pressure.
- \(\sigma = \sigma(x) = \) the contact pressure, \(x \in \Gamma_c\).
- \(V = \{v = (v_1,v_2,...,v_N) \in (H^1(\Omega))^N | \gamma(v_i) = 0 \text{ on } \Gamma_D, \quad 1 \leq i \leq N\};\)
  - here \(\gamma\) is the trace operator mapping \(H^1(\Omega)\) onto \(H^{1/2}(\Gamma)\), \(\Gamma\) being the boundary of \(\Omega\) and \(\Gamma_D\) that
portion of \( \Gamma \) on which the displacements are prescribed.

\[
\|v\|_1 = \text{norm on } V \text{ given by}
\]

\[
\|v\|_1 = \left[ \int_{\Omega} v_{i,j} v_{i,j} \, dx \right]^{1/2} \quad (\text{mes} \Gamma_D > 0)
\]  

(1.2)

\( Q = L^2(\Omega) \) (or, in certain instances, \( Q = L^2(\Omega) / \mathbb{R} \))

\( W = H^{1/2}(\Gamma_C) \), a Hilbert space with norm

\[
\|\phi\|_W = \inf \{ \|v\|_{H^1(\Omega)} | \phi = \gamma_C(v) \} \quad (1.3)
\]

\( \gamma_C \) being the trace operator mapping \( H^1(\Omega) \) onto \( H^{1/2}(\Gamma_C) \).

\( M = \{ \tau \in W' | \tau \leq 0 \} \) where \( W' \) is the dual of \( W \).

\( B(\cdot, \cdot) \) is a bilinear form from \( V \times V \) into \( \mathbb{R} \), defined by

\[
B(u,v) = 2\mu \int_{\Omega} \varepsilon_{ij}^D(u) \varepsilon_{ij}^D(v) \, dx
\]  

(1.4)

wherein \( \mu \) is the shear modulus of the material, a positive constant, and \( \varepsilon_{ij}^D(u) \) is defined as follows:

\[
\begin{align*}
\varepsilon_{ij}^D(u) &= \varepsilon_{ij}(u) - \frac{1}{N} \varepsilon_{kk}(u) \delta_{ij} \\
\varepsilon_{ij}(u) &= \frac{1}{2}(u_{i,j} + u_{j,i})
\end{align*}
\]  

(1.5)

Here \( \varepsilon_{ij}(u) \) are the components of the strain tensor produced by the displacement \( u \) \((1 \leq i,j \leq N)\) and
\( \varepsilon_{ij}^D(u) \) is the deviatoric part of the strain, i.e., that portion of \( \varepsilon_{ij}(u) \) not associated with a volume change:

\[
\varepsilon_{kk}^D(u) = \sum_{k=1}^{N} \varepsilon_{kk}^D(u) = 0 \; ; \; \varepsilon_{kk}^D(u) = u_{k,k} = \text{div } u \quad (1.6)
\]

\((\cdot, \cdot)\) = the inner product on \( L^2(\Omega) \)

\( \langle \cdot, \cdot \rangle = \text{duality pairing on } W' \times W \)

\( f(v) = \text{the potential energy of the external forces, with } f \text{ a continuous linear functional on } V \text{ given by} \)

\[
f(v) = \int_{\Omega} f_i v_i \, dx + \int_{\Gamma_F} S_i v_i \, ds \quad (1.7)
\]

wherein \( f_i \in L^2(\Omega) \) are the components of body force per unit volume and \( S_i \in L^2(\Gamma_F) \) are the components of surface traction per unit surface area given on a portion \( \Gamma_F \) of the boundary \( \Gamma \) of the body.

\( \gamma_n(v) = \gamma(v_i) n_i = \text{the normal trace of a vector } v \in V \text{ onto } H^{1/2}(\Gamma), n_i \) being the components of a unit vector outward and normal to \( \Gamma \).

\( s = \text{normalized initial gap between the body } \Omega \text{ and the foundation, given in } W \text{ as part of the data } (s \geq 0) \).

In (1.2) - (1.7) and throughout this paper, we employ the usual indicial notation and the summation convention: Latin indices \( i,j,k,\ldots \) take on values \( 1,2,\ldots,N \), repeated indices are summed from 1 to \( N \), and commas between indices denote partial differentiation with
respect to the coordinates \( x_i \) (e.g. \( \partial u_i / \partial x_j = u_{i,j} \), \( \partial^2 u_i / \partial x_j \partial x_k = u_{i,jk} \), etc.). We will also use interchangeably the notations \( \text{div } u \), \( u_{i,i} \), and \( \varepsilon_{ii}(u) \) for the divergence of the vector field \( u \).

We will also generally make the following assumptions concerning \( \Omega \) and its boundary \( \Gamma \):

\[\Omega \text{ is the open Lipschitzian domain in } \mathbb{R}^N \text{ with boundary } \Gamma, \]

\[\Gamma = \Gamma_C \cup \Gamma_D \cup \Gamma_F, \quad \Gamma_C \cap \Gamma_D = \emptyset, \quad \text{meas } \Gamma_C > 0, \quad \text{meas } \Gamma_D > 0, \quad \text{meas } \Gamma_F > 0 \]

\[(1.8)\]

\[\Gamma_C \cap \Gamma_D = \emptyset, \text{ and } \Gamma_C, \Gamma_D, \text{ and } \Gamma_F \text{ are smooth (e.g. } C^2 \text{) manifolds in } \mathbb{R}^{N-1}.\]

We will relax certain of these assumptions in our study of approximations of (1.1) later. We remark that if \( \Gamma_C \cap \Gamma_D \neq \emptyset \), it may be necessary to define the space \( W \) as the space \( H^{1/2}(\Gamma_C) \) of LIONS and MAGNENES [1] rather than \( H^{1/2}(\Gamma_C) \).

Note that problem (1.1) is inherently nonlinear. The portion \( \Gamma_C \) of \( \Gamma \) is the candidate contact surface; i.e. it contains the portion of the surface of the body which comes in contact with the foundation. Of course, the actual contact surface in \( \Gamma_C \) depends upon the solution \( u \) and \( p \) and is, therefore, unknown.

It is easily seen that the hydrostatic pressure \( p \) and the contact pressure \( \sigma \) can be interpreted as Lagrange multipliers associated
with the constraints

\[ \text{div } u = 0 \text{ in } \Omega \text{ and } u_i n_i - s \leq 0 \text{ on } \Gamma_C \] (1.9)

respectively. Indeed, the classical statement of this class of physical problems is

\[
\begin{align*}
\mu u_{i,jj} - p_i + f_i &= 0 \\
\text{in } \Omega \\
\mu u_{i,i} &= 0 \\
\mu u_i &= 0 \text{ on } \Gamma_D \\
2\mu e_{ij}^{D}(u)n_j - pn_i &= S_i \text{ on } \Gamma_F \\
\sigma &< 0, \ u_i n_i - s < 0 \\
\sigma(u_i n_i - s) &= 0 \text{ on } \Gamma_C \\
(2\mu e_{ij}^{D}(u) - p\delta_{ij})n_i - \sigma n_j &= 0
\end{align*}
\] (1.10)

with \( 1 \leq i, j \leq N \).

The direct finite-element approximation of (1.1) leads to a so-called mixed finite element method. For the special case of linear equality constraints (i.e. for problems without unilateral conditions),
several detailed studies of mixed methods have been made. The earliest were the works of ODEN \([1]\) and ODEN and REDDY \([1,2]\) and these were followed with more complete investigations by BABUSKA, ODEN, and LEE \([1,2]\), BREZZI \([1,2]\), THOMAS \([1]\), FORTIN \([1]\), and others. For problems with linear inequality constraints, the paper by BREZZI \([1]\) contains a general existence and approximation theory for mixed finite element methods. Mixed methods for linear inequality constraints were studied by BREZZI, HAGER, and RAVIART \([1]\). Additional references to literature on mixed methods can be found in the works cited here.

Unfortunately, some of the most natural mixed finite element approximations of \((1.1)\), with or without unilateral constraints, are numerically unstable. We explain why this is true in subsequent discussions. However, by "regularizing" the problem by incorporating the constraints as penalty terms in the formulation, elements which fail in mixed methods can be used successfully to extract good approximations of the solution of \((1.1)\). The overwhelming advantage of penalty formulations over mixed formulations, however, is the significant reduction in the number of unknowns in the discretized problem. In particular, in penalty formulations of \((1.1)\), the pressures \(p\) and \(\sigma\) do not appear as unknowns. We need only compute the approximate displacement field \(u\) and then compute a-posteriori approximations of \(p\) and \(\sigma\).

There is another advantage of the penalty formulation: it lends itself to a concrete and useful physical interpretation. Consider a *compressible* elastic body resting on an *elastic foundation* with modulus \(\varepsilon^{-1}\). This problem is governed by the variational principle
Find \( u_{\varepsilon} \in V \) such that

\[
B(u_{\varepsilon}, v) + \varepsilon_1^{-1} \langle \text{div } u_{\varepsilon}, \text{div } v \rangle + \varepsilon_2^{-1} \langle j(\gamma_n(u_{\varepsilon}) - s)\gamma_n(v) \rangle = f(v)
\]

\( \forall v \in V \) \hspace{1cm} (1.11)

where \( B(\cdot, \cdot) \) is given by (1.4), \( \varepsilon_1 \) and \( \varepsilon_2 \) are positive numbers, \( j \) is the Riesz map from \( W = H^{1/2}(\Gamma_\infty) \) into \( W' \) (i.e., \( (\phi, \psi)_{W'} = \langle j(\phi), \psi \rangle \), \( \|j(\psi)\|_{W'} = \|\psi\|_W \)), and \( (\gamma_n(u_{\varepsilon}) - s)_+ \) is the positive part of the function \( \gamma_n(u_{\varepsilon}) - s \), (e.g., \( \phi_+ = \max(\phi, 0), \phi \in \mathbb{R} \)). Alternatively, the classical version of (1.11) assumes the form

\[
\sigma_{ij}(u_{\varepsilon})_{,j} + f_i = 0 \quad \text{in } \Omega
\]

\( (u_{\varepsilon})_i = 0 \) \hspace{1cm} (1.12)

\[
\sigma_{ij}(u_{\varepsilon})n_j = S_i \quad \text{on } \Gamma_D
\]

\[
\sigma_{ij}(u_{\varepsilon})n_j = \varepsilon_1^{-1}((u_{\varepsilon})_i n_j - s_+)_+ \quad \text{on } \Gamma_C
\]

\[
\sigma_{ij}(u_{\varepsilon})n_j - \sigma_{\varepsilon i} = 0
\]

where \( \sigma_{ij}(u_{\varepsilon}) \) are the stress components.
In view of these results, it is clear that we should expect

\[ \sigma_{ij}(u_\varepsilon) = 2\mu \varepsilon^D_{ij}(u_\varepsilon) + \varepsilon^{-1}_1 \text{div} u_\varepsilon \delta_{ij} \quad (1.13) \]

Physically, the situation is this: if a compressible body is in a hydro-
static state of stress such that \( \sigma_{ij} = -p_\varepsilon \delta_{ij} \), \( p_\varepsilon \) being the scalar hydrostatic pressure, then, from (1.13),

\[ \sigma_{kk}(u_\varepsilon) = -Np_\varepsilon = N\varepsilon^{-1}_1 \text{div} u_\varepsilon , \]

so that

\[ p_\varepsilon = -\varepsilon^{-1}_1 \text{div} u_\varepsilon \quad (1.14) \]

The constant \( \varepsilon^{-1}_1 \) is, thus, the bulk modulus of the material. Finally, the boundary conditions on \( \Gamma_\varepsilon \) correspond to a situation in which a portion \( \Gamma_\varepsilon \) of the body is initially a (normalized) distance \( s \) from a network of uniformly distributed springs. If the body is displaced beyond the gap \( s \) and into this elastic foundation, a reactive force of intensity \( \sigma_\varepsilon \) per unit surface area is developed, normal to \( \Gamma_\varepsilon \) and in proportion to the amount of indentation, \( ((u_\varepsilon-i)_i, i - s)_+ \). The constant of proportionality is \( \varepsilon^{-1}_2 \), the foundation modulus:

\[ \sigma_\varepsilon = -\varepsilon^{-1}_2((u_\varepsilon-i)_i, i - s)_+ \quad (1.15) \]

In view of these results, it is clear that we should expect \( p_\varepsilon \) of (1.14) to approach the hydrostatic pressure of an incompressible material as the resistance of the material to volume changes increases;
i.e. as the bulk modulus \( \varepsilon_1^{-1} \) becomes larger. Thus, \( \varepsilon_1 \) plays the role of a penalty parameter for the constraint \( \text{div} \, \mathbf{u} = 0 \). Likewise, in the case of the boundary reactions \( \sigma_\varepsilon \) of (1.15), we would expect \( \sigma_\varepsilon \) to approach the contact pressure \( \sigma \) as the foundation becomes increasingly rigid; i.e. as the foundation modulus \( \varepsilon_2^{-1} \) becomes larger. Thus, \( \varepsilon_2 \) plays the role of a penalty parameter associated with the constraint \( \gamma_n(\mathbf{u}) - s \leq 0 \).

