Chapter 1

The Classical Variational Principles of Mechanics

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1.1 INTRODUCTION

The last twenty years have been marked by some of the most significant advances in variational mechanics of this century. These advances have been made in two independent camps. First and foremost, the entire theory of partial differential equations has been recast in a 'variational' framework that has made it possible to significantly expand the theory of existence, uniqueness, and regularity of solutions of both linear and nonlinear boundary-value problems. In this regard, the treatise of Lions and Magenes on linear partial differential equations, the works of Lions, Brezis, and Browder on nonlinear equations, and of Duvaut and Lions and Stampacchia on variational inequalities should be mentioned. Secondly, the underlying structure of classical variational principles of mechanics are better understood. The work of Vainberg led to important generalizations of classical variational notions related to the minimization of functionals defined on Banach spaces, and an excellent memoir on variational theory and differentiation of operators on Banach spaces was contributed by Nashed. Generalizations of Hamiltonian theory were described by Noble and Noble and Sewell and this led to generalization of the concept of complementary variational principles by Arthurs and others. Tonti showed that many of the linear equations of mathematical physics share a common structure that leads naturally to several dual and complementary variational principles, and these notions were further developed by Oden and Reddy. A quite general theory of complementary and dual variational principles for linear problems in mathematical physics was given in the monograph of Oden and Reddy, and a more complete historical account of the subject, together with additional references, can be found in that work.

This chapter deals with a general theory of variational methods for linear problems in mechanics and mathematical physics. This work is partly expository in nature, since one of its principal missions is to develop, in a rather tutorial way, complete with examples, a rather general mathematical
framework for both the modern theory of variational boundary-value problems and the 'classical' primal, dual, and complementary variational principles of mechanics. However, many of the ideas appear to be new, and generalized Green's formulas are given which make it possible to further generalize the recent theories of Oden and Reddy.\textsuperscript{15}

\section{1.2 MATHEMATICAL PRELIMINARIES}

\subsection{1.2.1 Transposes and adjoints of linear operators}

It is frequently important to distinguish between the transpose of a linear operator on Hilbert spaces, and its adjoint. We set the stage for this discussion by introducing some notation.

Let $\mathcal{U}$, $\mathcal{V}$ be Hilbert spaces with inner products $(u_1, u_2)$ and $(v_1, v_2)$, respectively.

Let $\mathcal{U}'$, $\mathcal{V}'$ be the topological duals of $\mathcal{U}$ and $\mathcal{V}$, i.e. the spaces of continuous linear functionals on $\mathcal{U}$ and $\mathcal{V}$, respectively.

Let $\langle \cdot, \cdot \rangle_\mathcal{U}$, $\langle \cdot, \cdot \rangle_\mathcal{V}$ denote duality pairings on $\mathcal{U} \times \mathcal{U}$ and $\mathcal{V} \times \mathcal{V}$, respectively, i.e. if $l \in \mathcal{U}$ and $q \in \mathcal{V}'$ we write

$$l(u) = \langle l, u \rangle_\mathcal{U} \text{ and } q(v) = \langle q, v \rangle_\mathcal{V}.$$  

We shall denote by $A$ a continuous linear operator from $\mathcal{U}$ into $\mathcal{V}$.

Now, following the standard arguments, we note that since $Au \in \mathcal{V}$, the linear functional $q(Au) = \langle q, Au \rangle_\mathcal{V}$ also identifies a continuous linear functional on $\mathcal{U}$, i.e. a correspondence exists between $q \in \mathcal{V}'$ and the elements of the dual space $\mathcal{U}$. We describe this correspondence by introducing an operator $A': \mathcal{V}' \to \mathcal{U}'$ such that

$$\langle A'q, u \rangle_\mathcal{U} = \langle q, Au \rangle_\mathcal{V} \text{ or } \langle A'q, u \rangle_\mathcal{U} = \langle q, Au \rangle_\mathcal{V} \quad (1.1)$$

The operator $A'$ is called the transpose of $A$; it is clearly linear and continuous from $\mathcal{V}'$ into $\mathcal{U}'$.