Finite element methods for cases in which only linear equality constraints (such as \( \text{div} \, \mathbf{u} = 0 \)) are imposed (no unilateral conditions) have been proposed by several authors. We mention, in particular, the works of FRIED [1], MALKUS [1], HUGHES [1,2], MALKUS and HUGHES [1], HUGHES, TAYLOR, and LEVY [1], REDDY [1], and ZIENKIEWICZ, TAYLOR, and TOO [1]. These authors have determined, on the basis of numerical experiment that it is necessary to use "reduced integration" of the penalty terms in order to obtain stable and convergent approximations. By "reduced integration" we mean the practice of using an approximate quadrature rule for integrating the penalty terms (such as \( (\text{div} \, \mathbf{u}_\varepsilon, \text{div} \, \mathbf{v}) \)) in the formulation which is of lower order than that required to integrate these terms exactly. The equivalence of penalty methods with certain mixed methods has been pointed out by MALKUS and HUGHES [1] and BERCOVIER [1]; also, the convergence of certain finite element methods based on penalty formulations of problems with linear equality constraints has been studied by BERCOVIER [1] and BERCOVIER and ENGLEMAN [1,2], but, unfortunately, under assumptions which do not hold for any of the methods of interest here. Finite elements with penalty for unilateral problems have been studied numerically by KIKUCHI and ODEN [1] and KIKUCHI and SONG [1,2];
We give particular attention to no information on the convergence or stability of these methods for problems of this type seems to be available.

In the present paper, we investigate finite element approximations of the penalized problem (1.11) and establish their relationship with direct mixed finite element methods for (1.1). We give particular attention to the role of reduced integration for penalty methods and show that the use of inexact quadrature rules is a key to the numerical stability of many finite element techniques based on penalty formulations. A principal question that arises in penalty-finite element methods for problems such as (1.1) is how to construct acceptable (stable) approximations to the multipliers $p$ and $\sigma$ from penalty approximations of the solution $u$ of (1.11). We resolve this question in Sections 5 and 6 of this paper, where we establish three conditions sufficient to guarantee the convergence of penalty approximations of $u$, $p$, and $\sigma$ to solutions of (1.1). A crucial step in this analysis is the identification of appropriate approximation spaces for $p$ and $\sigma$; we show that these spaces are defined naturally by one's choice of a numerical quadrature formula for integrating the penalty terms in the approximation. The stability of such methods is characterized by a compatibility condition between the approximation spaces $Q_h$ and $W_h$ for $p$ and $\sigma$ and the approximation space $V_h$ for displacements $u$; we refer to this condition as a "discrete LBB-condition."

In Section 6 of this paper, we develop a priori error estimates for our penalty-finite element methods. These reduce to estimates for
certain mixed methods as special cases. We find that the satisfaction of
the LBB-condition is, again, a key factor in determining asymptotic rates
of convergence for these methods.

A detailed study of the stability of these methods and the
results of many numerical experiments designed to test our theoretical
results is to be the subject of a companion paper.

2. SOME PRELIMINARIES

We will now record certain properties of the forms and operators
appearing in problems (1.1) and (1.11).

1. Let (1.8) hold. Then the space \( W = H^{1/2}(\Gamma_C) \) is well-
defined and is complete with respect to the norm (1.3). Moreover, the
normal trace operator \( \gamma_n \) is continuous and surjective from \( V \) onto
\( W \).

We note that a partial ordering "\( \leq \)" can be defined on \( W \) in a
natural way. For example, if \( w \in W \) and \( \{\psi_k\} \) is a sequence in \( C^\infty_0(\Gamma_C) \)
converging strongly to \( w \) in \( W \), then we write \( w \leq 0 \) if \( \psi_k(x) \leq 0 \)
for every \( k \) and every \( x \in \Gamma_C \). Similarly, if \( -w \leq 0 \) we write \( w \geq 0 \);
a linear functional \( \tau \in W' \) is \( \leq 0 \) if \( \tau(w) \leq 0 \) for all \( w \geq 0 \);
etc.

2. Recall that \( Q \) and \( W' \) are the spaces of candidate hydro-
static pressures and contact pressures, respectively. Usually, we identify
\( Q \) with its dual \( Q' \). We introduce the product space
and a norm on \( U' \) defined by

\[
\| (q, \tau) \|_{U'} = \| q \|_0 + \| \tau \|_{W'}
\]  

(2.2)

where \( \| q \|_0 = \| q \|_Q \). All the spaces \( U, V, W \) are Hilbert spaces.

Next, we introduce continuous linear operators

\[
\mathcal{B} : V \rightarrow U \text{ and } \mathcal{B}^* : U' \rightarrow V'
\]  

(2.3)

defined by

\[
\langle (q, \tau), Bv \rangle_{U' \times U} = \langle B^*(q, \tau), v \rangle_{V' \times V}
\]

\[
= (q, \text{div } v) + \langle \tau, \gamma_n(v) \rangle
\]  

(2.4)

We denote by \( X, Y, \) and \( Z \) the subspaces

\[
X = \ker \mathcal{B} = \{ v \in V | \langle (q, \tau), Bv \rangle_{U' \times U} = 0 \quad \forall (q, \tau) \in U' \}
\]

\[
Y = \ker \mathcal{B}^* = \{ (q, \tau) \in U' | \langle B^*(q, \tau), v \rangle_{V' \times V} = 0 \}
\]
that $B(-,-)$ is $Y$-elliptic: i.e. $B(-,.)$ is elliptic on the kernel of $(2.5)$

$$Z = U'/Y \approx \text{orthogonal complement of } Y \text{ in } U.$$ (2.5)

3. Now consider the bilinear form $B(\cdot,\cdot)$ defined in (1.4). According to Korn's inequality, there exists a constant $m_0 > 0$ such that

$$\int_{\Omega} \epsilon_{1j}^{(v)} \epsilon_{1j}^{(v)} \, dx \geq m_0 \|v\|_1^2 \quad \forall v \in V$$ (2.6)

However, (2.6) need not hold if $\epsilon_{1j}$ is replaced by $\epsilon_{1j}^D$. We conclude that $B(\cdot,\cdot)$ is $Y$-elliptic; i.e. $B(\cdot,\cdot)$ is elliptic on the kernel of $\text{div } (\cdot)$ but not on the whole space $V$.

However, consider the nonlinear form $B_\epsilon(\cdot,\cdot)$ in (1.11):

$$B_\epsilon(u,v) = B(u,v) + \epsilon_1^{-1}(\text{div } u, \text{div } v)$$

$$+ \epsilon_2^{-1}\left\langle j(Y_n(u)-s), Y_n(v) \right\rangle, \quad u,v \in V$$ (2.7)

Since

$$(\phi_+ - \psi_+)(\phi - \psi) \geq (\phi_+ - \psi_+)^2, \quad \phi, \psi \in \mathbb{R}$$

and since
\[
\int_\Omega \left( \varepsilon_{ij}^D \varepsilon_{ij}^D(v) + \frac{1}{N} \varepsilon_{i1}^1(v) \varepsilon_{j1}^1(v) \right) dx = \int_\Omega \varepsilon_{ij}^1(v) \varepsilon_{ij}^1(v) dx
\]

we use (2.6) to establish that

\[
B_\varepsilon(u_v, u_v - v) - B_\varepsilon(v_v, u_v - v) \geq m_0 \min(\mu, N\varepsilon_1^{-1}) \|u_v - v\|_1^2
\]

\forall u_v, v_v \in V \tag{2.8}

Thus, \(B_\varepsilon(\cdot, \cdot)\) (or the associated operator \(A_\varepsilon: V \rightarrow V'\) defined by \(\langle A_\varepsilon(u_v), v_v \rangle = B_\varepsilon(u_v, v_v)\)) is strongly monotone on \(V\).

4. Finally, we note that \(B(\cdot, \cdot)\) is continuous on \(V\): a constant \(M_0\) exists such that

\[
B(u_v, v_v) < M_0 \|u_v\|_1 \|v_v\|_1 \quad \forall u_v, v_v \in V \tag{2.9}
\]

However, from this fact and the fact that \(|\phi_+ - \psi_+| < |\phi - \psi|\), \(\phi, \psi \in \mathbb{R}\), it also follows that a constant \(C_0 = C_0(M_0, \mu, \varepsilon_1, \varepsilon_2)\) exists such that

\[
|B_\varepsilon(u_v, v_v) - B_\varepsilon(w_v, v_v)| \leq C_0 \|u_v - w_v\|_1 \|v_v\|_1
\]

\forall u_v, v_v, w_v \in V \tag{2.10}

These results are basic to the study of existence of solutions to (1.11) which is taken up in the next section.
3. ANALYSIS OF THE VARIATIONAL PROBLEM

We now establish several properties of the variational boundary-value problem (1.11) and its relationship with problem (1.1). Our first result is the following existence theorem:

**THEOREM 3.1** There exists a unique solution \( u_{\varepsilon} \) to (1.11) for any \( \varepsilon_1 > 0, \varepsilon_2 > 0 \). Moreover, for \( 0 < \varepsilon' < 1 \), where \( \varepsilon' = 2\varepsilon_1 \mu/N, \| u_{\varepsilon} \|_1 \) is bounded above by a constant independent of \( \varepsilon_1 \) and \( \varepsilon_2 \).

**PROOF:** From standard results in monotone operator theory (see, e.g. BROWDER [1] or LIONS [1]), it is known that an operator \( A : V \to V' \), \( V \) being a reflexive Banach space, is bijective if \( A \) is continuous and strongly monotone on \( V \). Let \( A_{\varepsilon} : V \to V' \) be defined by

\[
\langle A_{\varepsilon}(u), v \rangle = B_{\varepsilon}(u, v), \quad u, v \in V
\]

(3.1)

where \( B_{\varepsilon}(:, :) \) is defined in (2.7). Then (1.11) assumes the form

\[
A_{\varepsilon}(u_{\varepsilon}) = f \text{ in } V'
\]

(3.2)

The continuity and strong monotonicity of the operator \( A_{\varepsilon} \) from \( V \) to \( V' \) follow from (2.8) and (2.10). Thus, a unique solution \( u_{\varepsilon} \) exists to (3.2) for every \( f \in V' \).

Since \( u_{\varepsilon} \) is the solution of (3.2), we use (3.1) and (2.8) (with \( v = 0 \), \( u = u_{\varepsilon} \)) to obtain
\[ \| u_{\varepsilon} \|_1 \leq \frac{1}{m_0 \min(2\mu, N \varepsilon_1^{-1})} \| f \|_\ast \]

where \( \| \cdot \|_\ast \) is the norm on \( V' \). Thus, \( \| u_{\varepsilon} \|_1 \leq \| f \|_\ast / m_0 \mu \)

whenever \( \varepsilon_1 \leq 1 \).

According to the physical arguments given in Section 1 (recall (1.14) and (1.15)), it is natural to identify the functions

\[
\begin{align*}
q \in Q &; \quad q = -\varepsilon_1^{-1} \text{div} u_{\varepsilon} \\
\varepsilon \in W' &; \quad \varepsilon = -\varepsilon_2^{-1} j(\gamma_n(u_{\varepsilon})-s) + 
\end{align*}
\]

as possible approximations to the hydrostatic pressure \( p \) and the contact pressure \( \sigma \), respectively, in unilateral problems for incompressible materials, for \( \varepsilon_1 \) and \( \varepsilon_2 \) sufficiently small. Indeed, we hope that \( q \) and \( \varepsilon \) will converge in some sense to the functions \( p \) and \( \sigma \) in (1.1) as \( \varepsilon_1 \) and \( \varepsilon_2 \) tend to zero. Note that in (3.3), and later in this paper, we use the representation \( (\gamma_n(u)-s) = (\gamma_n(u)-s)_+ + (\gamma_n(u)-s)_- \),

where \( (\gamma_n(u)-s)_+(x) = \max(0,(\gamma_n(u)-s)(x)) \) and \( (\gamma_n(u)-s)_-(x) = \min(0,(\gamma_n(u)-s)(x)) \), a.e. on \( \Gamma_C \).

The success of the limit process described above will depend upon whether the sequences \( \{ q \} \) and \( \{ \varepsilon \} \) obtained from (3.3) as \( \varepsilon_1,\varepsilon_2 \to 0 \) are uniformly bounded in \( \varepsilon_1 \) and \( \varepsilon_2 \) in the appropriate norms. A condition sufficient to guarantee the uniform boundness of \( q \) in \( Q \) was first investigated by LADYSZHENSKAYA [1] in her studies of the Stokes problem. Similar conditions were developed for other problems with equality
constraints by BABUSKA [1] (see also BABUSKA and AZIZ [1] and BABUSKA, ODEN, and LEE [1]), and a detailed analysis of such "stability" conditions for linear saddle-point problems was contributed by BREZNI [1]. In view of this history, we will refer to such conditions as "LBB-conditions."

In the present problem, the LBB-condition assumes the following form:

There exists a positive constant $\alpha$ such that

$$\alpha \left( \|q\|_0 + \|\tau\|_{W'} \right) \leq \sup_{\gamma \in \gamma - \{0\}} \frac{|(q, \text{div } \gamma) + \langle \tau, \gamma_n \rangle|}{\|\gamma\|_1}$$

$$\forall (q, \tau) \in Z$$

(3.4)

As an example of conditions sufficient to guarantee that (3.4) holds, we have:

**LEMMA 3.1** Let (1.8) hold. Then (3.4) holds. ■

We give proof of this assertion in Appendix A of this paper.

The relationship between the penalized problem (1.11) and the saddle-point problem (1.1) is established in the following theorem.

**THEOREM 3.2** Let $\{u_\varepsilon\}$ be a sequence of solutions to (1.11) obtained for each $\varepsilon_1, \varepsilon_2$ as $\varepsilon_1, \varepsilon_2 \to 0$, with $\varepsilon_1 \leq N/2\mu$. Let $\{p_\varepsilon\}$ and $\{q_\varepsilon\}$ be sequences with entries defined by (3.3) and let the LBB-condition (3.4) hold. Then
(i) the sequence of triplets \( \{(u_{\varepsilon}, p_{\varepsilon}, \sigma_{\varepsilon})\} \subset V \times Q \times W' \) converges weakly to \( (u, p, \sigma) \subset V \times Q \times M \) as \( \varepsilon_1, \varepsilon_2 \to 0 \), where \( M = \{\tau \in W' | \tau \leq 0\} \).

(ii) the limit \( (u, p, \sigma) \) is the solution of the unilateral contact problem for incompressible materials given by (1.1), and

(iii) constants \( c_1, c_2 > 0 \) exist, independent of \( \varepsilon_1 \) and \( \varepsilon_2 \), such that

\[
\|u - u_{\varepsilon}\|_1 + \|p - p_{\varepsilon}\|_0 + \|\sigma - \sigma_{\varepsilon}\|_{W'} \leq c_1\varepsilon_1 + c_2\varepsilon_2
\]

\[(3.4)\]

PROOF: Without loss in generality, we take \( \varepsilon = \varepsilon_1 = \varepsilon_2 \leq N/2\mu \).

From THEOREM 3.1, \( \{u_{\varepsilon}\} \) is uniformly bounded in \( \varepsilon \). By (3.3), \( (p_{\varepsilon}, \sigma_{\varepsilon}) \subset Z \) and, from (1.11),

\[
\langle (p_{\varepsilon}, \text{div} \gamma) + \langle \sigma_{\varepsilon}, \gamma_n \rangle, \gamma \rangle \leq \|B(u_{\varepsilon}, \gamma)\| + \|f\|_1 \|\gamma\|_1
\]

\[
\leq C \|\gamma\|_1 \quad \forall \gamma \in V
\]

where \( C \) is a constant independent of \( \varepsilon \). Thus, according to (3.4), the sequences \( \{p_{\varepsilon}\} \) and \( \{\sigma_{\varepsilon}\} \) are uniformly bounded in \( \varepsilon \). Since \( V \times Q \times W' \) is a reflexive Banach space, it follows that there exists a subsequence \( \{(u_{\varepsilon}, p_{\varepsilon}, \sigma_{\varepsilon})\} \) which converges weakly to an element

\( (u, p, \sigma) \subset V \times Q \times W' \). We will show subsequently that, in fact, the
original sequence converges strongly to \((u,p,a)\) as \(\varepsilon \to 0\). We first establish that the weak limit satisfied (1.1). The proof of (ii) is given in four steps:

1. From the continuity of \(B(\cdot,\cdot), (\cdot,\cdot), j, \text{div},\) and \(\langle \cdot, \cdot \rangle\), we have, as \(\varepsilon \to 0\),

\[
\cdot f(v) = B(u_\varepsilon, v) - (p_\varepsilon, \text{div } v) - \langle q_\varepsilon, \gamma_n v \rangle + B(u, v) - (p, \text{div } v) - \langle q, \gamma_n v \rangle \quad \forall v \in V
\]

\[
\cdot (q, \text{div } u_\varepsilon) \to (q, \text{div } u) \quad \text{and}
\]

\[
\cdot \langle \tau, \gamma_n (u_\varepsilon) - s \rangle + \langle \tau, \gamma_n (u) - s \rangle \quad \forall q \in Q, \quad \forall \tau \in W'
\]

2. From (1.11) and (3.4), we can easily verify that

\[
\alpha \| \text{div } u_\varepsilon \|_0 + \alpha \| j(\gamma_n (u_\varepsilon) - s) \|_W' \leq C \varepsilon (\| u_\varepsilon \|_1 + \| f \|) + \| j(\gamma_n (u_\varepsilon) - s) \|_W',
\]

\(C\) being a constant independent of \(\varepsilon\). Thus, \(\| \text{div } u_\varepsilon \|_0 \to 0\) and \(\| j(\gamma_n (u_\varepsilon) - s) \|_W' \to 0\) as \(\varepsilon \to 0\). Therefore,

\[
(q, \text{div } u_\varepsilon) = 0 \quad \forall q \in Q \quad \text{and} \quad \gamma_n (u) - s \leq 0 \quad \text{in } W
\]

Indeed, \(\text{div } u = 0\) in \(Q\).
3. We next show that $\sigma \in M$. Let $\phi \in W$ satisfy $\phi \leq 0$.

Then

$$\langle \sigma, \phi \rangle = -\varepsilon^{-1} \langle j(\gamma_n (u_-) - s)_+, \phi \rangle \geq 0$$

so that

$$\langle \sigma, \phi \rangle = \lim_{\varepsilon \to 0} \langle \sigma, \phi \rangle \geq 0 \quad \forall \phi \in W \text{ such that } \phi \leq 0$$

But, by definition, this means that $\sigma \leq 0$; i.e. $\sigma \in M$.

4. From the definition (3.3) of $\sigma$, we have

$$\langle \sigma, \gamma_n (u_-) - s \rangle = \langle \sigma, (\gamma_n (u_-) - s)_+ \rangle + \langle \sigma, (\gamma_n (u_-) - s)_- \rangle$$

$$= \langle \sigma, (\gamma_n (u_-) - s)_+ \rangle + 0 \leq 0 \quad (3.6)$$

Thus, from (1.11),

$$0 \geq B(u_- , u_-) - (p, \text{div } u_-) - f(u_-) - \langle \sigma , \phi \rangle$$

Taking the lim inf of the above inequality yields

$$0 \geq B(u, u) - (p, \text{div } u) - f(u) - \langle \sigma , s \rangle = \langle \sigma , \gamma_n (u) - s \rangle$$

Thus,
\[ \langle \sigma, \gamma_n(u)-s \rangle \leq 0 \]

But \( \sigma \in M \) and \( \gamma_n(u)-s \leq 0 \) in \( W \). Therefore,

\[ \langle \sigma, \gamma_n(u)-s \rangle = 0 \quad (3.7) \]

It follows from (3.7) that

\[ \langle \tau - \sigma, \gamma_n(u)-s \rangle \geq 0 \quad \forall \tau \in M \quad (3.8) \]

Up to this point, we have shown that the limit \((u,p,\sigma)\) of the subsequence \( \{(u_\varepsilon,p_\varepsilon,\sigma_\varepsilon)\} \) is a solution of (1.1). The uniqueness of a solution \((u,p,\sigma) \in V/X \times Z\) to (1.1) follows easily from (3.4), the properties of \( B(\cdot,\cdot) \) listed in the previous sections, and the closedness of \( X \) and \( Y \). Note that if (1.8) holds, \( V/X = V \). It remains to be shown that condition (iii) in the statement of the theorem holds.

We now consider distinct penalty parameters \( \varepsilon_1 \) and \( \varepsilon_2 \). Subtracting (1.11) from (1.1) gives

\[ B(\omega_\varepsilon,\omega_\varepsilon,\varepsilon) = (p-p_\varepsilon, \text{div } \varepsilon) + \langle \sigma-\varepsilon_\varepsilon, \gamma_n(\varepsilon) \rangle \]

\[ \forall \omega_\varepsilon \in V \quad (3.9) \]

Since \((p-p_\varepsilon, \sigma-\varepsilon_\varepsilon) \in Z\), we can use (3.4) and (2.9) to obtain
\[ \| p - p_\varepsilon \|_0 + \| \sigma - \sigma_\varepsilon \|_{W^1} \leq \frac{1}{\alpha} \sup_{v \in V - \{0\}} \frac{|B(u - u_\varepsilon, v)|}{\| v \|_1} \]

\[ \leq (M_0 / \alpha) \| u - u_\varepsilon \|_1 \quad (3.10) \]

On the other hand, putting \( v = u - u_\varepsilon \) in (3.9) yields

\[ B(u - u_\varepsilon, u - u_\varepsilon) = (p - p_\varepsilon, \text{div} (u - u_\varepsilon)) + \langle \sigma - \sigma_\varepsilon, \gamma_n (u - u_\varepsilon) \rangle \quad (3.8) \]

We next note that

\[ - \langle \sigma - \sigma_\varepsilon, \gamma_n (u_\varepsilon) - s \rangle = - \langle \sigma - \sigma_\varepsilon, (\gamma_n (u_\varepsilon) - s) \rangle - \langle \sigma - \sigma_\varepsilon, (\gamma_n (u_\varepsilon) - s) \rangle \]

\[ = \langle \sigma - \sigma_\varepsilon, j^{-1}(\sigma_\varepsilon) \rangle \epsilon_2 - \langle \sigma - \sigma_\varepsilon, (\gamma_n (u_\varepsilon) - s) \rangle \]

\[ \leq \epsilon_2 \langle \sigma - \sigma_\varepsilon, j^{-1}(\sigma_\varepsilon) \rangle \]

Using this result, the definitions (3.3), and the fact that, according to (3.8), \( \langle \sigma - \sigma_\varepsilon, \gamma_n (u_\varepsilon) - s \rangle > 0 \), we have

\[ B(u - u_\varepsilon, u - u_\varepsilon) \leq \epsilon_1 (p - p_\varepsilon, p_\varepsilon) + \epsilon_2 \langle \sigma - \sigma_\varepsilon, j^{-1}(\sigma_\varepsilon) \rangle \quad (3.9) \]

\[ \leq - \epsilon_1 \| p - p_\varepsilon \|_0^2 + \epsilon_1 (p - p_\varepsilon, p) - \epsilon_2 \| \sigma - \sigma_\varepsilon \|_{W^1}^2 + \epsilon_2 \langle \sigma - \sigma_\varepsilon, j^{-1}(\sigma) \rangle \]
\[ \varepsilon_1 \| u_* - u \|_0 \leq \varepsilon_2 \| \sigma - \sigma_* \|_{W'} + \| \sigma \|_{W'}. \quad (3.11) \]

Our next objective is to find an estimate of \( \| \text{div} (u - u_\varepsilon) \|_0^2 \) in terms of \( \varepsilon_1 \). From (1.11),

\[ \| \text{div} (u - u_\varepsilon) \|_0^2 = \| \text{div} u_\varepsilon \|_0^2 = \varepsilon_1 [-B(u_* - u_\varepsilon) + \langle \sigma_\varepsilon - \sigma, \gamma_n (u_\varepsilon) \rangle + f(u_\varepsilon)]. \]