Now the fact that $\mathcal{U}$ and $\mathcal{V}$ are Hilbert spaces allows us to enter into a considerable amount of additional detail in describing $A'$ and other relationships between $\mathcal{U}$ and $\mathcal{V}$. Since $\mathcal{U}$ is a Hilbert space, we know from the Riesz representation theorem that for every linear functional $l \in \mathcal{U}'$ there exists a unique element $u_l \in \mathcal{U}$ such that

$$\langle l, u \rangle_\mathcal{U} = \langle u_l, u \rangle_\mathcal{U} \quad \forall u \in \mathcal{U}$$

Indeed, this relationship defines an isometric isomorphism $K_\mathcal{U} : \mathcal{U} \to \mathcal{U}'$ such that

$$\langle K_\mathcal{U}u_0, u \rangle_\mathcal{U} = \langle u_0, u \rangle_\mathcal{U} \quad \forall u_0, u \in \mathcal{U} \quad (1.2)$$
and \( K_\nu \) is called the Riesz map corresponding to the space \( \mathcal{U} \). Similarly, if \( K_\nu \) is the Riesz map corresponding to the space \( \mathcal{V} \), we have \( \langle K_\nu v, v \rangle_\nu = (v, v)_\nu \ \forall v \in \mathcal{V} \). In view of the definitions of the Riesz maps, we see that

\[
\langle K_\nu v, Au \rangle_\nu = (v, Au)_\nu \quad \forall v \in \mathcal{V}, u \in \mathcal{U}
\]

and

\[
\langle A'K_\nu v, u \rangle_\nu = (K_\nu^{-1}A'K_\nu v, u)_{\mathcal{U}} \quad \forall v \in \mathcal{V}, u \in \mathcal{U}
\]

Thus, we discover in a very natural way that for each continuous linear operator \( A : \mathcal{U} \to \mathcal{V} \) satisfying the above relations there corresponds an operator \( A^* : \mathcal{V} \to \mathcal{U} \) given by the composition

\[
A^* = K_\nu^{-1}A'K_\nu
\]

such that

\[
(A^*v, u)_\nu = (v, Au)_\nu
\]

The operator \( A^* \) is linear and continuous and is called the adjoint of \( A \).

The following theorem establishes some important properties of the transpose.

**Theorem 1.2.1** Let \( A' \in \mathcal{L}(\mathcal{V}', \mathcal{U}) \) (here \( \mathcal{L}(\mathcal{V}', \mathcal{U}) \) is the space of continuous linear operators mapping \( \mathcal{V}' \) into \( \mathcal{U} \)) denote the transpose of a continuous linear operator \( A \) mapping Hilbert spaces \( \mathcal{U} \) into \( \mathcal{V} \). Then the following hold:

(i)

\[
\mathcal{N}(A') = \mathcal{R}(A)^\perp
\]

where \( \mathcal{N}(A') \) is the null space of \( A' \) and \( \mathcal{R}(A)^\perp \) is the orthogonal complement of the range \( \mathcal{R}(A) \) of \( A \) in \( \mathcal{V}' \).

(ii) \( A' \) is injective if and only if \( \mathcal{R}(A) \) is dense in \( \mathcal{V} \).

**Proof:** This is a well-known theorem; see, for example, Dunford and Schwartz.\(^{16} \]

#### 1.2.2 Pivot Spaces

The Riesz map \( K_\nu : \mathcal{U} \to \mathcal{U} \) is an isometric isomorphism from \( \mathcal{U} \) onto its dual \( \mathcal{U}' \). Consequently, it is possible to identify \( \mathcal{U} \) with its dual. In many instances, particularly in the theory of linear boundary-value problems, we encounter collections of Hilbert spaces satisfying \( \mathcal{U}_n \subset \mathcal{U}_{n-1} \subset \ldots \subset \mathcal{U}_2 \subset \mathcal{U}_1 \subset \mathcal{U}_0 \), the inclusions being dense and continuous. When one member of this set is identified with its dual, say \( \mathcal{U}_0 \), we call it the **pivot space**, and write

\[
\mathcal{U}_0 = \mathcal{U}_0'
\]
The term 'pivot' arises from the fact that
\[ \ldots < \mathcal{U}_2 < \mathcal{U}_1 < \mathcal{U}_0 = \mathcal{U}_0' < \mathcal{U}_1' < \mathcal{U}_2' < \ldots \]
i.e. \( \mathcal{U}_0 \) provides a 'pivot' between the spaces \( \mathcal{U}_i, \ i \geq 0 \), and their duals.

1.2.3 Sobolev spaces

The notion of a Sobolev space is fundamental to the modern theory of boundary-value problems, and most of the applications of our theory can be developed within the framework of Sobolev spaces.