Applying the relations

\[ -B(u_* - u_\varepsilon) = -B(u_* - u_\varepsilon - v) - B(u_* - u_\varepsilon - u) - B(u_\varepsilon - u_\varepsilon) , \]

\[ B(u_* - u_\varepsilon) = \langle \sigma_\varepsilon - \sigma, \gamma_n (u_\varepsilon) \rangle, \quad \text{and} \]

\[-B(v_\varepsilon - u_\varepsilon) + f(u_\varepsilon) = -\langle \sigma_\varepsilon, \gamma_n (u_\varepsilon) \rangle - (\rho, \text{div} u_\varepsilon) , \]

we obtain

\[ \| \text{div} (u - u_\varepsilon) \|_0^2 \leq \varepsilon_1 [(-\rho, \text{div} u_\varepsilon) + \langle \sigma_\varepsilon - \sigma, \gamma_n (u_\varepsilon - u) \rangle] \]

In view of (3.6), the definition of \( \sigma_\varepsilon \), and the fact that \( \gamma_n (u_\varepsilon - u) \leq 0 \),

\[ \langle \sigma_\varepsilon, \gamma_n (u_\varepsilon - u) \rangle - \langle \sigma_\varepsilon, \gamma_n (u_\varepsilon - u) \rangle \leq 0 , \]

Therefore,
Adding (3.11) and (3.12) and using (2.6), we have

\[ \| \text{div} (u-u_\varepsilon) \|_0^2 \leq \varepsilon_1 ((p, \text{div} (u-u_\varepsilon)) + \langle \sigma, \gamma_n (u-u_\varepsilon) \rangle) \]

\[ \leq c\varepsilon_1 (\| p \|_0 + \| \sigma \|_{H^1\varepsilon}) \| u-u_\varepsilon \|_1 \]

(3.12)

Adding (3.11) and (3.12) and using (2.6), we have

\[ \hat{m}_0 \| u-u_\varepsilon \|_1^2 \leq B(u-u_\varepsilon, u-u_\varepsilon) + \| \text{div} (u-u_\varepsilon) \|_0^2 \]

\[ \leq (\| p-p_\varepsilon \|_0 + \| \sigma-\sigma_\varepsilon \|_{H^1\varepsilon}) (\| p \|_0 \varepsilon_1 + \| \sigma \|_{H^1\varepsilon} \varepsilon_2) \]

\[ + c\| u-u_\varepsilon \|_1 (\| p \|_0 + \| \sigma \|_{H^1\varepsilon}) \varepsilon_1 \]

where \( \hat{m}_0 \) is a positive constant independent of \( \varepsilon_1 \) and \( \varepsilon_2 \). Thus, it follows from (3.10) that

\[ \| u-u_\varepsilon \|_1 \leq \frac{\varepsilon_1}{\hat{m}_0} \left( \frac{M_0}{\alpha} + c \right) \| p \|_0 + \frac{1}{\hat{m}_0} \left( c\varepsilon_1 + \frac{M_0}{\alpha} \varepsilon_2 \right) \| \sigma \|_{H^1\varepsilon} \]

Therefore, from this result and (3.10), condition (iii) follows.

REMARK 3.1) Generalizations of Theorem 3.2 to penalty methods for nonlinear variational inequalities, including error estimates in terms of \( \varepsilon \), can be found in KIKUCHI [1].
4. APPROXIMATIONS

4.1 Penalty–Finite Element Approximations. We now construct a family of finite-dimensional subspaces \( \{ V_h \} \) of the space \( V \) using conforming finite elements on a suitable discretization \( \Omega_h \) of \( \Omega \). The index \( h \) is, as usual, the mesh parameter. We will generally assume that the family \( \{ V_h \} \) is generated by regular refinements of the mesh and that \( \Omega_h \) coincides with \( \Omega \). More will be said about the approximation properties assumed for \( V_h \) later.

In anticipation of some numerical difficulties to be addressed later, we will use numerical quadrature to evaluate certain integrals appearing in our approximation of the penalized problem. In particular, for \( f \in C^0(\Omega) \), we use integration rules of the type

\[
I(f) = \sum_{e=1}^{E} I_{G_1}^e(f) \approx \int_{\Omega} f \, dx
\]

(4.1)

\[
I_{G_1}^e(f) = \sum_{j=1}^{G_1} w_j^e f(\xi_j^e) , \quad \xi_j^e \in \bar{G}_e
\]

Here \( E \) is the total number of elements in the mesh and \( w_j^e \) and \( \xi_j^e \) are the quadrature weights and points within an element \( \bar{G}_e \), \( 1 \leq j \leq G_1 \). Thus, \( I_{G_1}^e(f) \approx \int_{\bar{G}_e} f \, dx \), \( 1 \leq e \leq E \). Likewise, for \( f \in C^0(\bar{G}_C) \), we define
\[ J(f) = \sum_{e=1}^{E'} J^e_G(f) \approx \int_{\Gamma_C} f \, dx \]

\[ J^e_G(f) = \sum_{j=1}^{G_2} Q_j^e f(\eta_j^e), \quad \eta_j^e \in \overline{Q}_e \cap \overline{\Gamma}_C \]

where \( E' \) is the number of elements with boundaries \( \Gamma^e \cap \Gamma_C \neq \emptyset \), \( Q_j^e \) and \( \eta_j^e \) are the quadrature weights and integration points, respectively, \( 1 \leq j \leq G_2 \), and \( J^e_G(f) \approx \int_{\Gamma^e \cap \Gamma_C} f \, ds \). Clearly, \( G_1 \) and \( G_2 \) can, in most cases, be chosen so that the integration rules are exact for polynomials \( f \) of a given degree.

Our approximation of the penalized variational problem (1.11) assumes the following form:

For given \( \varepsilon_1, \varepsilon_2 > 0 \), find \( u_h^\varepsilon \in V_h \) such that

\[ \begin{align*}
B(u_h^\varepsilon, v^\varepsilon) + \varepsilon_1^{-1}I(\text{div} u_h^\varepsilon, \text{div} v^\varepsilon) \\
+ \varepsilon_2^{-1}J((\gamma_n u_h^\varepsilon)_\varepsilon + \gamma_n v^\varepsilon) = f(v^\varepsilon) \quad \forall v^\varepsilon \in V_h
\end{align*} \]  

(4.3)

\[ 4.2 \text{ Existence of Approximate Solutions.} \]

We cannot conclude from Theorem 3.1 that there exist solutions to the approximate problem (4.3). The proof of existence of solutions to the "continuous" problem (1.11) was based on Korn's inequality (2.6), which may not hold for the approximate forms in (4.3) due to the possible inexact inte-
gration rules $I$ and $J$ of the penalty terms. What is needed is to choose integration formulas that will preserve the ellipticity of these forms on $V_h$. In particular, let $||| \cdot ||| : V_h \to \mathbb{R}$ be defined by

$$|||v_h^h||| = \frac{1}{2\mu} B(v_h^h, v_h^h) + \frac{1}{N} I[(\text{div } v_h^h)^2]$$ (4.4)

Then we easily establish the following:

**THEOREM 4.1** Let the quadrature weights $w_{ij}^e$ and $q_{ij}^e$ of (4.1) and (4.2) be positive. Moreover, let there exist a positive constant $C$, independent of $v_h^h$, such that

$$|||v_h^h||| \geq C ||v_h^h||_1 \quad v_h^h \in V_h$$ (4.5)

Then, for every $\varepsilon_1, 0 < \varepsilon_1 \leq N/2\mu$, and every $\varepsilon_2 > 0$, there exists a unique solution $u_{\varepsilon}^h \in V_h$ to problem (4.3). Moreover, the solution $u_{\varepsilon}^h$ is uniformly bounded in $\varepsilon_1, \varepsilon_2$ in $V$: $||u_{\varepsilon}^h||_1 \leq C$, $C$ a constant independent of $\varepsilon_1$ and $\varepsilon_2$. \hfill \blacksquare

Condition (4.5) must be verified for each choice of finite element. However, we are particularly interested here in the use of quadrilateral elements for two-dimensional problems because of their popularity in applications and because their behavior in approximations such as (4.3) is considerably more delicate than that of triangular elements. Typical examples of such elements are the bilinear $(Q_1)$-elements with four nodes and the nine-node bi quadratic $(Q_2)$-elements.
shown in Fig. 1. For these choices, we can prove the next result.

**LEMMA 4.1** Let $\Omega_h$ be a rectangular mesh in $\mathbb{R}^2$ constructed using either $Q_1$ or $Q_2$ elements. Let the approximate penalty functional $I$ of (4.1) be computed using the following quadrature rules:

- $Q_1 : I(f) = 2 \times 2$ or 1-point Gaussian quadrature
- $Q_2 : I(f) = 3 \times 3$, $2 \times 2$, or 1-point Gaussian quadrature

Then there exists a constant $C$, independent of $h$, such that (4.5) holds on spaces $V_h \subset V$ constructed using these elements.

**PROOF:** We prove this result in Appendix B. •

**REMARK 4.1** While (4.5) holds for $Q_1$-elements with $2 \times 2$ Gauss and $Q_2$-elements with $3 \times 3$ Gauss quadrature (exact integration), both of these methods are unsatisfactory. The approximation of the penalty terms (such as $\|\text{div} u\|^2_0$) is positive definite, which means that the approximate solution is "locked" (i.e. over-constrained). The approximate displacement field $u^h_{\varepsilon}$ is then identically zero. To overcome such difficulties, "reduced integration" (inexact quadrature) must be used. We elaborate on these points in Section 5. •

**4.3 An Algorithm.** Owing to the presence of the term $(\gamma_n (u^h_{\varepsilon}-s)+$ in (4.3), the problem is nonlinear. While there are many numerical
\( \odot \): nodal point

\( \times \): Gauss-points for \( I \)

**FIGURE 1.** Bilinear and biquadratic elements and location of Gaussian-quadrature points for computing the functional \( I \).
schemes that could be used to solve this nonlinear system, a simple successive-iteration scheme has proved to be very effective in all of the applications we have considered. The algorithm consists of the steps

1. Set the first iterate \( u_{\varepsilon}^{h,1} = 0 \)

2. Define

\[
\beta_{t+1}(u_{\varepsilon}^h, t) = \begin{cases} 
\gamma_n(u_{\varepsilon}^h, t) - s, & \text{if } \gamma_n(u_{\varepsilon}^h, t) - s > 0 \\
0, & \text{if } \gamma_n(u_{\varepsilon}^h, t) - s \leq 0 
\end{cases} \quad (4.6)
\]

3. Obtain the \( t \)-th iterate \( u_{\varepsilon}^{h,t} \) by solving the linear system,

\[
B(u_{\varepsilon}^h, t, \nabla^h) + \varepsilon_1^{-1} I(\text{div} u_{\varepsilon}^h, \text{div} \nabla^h) + \varepsilon_2^{-1} J[\beta_{t}(u_{\varepsilon}^h, t) \gamma_n(\nabla^h)] \\
= f(\nabla^h) \quad \forall \nabla^h \in V_h
\]

\[
(4.7)
\]

4. Repeat the process until the relative error,

\[
e_{R}^{t+1} = \frac{\|u_{\varepsilon}^{h, t+1} - u_{\varepsilon}^{h, t}\|_1}{\|u_{\varepsilon}^{h, t+1}\|_1}
\]

is less than a preassigned tolerance \( \varepsilon_r \).

Various sparse matrix routines can be used to solve (4.7).
5. CONVERGENCE TO A MIXED FORMULATION (h fixed)

The fact that (4.3) is solvable for \( u_\varepsilon^h \) is no indication that (4.3) represents an adequate approximation of (1.11). Moreover, we have not indicated a method for extracting approximations of the pressures \( p \) and \( \sigma \) from (4.3). The most obvious method for approximating these pressures is suggested by (3.3): set

\[
\rho_\varepsilon^h = -\varepsilon_1^{-1} \text{div} u_\varepsilon^h \quad \text{and} \quad \mu_\varepsilon^h = -\varepsilon_2^{-1} (\gamma_n (u_\varepsilon^h) - s)_+ \tag{5.1}
\]

and hope that \( \rho_\varepsilon^h \rightarrow p \) and \( \mu_\varepsilon^h \rightarrow \mu \) in appropriate norms as \( \varepsilon_1, \varepsilon_2 \rightarrow 0 \). Unfortunately, such approximations are, in general, unstable and, at best, result in a delicate conditionally stable scheme with suboptimal rates of convergence.

Key points in arriving at acceptable approximations of \( p \) and \( \sigma \) lie in the observations that

1. whatever form the approximations \( (u_\varepsilon^h, p_\varepsilon^h, \sigma_\varepsilon^h) \) take, they should produce an acceptable mixed finite element approximation of (1.1) in the limit as \( \varepsilon_1, \varepsilon_2 \rightarrow 0 \) for each fixed \( h \),

2. any mixed approximation of (1.1) will naturally involve three finite-dimensional approximation spaces: the space \( V_h \) of displacements \( v_h \), and spaces \( Q_h \) and \( W_h \) of approximate hydrostatic and contact pressures, respectively,

3. the penalty formulation (4.3) is described in terms of a single approximation space \( V_h \) for the displacements, and
4. (in view of (1), (2), and (3)) the structure of the spaces \( Q_h \) and \( W_h' \) in the mixed formulation must be intrinsically related to how we define the approximate pressures \( p_{e}^{h} \) and \( q_{e}^{h} \) and to the form of the numerical quadrature rules \( I \) and \( J \) used in (4.3).

With these observations in mind, we now lay down the conditions which will serve to characterize the spaces \( Q_h \) and \( W_h' \). We shall then show that these provide sufficient conditions for the penalty formulation (4.3) to produce an acceptable mixed finite element method as \( \varepsilon_1, \varepsilon_2 \to 0 \).

Let the mesh \( \Omega_h = \Omega \) consist of \( E \) finite elements \( \Omega_e \), \( e = 1, 2, \ldots, E \), with boundaries \( \Gamma_e \) and let \( \Gamma^h_C = \Gamma_C \) denote the intersection of \( \Gamma_C \) with \( \Omega_h \). We seek families of finite dimensional spaces \( Q_h \subset Q \cap \mathcal{C}^0(\Omega_h) \) and \( W_h' \subset W' \cap \mathcal{C}^0(\Gamma^h_C) \), where

\[
\mathcal{C}^0(\Omega_h) = \{ f \in L^2(\Omega_h) \mid f|_{\Omega_e} \in \mathcal{C}^0(\Omega_e) \} \tag{5.2}
\]

\[
\mathcal{C}^0(\Gamma^h_C) = \{ f \in L^2(\Gamma_C) \mid f|_{\Gamma_e \cap \Gamma^h_c} \in \mathcal{C}^0(\Gamma_e \cap \Gamma^h_c) \}
\]

such that the following conditions are satisfied:

A. The numerical quadrature rules \( I \) and \( J \) of (4.1) and (4.2) and the spaces \( Q_h \) and \( W_h' \) are such that

\[
\begin{align*}
(q^h, \hat{q}^h) &= I(q^h, q^h) \quad \forall q^h, \hat{q}^h \in Q_h \\
\langle \tau^h, \hat{\tau}^h \rangle &= J(\tau^h, \tau^h) \quad \forall \tau^h, \hat{\tau}^h \in W_h'
\end{align*}
\tag{5.3}
\]
B. There exist unique elements $p^h_\varepsilon \in Q_h$ and $\sigma^h_\varepsilon \in W_h$ such that

$$p^h_\varepsilon(\xi^e_1) = -\frac{1}{c_1} \text{div} u^h_\varepsilon(\xi^e_1)$$
$$\sigma^h_\varepsilon(\eta^e_2) = -\frac{1}{c_2} (\gamma_n(u^h_\varepsilon) - s)(\eta^e_2)$$

(5.4)

where $u^h_\varepsilon$ is the solution of (4,3) for fixed $\varepsilon_1$, $\varepsilon_2$, and $h$ and $\xi^e_1$ and $\eta^e_2$ are the quadrature points in the numerical integration rules $I$ and $J$, respectively.

C. Let

$$B_h : V_h \to Q'_h \times W_h, \quad B^*_h : Q_h \times W'_h \to V'_h$$

with $Q_h$ identified with $Q'_h$, be the discrete operators defined by

$$\langle (q^h, \tau^h), B_h v^h \rangle = \langle B^*_h(q^h, \tau^h), v^h \rangle$$

$$= I(q^h \text{div} v^h) + J(\tau^h \gamma_n(v^h))$$

with $v^h \in V_h$, $q^h \in Q_h$, $\tau^h \in W'_h$. 


Then there exist constants $\alpha_h, \beta_h > 0$ such that

$$
\alpha_h \| q^h \|_0 + \beta_h \| \tau^h \|_{0, \Gamma} \leq \sup_{y^h \in V^h - \{0\}} \frac{| \langle b^*(q^h, \tau^h), y^h \rangle |}{\| y^h \|_1}
$$

$$
\forall \ (q^h, \tau^h) \in Z_h
$$

wherein

$$
Z_h = (Q^h_h \times W^h_h)/(\ker B^*_h)
$$

**REMARKS:**

5.1) Note that condition A endows the spaces $Q^h_h$ and $W^h_h$ with the property that the quadrature rules $I(\cdot)$ and $J(\cdot)$ provide exact integration in $Q^h_h$ and $W^h_h$, respectively, even though $I(\cdot)$ and $J(\cdot)$ will not integrate $(\text{div} \, y^h, \text{div} \, y^h)$ or $\int_{\Gamma} \gamma_n (y^h)^2 \, ds$ exactly.

5.2) According to (5.4), the approximate pressures $p^h$ and $\phi^h$ are defined by applying (3.3) at the quadrature points of the integration formulas (4.1) and (4.2) rather than as in (5.1). If these quadrature points are not situated on element boundaries (which is often the case), the spaces $Q^h_h$ and $W^h_h$ will contain "nonconforming" finite elements, in the sense that $q^h$ and $\tau^h$ will suffer discontinuities across interelement boundaries.

5.3) Conditions A and B essentially determine the spaces $Q^h_h$ and $W^h_h$. If $Q_2$-elements are used to construct $V^h_h$ and $3 \times 3$ (exact) Gaussian integration is used in formula I, then $Q^h_h$ consists of piecewise biquadratic elements with nodal points located at, for example,
each of the nine Gauss points in an element. If the rule $I$ of (4.1) is 2x2-Gaussian quadrature, $Q_h$ will consist of piecewise bilinear functions on elements with nodes at each of the four Gauss points. If $Q_1$-elements are used to construct $V_h$, and $I$ is defined by a 1-point rule, then $Q_h$ consists of functions which are piecewise constants (constant over each element in the mesh). Similarly, if $Q_2$-elements are used for constructing $V_h$, and Simpson's rule is used for the integration formula $J$ of (4.2), then $W^i_h$ becomes the space of continuous, piecewise quadratic functions defined on $\Gamma_C$. We elaborate on these points later.

5.4) The parameters $\alpha_h$ and $\beta_h$ may depend upon the mesh size $h$. The fact that (5.7) need hold only on $Z_h$ and the fact that $\ker B_h^*$ may not contain $\ker B_h^*$ has apparently been overlooked by many others who have studied mixed finite element methods for problems of this type. In view of our remarks in Section 3, we will refer to (5.7) as a discrete LBB-condition.

**THEOREM 5.1** Let conditions A, B, and C above and the hypothesis of THEOREM 4.1 hold for $s \in C^0(\Gamma_C)$. For fixed $h$, let $\{u^h\}$ be a sequence of solutions of (4.3) obtained as $\varepsilon_1, \varepsilon_2 \to 0$ and $\{p^h\}$ and $\{\sigma^h\}$ sequences constructed using (5.4). Then these sequences converge to $u^h \in V_h$, $p^h \in Q_h$, and $\sigma^h \in W^i_h$, respectively as $\varepsilon_1, \varepsilon_2 \to 0$. Moreover, the limit $(u^h, p^h, \sigma^h)$ satisfies

$$B(u^h, v^h) - I(p^h \text{div } v^h) - J[\sigma^h \gamma_h(v^h)] = f(v^h) \quad \forall v^h \in V_h$$
\[ I(q^h \text{div} \ u^h) = 0 \quad \forall \ q^h \in Q_h \] (5.9)

\[ J[(\tau^h - \sigma^h)(\gamma_n(u^h) - s)] \geq 0 \quad \forall \ \tau^h \in M_h \]

and

\[ (u^h, p^h, \sigma^h) \in K_h \times Q_h \times M_h \] (5.10)

where

\[ K_h = \{ v^h \in V_h \mid \text{div} v^h(\xi^e) = 0, 1 \leq e \leq E, 1 \leq i \leq G_1 \}, \] and

\[ (\gamma_n(v^h) - s)(n^e_j) \leq 0, \ 1 \leq e \leq E, \] (5.11)

\[ 1 \leq j \leq G_2 \}

\[ M_h = \{ \tau^h \in W_h \mid \tau^h(n^e_j) \leq 0, \ 1 \leq e \leq E', \] (5.11)

\[ 1 \leq j \leq G_2 \}

**Proof**: (Uniform boundedness of \( u^h, p^h, \) and \( \sigma^h \)). From THEOREM 4.1, \( u^h \) is uniformly bounded in \( V \) with respect to \( \varepsilon \) for fixed \( h \). Applying condition C, (4.3) implies

\[ \alpha_h \| p^h \|_0 + \beta_h \| \sigma^h \|_0, \gamma \leq C(\| u^h \|_1 + \| f \|_1^*) \]
are also uniformly bounded in $\mathcal{L}$. Therefore, there exist convergent subsequences of $u_{h,\epsilon}^h, p_{h,\epsilon}^h, \text{ and } \sigma_{h,\epsilon}^h$, still denoted $u_{h,\epsilon}^h, p_{h,\epsilon}^h, \text{ and } \sigma_{h,\epsilon}^h$. We note that $\sigma_{h,\epsilon}^h(n_{\epsilon}^h) \to \sigma_h^h$ as $\epsilon \to 0$, from the definition of $J[\sigma_{h,\epsilon}^h(\nu_{\epsilon}^h)]$, since $\sigma_{h,\epsilon}^h$ converges to $\sigma_h^h$ in $\mathcal{C}^0(T^h)$ as $\epsilon \to 0$. Passing to the limit as $\epsilon_1, \epsilon_2 \to 0$ in the penalty-finite element formulation (4.3), we have

$$B(u_{\infty}^h, \psi_{\infty}^h) - I(p_{\infty}^h \text{div } \psi_{\infty}^h) - J[\sigma_{\infty}^h(\nu_{\infty}^h)] = f(\psi_{\infty}^h), \quad \psi_{\infty}^h \in \psi_{h_{\infty}}$$

Next we will prove that $u_{\infty}^h \in K_h$. To this end, we first note that equation (4.3) can be represented by the inequality

$$B(u_{\infty}^h, \psi_{\infty}^h - u_{\infty}^h) + \frac{1}{\epsilon_1} I(\text{div } u_{\infty}^h \text{div } (\psi_{\infty}^h - u_{\infty}^h))$$

$$+ \frac{1}{\epsilon_2} J[(\gamma_n(u_{\infty}^h) - s) + \gamma_n(\psi_{\infty}^h - u_{\infty}^h)] \geq f(\psi_{\infty}^h), \quad \psi_{\infty}^h \in \psi_{h_{\infty}}$$

Taking $\psi_{\infty}^h \in K_h$ yields

$$B(u_{\infty}^h, \psi_{\infty}^h - u_{\infty}^h) - f(\psi_{\infty}^h)$$

$$\geq \frac{1}{\epsilon_1} I(\text{div } u_{\infty}^h \text{div } u_{\infty}^h) + \frac{1}{\epsilon_2} J[(\gamma_n(u_{\infty}^h) - s) + (\gamma_n(u_{\infty}^h) - s)]$$

If $\epsilon = \epsilon_1 = \epsilon_2$, then

$$\epsilon(B(u_{\infty}^h, \psi_{\infty}^h) - f(\psi_{\infty}^h))$$

$$\geq I(\text{div } u_{\infty}^h \text{div } u_{\infty}^h) + J[(\gamma_n(u_{\infty}^h) - s) + (\gamma_n(u_{\infty}^h) - s)]$$
In the above inequality, we have
\[ \text{div } u^h(\xi^e) = 0 \quad \text{and} \quad (\gamma_n(u^h) - s)(\eta^e) \leq 0, \]
that is, \( u^h \in \mathcal{K}_h \) and that this immediately implies that
\[ I(q^h \text{div } u^h) = \sum_{e=1}^{E} \sum_{i=1}^{G_1} W^e_i q^h(\xi^e_{i}) \text{div } u^h(\xi^e_{i}) = 0 \]
for every \( q^h \in Q_h \).