Let \( \Omega \) be a smooth,† open, bounded domain in \( \mathbb{R}^n \). We shall denote by \( H^m(\Omega) \) the Sobolev space of order \( m \), which is a linear space of functions (or equivalence classes of functions) defined by

\[ H^m(\Omega) = \{ u : u \text{ and all of its distributional derivatives } D^\alpha u \text{ of order } \leq m \text{ are in } L^2(\Omega), \ m \geq 0 \} \]  (1.5)

Here we employ standard multi-index notations (see, Oden and Reddy19). Clearly, \( H^0(\Omega) = L^2(\Omega) \).

Now the Sobolev spaces (1.5) are Hilbert spaces. Indeed, \( H^m(\Omega) \) is complete with respect to the norm associated with the inner product

\[ (u, v)_{H^m(\Omega)} = \int_{\Omega} \sum_{|\alpha| \leq m} D^\alpha u \overline{D^\alpha v} \, dx \]  (1.6)

and the norm on \( H^m(\Omega) \) is

\[ \| u \|_{H^m(\Omega)} = \sqrt{(u, u)_{H^m(\Omega)}} = \left\{ \sum_{|\alpha| \leq m} \| D^\alpha u \|_{L^2(\Omega)}^2 \right\}^{1/2} \]  (1.7)

The subspace of \( H^m(\Omega) \) consisting of those functions in \( H^m(\Omega) \) whose derivatives of order \( \leq m - 1 \) vanish on the boundary \( \partial \Omega \) of \( \Omega \) is denoted \( H^m_0(\Omega) \):

\[ H^m_0(\Omega) = \{ u \in H^m(\Omega) : D^\alpha u|_{\partial \Omega} = 0, |\alpha| \leq m - 1 \} \]  (1.8)

Equivalently, \( H^m_0(\Omega) \) can be defined as the completion of the space \( C^\infty_c(\Omega) \) of infinitely differentiable functions with compact support in \( \Omega \) with respect to the \( H^m \)-Sobolev norm defined in (1.8).

The so-called negative Sobolev spaces are defined as the duals of the \( H^m_0(\Omega) \) spaces:

\[ H^{-m}(\Omega) = (H^m_0(\Omega))^*, \quad m \geq 0 \]  (1.9)

† We shall assume throughout that \( \Omega \) is simply-connected with a \( C^\infty \)-boundary \( \partial \Omega \). However, most of the results and examples we cite subsequently hold if \( \partial \Omega \) is only Lipschitzian. See, for example, Nečas17 or Adams18.
The Sobolev spaces are important in making precise the 'degree of smoothness' of functions. The following list summarizes some of their most important properties:

(i) \( H^m(\Omega) \subset H^k(\Omega), \quad m \geq k \geq 0 \) \hspace{1cm} (1.10)

Indeed,

\[ H^m(\Omega) \subset H^{m-1}(\Omega) \subset \ldots \subset H^2(\Omega) \subset H^1(\Omega) \subset H^0(\Omega) \] \hspace{1cm} (1.11)

The remarkable fact is that these inclusions are continuous and dense. Indeed, \( H^m(\Omega) \) is dense in \( H^k(\Omega), \ m \geq k \geq 0 \), and \( \|u\|_{H^k(\Omega)} \leq \|u\|_{H^m(\Omega)} \)

(ii) Obviously, if \( m \) is sufficiently large, the elements of \( H^m(\Omega) \) can be identified with continuous functions. Just how large \( m \) must be in order that each \( u \in H^m(\Omega) \) be continuous is determined by one of the so-called Sobolev embedding theorems: If \( \Omega \) is smooth (e.g. if \( \Omega \subset \mathbb{R}^n \) satisfies the cone condition) and if \( m > n/2 \), then \( H^m(\Omega) \) is continuously and compactly embedded in the space \( C^0(\bar{\Omega}) \) of continuous functions.

(iii) Sobolev spaces \( H^i(\partial \Omega) \) can be defined for classes of functions whose domain is the boundary \( \partial \Omega \). There is an important relation between these boundary spaces and those containing functions defined on \( \Omega \). We define the trace operators \( \gamma_i \) as

\[ \gamma_i u = \frac{\partial^i u}{\partial n^i} \Big|_{\partial \Omega}, \quad 0 \leq i \leq m - 1 \] \hspace{1cm} (1.12)

i.e. they are the normal derivatives at \( \partial \Omega \) of order \( \leq m - 1 \). Let \( \varphi \in L_2(\partial \Omega) \) and define the norm\(^\dagger\)

\[ \|\varphi\|_{H^{m - i - 1/2}(\partial \Omega)} = \inf_{u \in H^m(\Omega)} \{ \|u\|_{H^m(\Omega)} ; \varphi = \gamma_i u \} \] \hspace{1cm} (1.13)