Since \( u^h \in \mathcal{K}_h \), for every \( \tau^h \in M_h \),
\[ J[(\tau^h - \sigma^h)(\gamma_n(u^h) - s)] \geq \sum_{e=1}^{E} \sum_{j=1}^{G_2} \Phi^h_j(\eta^e_j)(\gamma_n(u^h) - s)(\eta^e_j) \geq 0 \]
Passing to the limit \( \varepsilon \to 0 \) gives
\[ J[(\tau^h - \sigma^h)(\gamma_n(u^h) - s)] \geq \sum_{e=1}^{E} \sum_{j=1}^{G_2} \Phi^h_j(\eta^e_j)(\gamma_n(u^h) - s)(\eta^e_j) \geq 0 \]
Since \( \sigma^h(e_j) \leq 0 \), we must have \( \sigma^h \in M_h \).

The next objective is to find an estimate of the approximation \( u^h \) to \( u^h \) in terms of the penalty parameters \( \varepsilon_1 \) and \( \varepsilon_2 \).

**Theorem 5.2** Let the hypothesis of **Theorem 5.1** hold, let
\( (u^h, p^h, s^h) \) be the solution of (4.3) with \( p^h \) and \( s^h \) defined by (5.4),
and let \( (u^h, p^h, s^h) \) be the solution of (5.9). Then, for \( \varepsilon_1 = \varepsilon_2 = \varepsilon \),
\[ B(u_h^{1/2} - u_{h_{1/2}}^{1/2}, u_h^{1/2} - u_{h_{1/2}}^{1/2}) \leq \varepsilon_1 \| p_h - p_{h\varepsilon} \|_0 \| p_h \|_0 + \varepsilon_2 \| \sigma_h - \sigma_{h\varepsilon} \|_{0, \Gamma_C} \| \sigma_h \|_{0, \Gamma_C} \] (5.12)

\[ I[(\text{div} (u_h^{1/2} - u_{h_{1/2}}^{1/2}), u_h^{1/2} - u_{h_{1/2}}^{1/2})] \leq \varepsilon_1 \left\{ I[(\text{div} (u_h^{1/2} - u_{h_{1/2}}^{1/2}))^2] \right\}^{1/2} \| p_h \|_0 \]

\[ + \| \sigma_h \|_{0, \Gamma_C} \left\{ J[(\gamma_n(u_h^{1/2} - u_{h_{1/2}}^{1/2}))^2] \right\}^{1/2} \] (5.13)

If a constant \( C \), independent of \( h \), exists such that

\[ J[(\gamma_n(v_h^{1/2}))^2] \leq C \| v_h^{1/2} \|_1 \quad \forall v_h^{1/2} \in V_h \] (5.14)

Then

\[ \| p_h - p_{h\varepsilon} \|_0 \leq \frac{M_o}{\alpha_h} \| u_h - u_{h\varepsilon} \|_1 \] (5.15)

\[ \| \sigma_h - \sigma_{h\varepsilon} \|_{0, \Gamma_C} \leq \frac{M_o}{\beta_h} \| u_h - u_{h\varepsilon} \|_1 \]

\[ \| u - u_{\varepsilon} \|_1 \leq \frac{\varepsilon_1}{m_{1/2}} (1 + \frac{M_o}{\alpha_h}) \| p_h \|_0 + \frac{\varepsilon_2}{m_{1/2}} \left( c \varepsilon_1 + \frac{M_o}{\beta_h} \right) \| \sigma_h \|_{0, \Gamma_C} \] (5.16)

**PROOF:** Subtracting (4.3) from (5.9) and setting \( v_h^{1/2} = u_h - u_{h_{1/2}}^{1/2} \)

gives

\[ B(u_h - u_{h_{1/2}}^{1/2}, u_h - u_{h_{1/2}}^{1/2}) = I[(p_h - p_{h\varepsilon}) \text{div} (u_h - u_{h_{1/2}}^{1/2})] + J[(\sigma_h - \sigma_{h\varepsilon}) \gamma_n(u_h - u_{h_{1/2}}^{1/2})] \] (5.17)
Using (5.3) and (5.4), we easily verify that

\[ I((p^h - p^h_\varepsilon) \text{div} (u^h - u^h_\varepsilon)) \leq \varepsilon_1 \| p^h - p^h_\varepsilon \|_0 \| p^h \|_0 \]  

(5.18)

and, together with (5.9), that

\[ J[(\sigma^h - \sigma^h_\varepsilon) \gamma_n (u^h - u^h_\varepsilon)] \leq J[(\sigma^h - \sigma^h_\varepsilon) \gamma_n (u^h_\varepsilon - s)] \]

\[ = -\varepsilon_2 J[(\sigma^h_\varepsilon - \sigma^h)(\sigma^h_\varepsilon) + J[(\sigma^h - \sigma^h_\varepsilon) \gamma_n (u^h_\varepsilon - s)]] \]

\[ \leq -\varepsilon_2 J[(\sigma^h - \sigma^h_\varepsilon) (\sigma^h - \sigma^h_\varepsilon)] \]

\[ \leq -\varepsilon_2 J[(\sigma^h_\varepsilon - \sigma^h_\varepsilon)] \]

\[ \leq \varepsilon_2 \| \sigma^h_\varepsilon - \sigma^h \|_0, \Gamma_C \| \sigma^h \|_0, \Gamma_C \]  

(5.19)

Inequality (5.12) follows from (5.17), (5.18), and (5.19).

To derive (5.13), note that

\[ I(\text{div} (u^h - u^h_\varepsilon) \text{div} (u^h - u^h_\varepsilon)) = I(\text{div} u^h \text{div} u^h_\varepsilon) \]

\[ = \varepsilon_1 \{-B(u^h, u^h_\varepsilon) + J[\sigma^h_\varepsilon \gamma_n (u^h_\varepsilon)] + f(u^h_\varepsilon)\} \]
Using the relations

\[-B(u_h^+, u_h^-) = -B(u_h^+, u_h^- - u_h^+) - B(u_h^-, u_h^-) - B(u_h^-, u_h^+),\]

\[-B(u_h^+, u_h^-) = -(p_h^+, \text{div } u_h^-) - J[\sigma_h^+ \gamma_n(u_h^-)] - f(u_h^-),\]

\[-B(u_h^-, u_h^-)' = J[(\sigma_h^- - \sigma_h^+)\gamma_n(u_h^-)],\]

we obtain

\[I(\text{div } u_h^+, \text{div } u_h^-) \leq \varepsilon \{ (p_h^-, \text{div } u_h^-) + J[(\sigma_h^- - \sigma_h^+)\gamma_n(u_h^-)] \} \]

Inequality (5.13) follows from this result and the fact that

\[-J[\sigma_h^+ \gamma_n(u_h^-)] + J[\sigma_h^- \gamma_n(u_h^-)] \leq 0\]

The estimate (5.15) follows immediately from the discrete LBB-condition (5.7) and (5.12). To obtain (5.16), we add (5.12) and (5.13) and then use (4.5) and the Schwartz inequality. \[\square\]
6. CONVERGENCE OF THE METHOD

Error estimates for our "reduced integration mixed method" (5.9), which is obtained from the penalized problem in the limit as \( \epsilon_1, \epsilon_2 \to 0 \), are given in the following theorem:

**THEOREM 6.1.** Let \((u, p, \sigma) \in V \times Q \times M\) and \((u_h, p_h, \sigma_h) \in V_h \times Q_h \times M_h\) be solutions of (1.1) and (5.9), respectively. Let \( s \in C^0(\Gamma_C) \). Then there exists a constant \( C > 0 \), independent of \( h \), such that the following estimates hold:

\[
\begin{align*}
\| u_{\text{r}} \|_1 & \leq C\left( (1 + \frac{1}{\alpha_h}) \| u_{\text{r}} \|_1 + \frac{1}{eta_h} \| \gamma_n (u_{\text{r}}) \|_0, \Gamma_C \right) \\
& \quad + \| p_{\text{r}} \|_0 + \| \sigma_{\text{r}} \|_{W'} + |\langle \tau_{\text{r}}, \gamma_n (u_{\text{r}}) \rangle|^{1/2} \\
& \quad + |\langle \tau_{\text{r}}, \gamma_n (u_{\text{r}}) \rangle|^{1/2} + \xi(E)^{1/2} + \zeta(E) \right) \quad (6.1) \\
\| p_{\text{r}} \|_0 & \leq (1 + \frac{1}{\alpha_h}) \| p_{\text{r}} \|_0 + \frac{1}{\sigma_h} \{ M_o \| \gamma_n (u_{\text{r}}) \|_1 \\
& \quad + \| \sigma_{\text{r}} \|_{W'} + \xi(E) \right) \} \quad (6.2) \\
\| \sigma_{\text{r}} \|_{0, \Gamma_C} & \leq \| \sigma_{\text{r}} \|_{0, \Gamma_C} + \frac{1}{\beta_h} \{ \| \sigma_{\text{r}} \|_{W'} \\
& \quad + M_o \| \gamma_n (u_{\text{r}}) \|_1 + \| p_{\text{r}} \|_0 + \zeta(E) \right) \} \quad (6.3)
\end{align*}
\]
for every $(v^h, q^h, T^h) \in V_h \times Q_h \times M_h$. Here

$$
\zeta(E) = \sup_{v^h \in V_h} \frac{|E_1(p \div v^h)|}{\|v^h\|_1} + \sup_{v^h \in V_h} \frac{|E_1(\sigma \gamma_n(v^h))|}{\|v^h\|_1}
$$

(6.4)

$$
\zeta(E) = |E_1((\div u^h)^2)| + |E_J(\tau^h(\gamma_n(u) - s))|
+ |E_J(\sigma^h(\gamma_n(v^h) - s))|
$$

(6.5)

with

$$
E_1(f) = \left\{ \begin{array}{l}
\int_{\Omega} f \, dx - I(f) , \ f \in C^0(\Omega) \\
\end{array} \right\}
$$

$$
E_J(f) = \left\{ \begin{array}{l}
\int_{\Gamma_C} f \, ds - J(f) , \ f \in C^0(\Gamma_C) \\
\end{array} \right\}
$$

(6.6)

**PROOF.** From (1.1) and (5.9), we obtain the following identity

$$
B(u-u, u-u) = B(u-u, u-v) + (p, \div (v-u))
$$

Furthermore, we have, after some algebraic manipulations,

$$
(p, \div (v-u)) - I(p^h \div (v^h-u^h)) = (p-p^h, \div (v^h-u))
+ (p-q^h, \div (u-u^h))
$$
\[
\langle 0, \gamma_n (v^h - u^h) \rangle - J(\sigma_n (v^h - u^h)) = \langle \sigma - \sigma^h, \gamma_n (v^h - u) \rangle + \langle \tau^h - \sigma, \gamma_n (u) - s \rangle
\]
\[
+ \langle \tau - \sigma^h, \gamma_n (u) - s \rangle + E_J(\sigma^h (\gamma_n (v^h) - s)) - E_J(\tau^h (\gamma_n (u^h) - s))
\]

Also, since \( \text{div} \ u = 0 \),

\[
\|\text{div} \ (u - u^h)\|_0^2 = \|\text{div} \ u^h\|_0^2
\]  

(6.7)

Incorporating the above results into (2.6), we obtain

\[
m_0 \| u - u^h \|^2_1 \leq B(u - u^h, u - u^h) + \| \text{div} \ (u - u^h)\|_0^2
\]

\[
\leq M_0 \| u - u^h \|_1 \| u - v^h \|_1 + \| p - p^h \|_0 \| v^h - u \|_1 + \| p - q^h \|_0 \| u - u^h \|_1
\]

\[
+ \| \sigma - \tau^h \|_{W'} \| v^h - u \|_1 + \| \sigma - \tau^h \|_{W'} \| u - u^h \|_1
\]

\[
+ \| \tau^h - \sigma^h \|_{0, \Gamma_C} \| \gamma_n (v^h - u) \|_{0, \Gamma_C} + \| \text{div} \ u^h \|_0^2 + | \langle \tau^h - \sigma, \gamma_n (u) - s \rangle |
\]

\[
+ | \langle \tau - \sigma^h, \gamma_n (u) - s \rangle | + | E_J(\sigma^h (\gamma_n (v^h) - s)) | + | E_J(\tau^h (\gamma_n (u^h) - s)) |
\]  

(6.8)

Next we need to estimate \( \| p - p^h \|_0 \) and \( \| \tau^h - \sigma^h \|_{0, \Gamma_C} \) in (6.8). Again, from (1.1) and (5.9) we can obtain the following bound:
Under the conditions of Theorems 5.2 and 6.1, the final error estimates for our reduced-integration penalty methods now follow from Theorems 5.2 and 6.1, and the triangle inequality.

The results are recorded as follows:

**Theorem 6.2.** Under the conditions of Theorems 5.2 and 6.1, the errors in the reduced-integration penalty method characterized by (4.3) satisfy the following estimates:
\[ \|u_u\|_1 \leq C \left\{ \inf_{v_h} \left[ \left( 1 + \frac{1}{\alpha_h} \right) \|u-u^h\|_1 + \frac{1}{\beta_h} \|\gamma_n(u-u^h)\|_{0,G_C} \right] \right\} \]

\[ + \inf_{q_h \in Q_h} \|p-q^h\|_0 + \inf_{\tau^h \in M_h} |<\tau^h, \gamma_n(u-u^h)>|^{1/2} \]

\[ + |<\tau^h, \gamma_n(u-u^h)>|^{1/2} \]

\[ + \varepsilon_1 (1 + \frac{1}{\alpha_h}) \|p^h\|_0 + (\varepsilon_1 + \frac{\varepsilon_2}{\beta_h}) \|\sigma^h\|_{0,G_C} + \zeta(E) \]

\[ + \xi(E)^{1/2} \]  

(6.10)

\[ \|p-p^h\|_0 + \|\sigma-\sigma^h\|_0,G_C \leq \left( \frac{1}{\alpha_h} + \frac{1}{\beta_h} \right) \left[ M_0 \left( \|u-u^h\|_1 + \|u^h-u_n^h\|_1 \right) \right] \]

\[ + \zeta(E) + (1 + \frac{1}{\alpha_h} + \frac{1}{\beta_h}) \inf_{q_h \in Q_h} \|p-q^h\|_0 \]

\[ + \inf_{\tau^h \in M_h} \|\sigma-\tau^h\|_{0,G_C} + \left( \frac{1}{\alpha_h} + \frac{1}{\beta_h} \right) \|\sigma-\tau^h\|_{W_1} \]  

(6.11)

wherein the estimates (5.16), (6.1) for \( \|u^h-u_n^h\|_1 \) and \( \|u-u^h\|_1 \)

hold. Here \( \zeta(E) \) and \( \xi(E) \) are given by (6.3) and (6.4).
REMARKS 6.1) A key point here is that estimate (6.1) holds for any \( \psi^h \in V_h, \tau^h \in M_h \) (and not \( \psi^h \in K_h \) as in, for example, FORTIN [1]).

6.2) Owing to the way we constructed the spaces \( Q_h \) and \( W'_h \) in Section 5, the errors \( E_I \) and \( E_J \) due to inexact integration of the penalty terms in (1.11) generally play an insignificant role in determining the rate-of-convergence of our method. For example, suppose \( V_h \) is constructed using \( Q_2 \)-elements (biquadratic) and Simpson's rule is used in the definition of the quadratic formula \( J \). Then \( W'_h \) is spanned by \( C^0 \)-piecewise quadratic functions and the trace of \( V_h \) on \( \Gamma_C \) is \( W'_h \). Then \( J \) approximates the integration of quartics, and \( |E_J| = O(h^4) \). For \( Q_1 \)-elements for \( V_h \) and the trapezoid rule on \( J \), \( |E_J| = O(h^2) \). Similar results are obtained for Gaussian quadrature on \( I \) for such elements. This remarkable result indicates that reduced integration for our penalty methods may be used to produce a stable scheme without loss in accuracy (rates of convergence) due to inexact integration.

6.3) The influence of terms such as \( \sqrt{<\tau^h - \sigma, \gamma_n (u) - s>} \) in (6.11) depends strongly on the location of nodal points on \( \Gamma_C \) relative to the actual "separation point" \( x_C \) on the contact surface (where the body comes in contact with the foundation). If we locate element nodes for conforming elements in \( W'_h \) so that they fall exactly on the surface of separation points, then \( <\tau^h - \sigma, \gamma_n (u) - s> = 0 \).

Examples of convergence rates predicted by the estimates in Theorem 6.2 for various choices of elements and integration rules are given.
in Figs 2 and 3. Here we have assumed that $\alpha_h = \alpha_0 h^\delta, \beta_h = \beta_0 h^\lambda$. We show that $\delta = 0$ for schemes 1), 2), 4), and 5) in a second part to this paper which is forthcoming by SONG, ODEN, and KIKUCHI [1]. The estimate for $\beta_h$ given in Fig. 3 is also proved in it. No sharp estimates of $\delta$ are available for schemes 3) and 6). For certain boundary conditions we are able to show that $\delta = 1$ for these schemes, but numerical experiments seem to indicate that $\delta = 0$. The proof that $\delta = 0$ for schemes 3) and 6) remains open. The rates of convergence indicated in these figures are computed using standard interpolation properties of finite-element subspaces (see, e.g. CIARLET [1]) and assuming regular refinements of a regular, uniform finite-element mesh, and that the solution $(u, p, \sigma)$ is regular, i.e., $u \in H^3(\Omega), p \in H^2(\Omega), \sigma \in H^{1.5}(\Omega)$. We shall describe estimates of $\delta$ and $\lambda$ for various elements in a forthcoming paper.

7. CONCLUDING COMMENTS

We have shown that the LBB-condition plays a fundamental role in establishing convergence of penalty solutions to appropriate multipliers for problems in elasticity in which displacement fields are constrained to be divergence free and to satisfy unilateral conditions. For finite element discretizations of such penalty formulations, the discrete LBB-condition emerges as a condition on the numerical stability of these methods. Indeed, stability considerations lead one to use reduced integration in evaluating penalty terms. In turn, the use of inexact integration rules produces penalty methods which are equivalent to certain non-conforming
- Convergence Rates for Incompressibility Constraint Condition.

- Penalty parameter: $\epsilon_1$
- LBB constant: $\alpha_h = \alpha_0 h^\delta$

- Integration points for I: [Diagram]

<table>
<thead>
<tr>
<th>Finite Element</th>
<th>Numerical Integration I</th>
<th>$|u-u_0^h|_1$</th>
<th>$|P-P_0^h|_0$</th>
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<tbody>
<tr>
<td>1)</td>
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<td>$h+\epsilon_1$</td>
<td>$h+\epsilon_1$</td>
</tr>
<tr>
<td>2)</td>
<td></td>
<td>$h^2+\epsilon_1$</td>
<td>$h^2+\epsilon_1$</td>
</tr>
<tr>
<td>3)</td>
<td></td>
<td>$h+h^{1-\delta}$ + $\epsilon_1(1+h^{-\delta})$</td>
<td>$h+h^{1-\delta}+h^{1-2\delta}$ + $\epsilon_1(1+h^{-\delta})$</td>
</tr>
<tr>
<td>4)</td>
<td></td>
<td>$h+\epsilon_1$</td>
<td>$h+\epsilon_1$</td>
</tr>
<tr>
<td>5)</td>
<td></td>
<td>$h+\epsilon_1$</td>
<td>$h+\epsilon_1$</td>
</tr>
<tr>
<td>6)</td>
<td></td>
<td>$h^2+h^{2-\delta}$ + $\epsilon_1(1+h^{-\delta})$</td>
<td>$h^2+h^{2-\delta}+h^{2-2\delta}$ + $\epsilon_1(1+h^{-\delta})$</td>
</tr>
</tbody>
</table>

FIGURE 2. Convergence Rates for Incompressibility Constraint Condition.
- Penalty parameter: $\epsilon_2$

- LBB constant: $\beta_h = \beta_0 h^\lambda$, $\lambda = 1/2$

- Integration point for $J$: ▲

<table>
<thead>
<tr>
<th>Finite Elements</th>
<th>Numerical Integration $J$</th>
<th>$|u-u_h|_1$</th>
<th>$|\sigma-\sigma_h|_{0\Gamma_C}$</th>
</tr>
</thead>
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<td>1-pt. Gaussian rule</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td><img src="image2.png" alt="Image" /></td>
<td>Trapezoid rule</td>
<td>$h+\epsilon_2 h^{-1/2}$</td>
<td>$h^{1/2}+\epsilon_2 h^{-1}$</td>
</tr>
<tr>
<td><img src="image3.png" alt="Image" /></td>
<td>1-pt. Gaussian rule</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td><img src="image4.png" alt="Image" /></td>
<td>2-pt. Gaussian rule</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td><img src="image5.png" alt="Image" /></td>
<td>Simpson's rule</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

X: fails

mixed methods. In particular, for fixed mesh parameter \( h \), our penalty approximations converge to a mixed finite element method as penalty parameters \( \epsilon_1, \epsilon_2 \) approach zero.

The error estimates derived in Section 6 indicate that the behavior of the parameters \( \alpha_h \) and \( \beta_h \) as \( h \to 0 \) may govern the asymptotic rate-of-convergence of our penalty methods. If 1-point integration rules are used in evaluating the penalty term \( I(\text{div} \, v^h \cdot \text{div} \, u^h) \), we have seen that a reduction of one order in the rate of convergence in \( || \cdot ||_1 \) results, even though an independent calculation reveals that such integration rules are usually stable in that they lead to LBB-constants \( \alpha_h \) which are independent of \( h \). However, if a higher-order integration rule is used, an increase in the order of error due to numerical quadrature results but the stability of the method suffers; e.g. \( \alpha_h \) then may depend upon \( h \). Thus, these methods may frequently lead to suboptimal rates of convergence.

We plan to discuss the results of a more detailed study of the LBB-condition, together with the results of extensive numerical experiments in a forthcoming paper.

Acknowledgement. The work was completed during the course of a research program sponsored by the U. S. Air Force Office of Scientific Research under Contract F-49620-78-C-0083.

REFERENCES


APPENDIX A

PROOF OF LEMMA 3.1

We use the notations defined in (2.4) and (2.5). Suppose, for the moment, that $B$ is surjective. Then the range $\text{Rg}(B)$ of $B$ is closed, $\text{Rg}(B) = (\ker B^*)^\perp$, so that $\ker B^* = \{(0,0)\}$ and $B^*$ is injective. It then follows from the Banach theorem that $\text{Rg}(B^*)$ is closed and that, therefore, $B^*$ is bounded below. But this means that if $\psi = j^{-1}(\tau)$,

$$
\|B^*(q,\psi)\|_* = \sup_{\psi \in V, \psi \neq 0} \left| \langle B^*(q,\psi), \psi \rangle \right| = \sup_{\psi \in V, \psi \neq 0} \frac{|(q, \text{div} \psi) + \langle \tau, \gamma_n(\psi) \rangle|}{\|\psi\|_1}
$$

$$
\geq c_o \|q, j^{-1}(\tau)\|_U = c_o \left( \|q\|_o + \|\tau\|_{W^1_0} \right)
$$

where $c_o$ is a positive constant and $\| \cdot \|_*$ is the norm on $V'$. But this last inequality is precisely (3.4) which was to be proved. Hence, it suffices to show that $B$ is surjective.

Toward this end, we will make use of the following result which can be found in the book of TEMAM [1, pp. 31, 32]: Let $\Omega$ be an open bounded set of class $C^2$ in $\mathbb{R}^n$ and let there be given $f_1 \in H^{-1}(\Omega), q \in L^2(\Omega), \psi \in H^{1/2}(\Gamma))$ such that

$$
\int_{\Omega} q \, dx = \int_{\Gamma} \psi \cdot n \, ds \tag{A.1}
$$

Then there exist unique $u \in H^1(\Omega)$ and $\tilde{p} \in L^2(\Omega)/\mathbb{R}$ such that
\[-\Delta \tilde{u} + \text{grad} \tilde{\psi} = f \quad \text{in} \quad \Omega \]

\[\text{div} \, u = q \quad \text{in} \quad \Omega \]

\[\gamma(u) = \psi \quad \text{on} \quad \Gamma \]

Moreover, if \(\psi\) is given as the trace of a function \(\tilde{\psi} \in (H^1(\Omega))^3\) such that \(\int_{\Omega} q \, dx = \int_{\Omega} \text{div} \tilde{\psi} \, dx\) and \(u - \Phi \in (H^1_0(\Omega))^3\), then \(\Omega\) need only be Lipschitzian.

Of course, rather than to employ fully this result, we need only show that for every \((q,\psi) \in \mathcal{U}\), with \(\psi = j^{-1}(\tau)\), there exists a \(v \in \mathcal{V}\) such that \(B(v) = (q,\tau)\); i.e.

\[\text{div} \, \tilde{v} = q \quad \text{in} \quad \Omega \quad \gamma_n(\tilde{v}) = j^{-1}(\tau) \quad \text{on} \quad \Gamma^c\]

The problem that must be dealt with here is the compatibility condition (A.1)

Owing to our assumptions on \(\Omega\) and \(\Gamma\), we can identify a bounded open domain \(\Omega\) of class \(C^2\) such that \(\Omega_0 \supseteq \Omega\), with boundary \(\Gamma_0\) such that \(\Gamma_C \subseteq \Gamma_0\), \(\Gamma_D \subseteq \Gamma_0\), and \(\Gamma_0 = \Gamma_C \cup \Gamma_D \cup \Gamma_F^0\); the domain \(\Omega_0\) is envisioned as any smooth domain containing \(\Omega\) such that \(\Gamma_0\) is a smooth \(C^2\) connection of smooth extensions of \(\Gamma_0\) and \(\Gamma_D\). Let \(K \subseteq \Gamma_0\) be a set such that \(\text{meas}(K - \Gamma_C^0 - \Gamma_D^0) = \varepsilon_0 > 0\) and define a function \(\tilde{\psi}\) on \(\Gamma_0\) such that
\[ \psi = \varphi n \in (H^{1/2}(\Gamma))^3 , \]

\[ \psi = \begin{cases} \hat{\psi} & \text{on } K \\ \mu \hat{\psi} & \text{on } 0 - K \end{cases} ; \quad \hat{\psi} = \begin{cases} 0 & \text{on } \Gamma_D \\ \psi = j^{-1}(r) & \text{on } \Gamma_C \end{cases} \]

where \( \hat{\psi} \in H^{1/2}(K) \), \( \mu \in \mathbb{R} \), and

\[ \int_{\Gamma_0^-K} \hat{\psi} \, ds = 1 \]

With this construction, we may take, for any \( (q, \psi) \in \mathcal{U} \),

\[ \mu = \mu(q, \psi, K) = \int_{\Omega} q \, dx - \int_{K} \hat{\psi} \, ds \]

Then \( \psi \) is such that

\[ \int_{\Omega} \psi \cdot n \, ds = \int_{\Gamma_C} \psi \, ds = \int_{\Omega} q \, dx , \]

i.e. (4.7) holds. Moreover, \( \text{div} : (H^1_0(\Omega))^3 \to L^2(\Omega)/\mathbb{R} \) and \( \gamma : H^1(\Omega) \to H^{1/2}(\Gamma) \) are surjective; hence, \( \psi \) is the trace of a function \( \hat{\psi} \in (H^1(\Omega))^3 \) satisfying \( \int_{\Omega} q \, dx = \int_{\Omega} \text{div} \, \phi \, dx \) and \( u - \hat{\psi} \in (H^1_0(\Omega))^3 \). Thus, we have established the existence of a \( \tilde{u} \in V \) such that \( \text{div} \, \tilde{u} = q \) in \( \Omega \) and

\[ \gamma(\tilde{u}) = \gamma(\hat{\psi}) = \psi \]
for any \( q \in L^2(\Omega) \). But this latter result is equivalent to

\[
\gamma_n(u) = \psi = j^{-1}(\tau) \text{ on } \Gamma_C, \quad \gamma(u) = 0 \text{ on } \Gamma_D
\]

for any \( \psi \in W \). Hence, \( B \) is surjective.

**Remark.** Thanks to P. LETALLEC [1], we can use an alternate and somewhat more direct construction to satisfy the conditions of this theorem in TEMAM [1]. The idea rests on the fact that \( \Gamma_F \) has a subset which is a \( C^1 \)-submanifold on \( \mathbb{R}^{N-1} \) and, as a consequence, one can show the existence of a function \( \varrho \) such that

\[
\int_{\Omega} \text{div } \varrho \ dx \neq 0,
\]

where \( \Omega \) is an open subset of \( \mathbb{R}^n \) such that \( \Omega \cap \Gamma \subseteq \Gamma_F \) and \( \mu \) is a subset of \( \mathbb{R}^n \) such that \( \Omega \cap \Gamma \subseteq \Gamma_F \) and \( \mu \) is a subset of \( \mathbb{R}^n \) such that

\[
\gamma_n(\varphi) = j^{-1}(\tau) \text{ on } \Gamma_C.
\]

If we choose \( \mu \in \mathbb{R} \) such that

\[
\mu = \frac{\int_{\Omega} (q - \text{div } \varphi) \ dx}{\int_{\Omega} \text{div } \varrho \ dx}
\]

the result quoted from TEMAM implies that existence of a \( \varphi_0 \in V \) such that

\[
\text{div } \varphi_0 = q - \text{div } \varphi - \mu \text{ div } \varrho
\]

\[
\gamma(\varphi_0) = 0 \text{ on } \Gamma.
\]
Setting

\[ \psi = \psi_0 + \phi + \mu \theta \]

we have \( \nabla \psi = q \) and, since \( \text{supp} \ \theta \subseteq \Omega \), \( \Omega \cap \Gamma \subseteq \Gamma_F \), we have \( \theta = 0 \) on \( \Gamma_C \cup \Gamma_D \), so that

\[ \gamma_n(u) = \gamma_n(\psi) = j^{-1}(\tau) \text{ on } \Gamma_C; \ \gamma(\psi) = 0 \text{ on } \Gamma_D \]

as required. \( \square \)

**APPENDIX B**

**PROOF OF LEMMA 4.1**

It suffices to show that

\[ c_1 \left\| \nabla_h \psi \right\|_{1, \hat{\Omega}_e} \leq \left\| \nabla_h \psi \right\|_{\hat{\Omega}_e} \leq c_2 \left\| \nabla_h \psi \right\|_{1, \hat{\Omega}_e}, \quad \nabla_h \psi \in V_h \]

for the master element of \( Q_1 \)- and \( Q_2 \)-elements, where

\[ \left\| \nabla_h \psi \right\|_{1, \hat{\Omega}_e} = \left\{ \int_{\hat{\Omega}_e} \sum_{i,j} (\varepsilon_{ij}(\nabla_h \psi)\varepsilon_{ij}(\nabla_h \psi) \right\}^{1/2}, \]

\[ \left\| \nabla_h \psi \right\|_{\hat{\Omega}_e} = \left\{ \int_{\hat{\Omega}_e} \sum_{i,j} (\varepsilon_{ij}(\nabla_h \psi)\varepsilon_{ij}(\nabla_h \psi) + \frac{1}{2} I(\text{div} \nabla_h \psi \text{div} \nabla_h \psi) \right\}^{1/2}, \]
and $\hat{\Omega}_e = (-1,1) \times (-1,1)$. Recall that

$$\varepsilon_{ij} \varepsilon_{ij} = \varepsilon_{ij}^D \varepsilon_{ij}^D + \frac{1}{2} (\varepsilon_{kk})^2$$

($Q_1$-elements) We first note that

$$\varepsilon_{ij}^D \varepsilon_{ij}^D = \frac{1}{2} (\varepsilon_{11} - \varepsilon_{22})^2 + 2 \varepsilon_{12}$$

for $N = 2$. By direct expansion of terms, we can obtain

$$\frac{1}{2} \int_{-1}^{1} \int_{-1}^{1} (\varepsilon_{11} - \varepsilon_{22})^2 \, d\xi d\eta = \frac{1}{8} (A-C)^2 + \frac{1}{12} (B^2 + D^2)$$

$$2 \int_{-1}^{1} \int_{-1}^{1} \varepsilon_{12}^2 \, d\xi d\eta = \frac{1}{8} (E-F)^2 + \frac{1}{12} (B^2 + D^2)$$

$$\frac{1}{2} \int_{-1}^{1} \int_{-1}^{1} (\varepsilon_{11} + \varepsilon_{22})^2 \, d\xi d\eta = \frac{1}{8} (A+C)^2 + \frac{1}{12} (B^2 + D^2)$$

$$\frac{1}{2} \left| \left. I(\text{div} \, \mathbf{v}^h \cdot \text{div} \, \mathbf{v}^h) \right|_{\hat{\Omega}_e} \right. = \frac{1}{8} (A+C)^2,$$

where
\[
\begin{align*}
A &= -u_1 + u_2 + u_3 - u_4 \\
B &= u_1 - u_2 + u_3 - u_4 \\
C &= -v_1 - v_2 + v_3 + v_4 \\
D &= v_1 - v_2 + v_3 - v_4 \\
E &= -u_1 - u_2 + u_3 + u_4 \\
F &= v_1 - v_2 - v_3 + v_4
\end{align*}
\]

Then
\[
\|v^h\|_{1, \hat{\Omega}_e} = \left\{ \frac{1}{4}(A^2 + C^2) + \frac{1}{8}(E-F)^2 + \frac{1}{4}(B^2 + D^2) \right\}^{1/2}
\]

\[
\|v^h\|_{\hat{\Omega}_e} = \left\{ \frac{1}{4}(A^2 + C^2) + \frac{1}{8}(E-F)^2 + \frac{1}{6}(B^2 + D^2) \right\}^{1/2}
\]

Therefore,
\[
\frac{\sqrt{2}}{\sqrt{3}} \|v^h\|_{1, \hat{\Omega}_e} \leq \|v^h\|_{\hat{\Omega}_e} \leq \|v^h\|_{1, \hat{\Omega}_e}
\]

(Q₂-Element with 1-point Integrations) We shall next consider the case in which I is one-point Gaussian integration for the Q₂-element. Within the master element (-1,1) × (-1,1) of the Q₂-element, we can obtain the following identities: Let
Then we can show that

\[
\int_{-1}^{1} \int_{-1}^{1} (\varepsilon_{11} - \varepsilon_{22})^2 \, d\xi \, d\eta
\]

\[
= 4(A_0 - B_0)^2 + \frac{4}{3}[(B_1 + 2u_9)^2 - 2(A_0 - B_0)(A_2 + B_2) + (A_1 + 2v_9)^2]
\]

\[
+ \frac{4}{5}(A_2 + B_2) + \frac{4}{9}[4(A_{11} - B_{11})^2 - 4(B_1 + 2u_9)A_3 - 4(A_1 + 2u_9)B_3 - 2A_2 B_2]
\]

\[
+ \frac{16}{15}(A_3 + B_3)^2
\]

\[
4 \int_{-1}^{1} \int_{-1}^{1} \varepsilon_{12}^2 \, d\xi \, d\eta
\]

\[
= 4(E_0 + F_0)^2 + \frac{4}{3}[(A_1 - 2v_9)^2 + 2(E_0 + F_0)(A_{11} + B_{11}) + (B_1 - 2u_9)^2]
\]
\[\begin{align*}
+ \frac{4}{5}(A_{11}^2 + B_{12}^2) + \frac{4}{9}[4(A_2 + B_2)^2 + 4(A_1 - 2v_9)B_3 + 4A_3(B_1 - 2u_9) + 2A_{11}B_{11}] \\
+ \frac{16}{15}(A_3^2 + B_3^2) \\
\int_{-1}^{1} \int_{-1}^{1} (\varepsilon_{11} + \varepsilon_{22})^2 \, d\xi d\eta \\
= 4(A_o + B_o)^2 + \frac{4}{3}[(B_1 - 2u_9)^2 + 2(A_o + B_o)(A_2 + B_2) + (A_1 - 2v_9)^2] \\
+ \frac{4}{5}(A_2^2 + B_2^2) + \frac{4}{9}[4(A_{11} + B_{11})^2 + 4(B_1 - 2u_9)A_3 + 4(A_1 - 2v_9)B_3 + 2A_2B_2] \\
+ \frac{16}{15}(A_3^2 + B_3^2) \\
\int_{-1}^{1} \int_{-1}^{1} (\varepsilon_{11} + \varepsilon_{22})^2 \, d\xi d\eta = 4(A_o + B_o)^2
\end{align*}\]

With these results and the technique described earlier, we can show by a direct computation that

\[\frac{1}{2} \left\| v^h \right\|_{1, \hat{\Omega}_e}^2 \leq \left\| v^h \right\|_{\hat{\Omega}_e}^2 \leq \left\| v^h \right\|_{1, \hat{\Omega}_e}^2.\]

For other choices of numerical integration, we can obtain similar inequalities by similar arguments. \(\blacksquare\)