The completion of \( L_2(\partial \Omega) \) with respect to this norm is a Hilbert space of boundary functions denoted

\[ H^{m - i - 1/2}(\partial \Omega), \quad 0 \leq i \leq m - 1 \] \hspace{1cm} (1.14)

Two important properties of these spaces are called the trace properties of Sobolev spaces:

(iii.1) The operators \( \gamma_i \) can be extended to continuous linear operators mapping \( H^m(\Omega) \) onto \( H^{m - i - 1/2}(\partial \Omega) \), i.e. there exist constants \( C_i > 0 \) such that

\[ \|\gamma_i u\|_{H^{m - i - 1/2}(\partial \Omega)} \leq C_i \|u\|_{H^m(\Omega)}, \quad 0 \leq i \leq m - 1 \] \hspace{1cm} (1.15)

\(^\dagger\) There are other more constructive ways of defining these spaces. See, for example, Oden and Reddy.\(^x\)
(iii.2) The kernel of the collection of trace operators \( \gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{m-1}) \) is \( H_0^m(\Omega) \) and \( H_0^m(\Omega) \) is dense in \( H''(\Omega) \):

\[
\gamma_j(H_0^m(\Omega)) = 0, \quad j = 0, 1, \ldots, m - 1
\]

1.3 GREEN'S FORMULAE FOR OPERATORS ON HILBERT SPACES

1.3.1 A general comment

One of the most important applications of the notion of adjoints of linear operators involves cases in which it makes sense to distinguish between spaces of functions defined on the interior of some domain and spaces of functions defined on the boundary of a domain. The introduction of boundary values is obviously essential in the study of boundary value problems in Hilbert spaces, and it leads us to the idea of an abstract Green's formula for linear operators.

In this section, we develop a general and abstract Green's formula which extends those given previously in the literature. Our format resembles that of Aubin, who developed Green's formulae for elliptic operators of even order. Our results involve formal operators associated with bilinear forms \( B : H \times G, H \) and \( G \) being Hilbert spaces (see Section 1.3.3) and reduce to those of Aubin when collapsed to the very special case, \( G = H \).

1.3.2 Abstract trace property

An abstraction of the idea of boundary values of elements in Hilbert spaces is embodied in the concept of spaces with a trace property. A Hilbert space \( H \) is said to have the trace property if and only if the following conditions hold:

(i) \( H \) is contained in a larger Hilbert space \( U \) which has a weaker topology than \( H \).

(ii) \( H \) is dense in \( U \) and \( U \) is a pivot space, i.e.

\[
H \subset U = U \subset H
\]

(iii) There exists a linear operator \( \gamma \) that maps \( H \) onto another Hilbert space \( H' \) such that the kernel \( H_0 \) of \( \gamma \) is dense in \( U \), i.e.

\[
\ker \gamma = H_0 \subset H : H_0 \subset U = U \subset H'
\]

The space \( \partial H \) corresponds to a space of boundary values, and the operator \( \gamma \) extends the elements of \( H \) which can be thought of as functions
defined on the interior of some domain, onto the space of boundary values \( \partial \Omega \). The operator \( \gamma \) is sometimes called the trace operator.

The spaces \( H^m(\Omega) \) and the operators \( \gamma_i \) in (1.12)-(1.16) are examples: \( H^m(\Omega) \) is dense in \( L_2(\Omega) \), \( m \leq 0 \). \( L_2(\Omega) \) can be identified with its dual, extensions of the trace operators \( \gamma_i \) of (1.12) map \( H^m(\Omega) \) onto the boundary spaces \( H^{m-j-\frac{1}{2}}(\partial \Omega) \), \( 0 \leq j \leq m - 1 \), and \( \ker \gamma = \ker (\gamma_0, \gamma_1, \ldots, \gamma_{m-1}) = H^m_0(\Omega) \) is dense in \( L_2(\Omega) \).

### 1.3.3 Bilinear forms and associated operators

Let \( \mathcal{H} \) and \( \mathcal{G} \) denote two real Hilbert spaces (the extension of our results to complex spaces is trivial), and let both \( \mathcal{H} \) and \( \mathcal{G} \) have the trace property, i.e.

\[
\begin{align*}
\mathcal{H} \subset \mathcal{U} = \mathcal{U}' \subset \mathcal{H}, \quad \mathcal{G} \subset \mathcal{V} = \mathcal{V}' \subset \mathcal{G} \\
\gamma : \mathcal{H} \rightarrow \partial \mathcal{H}, \quad \gamma^* : \mathcal{G} \rightarrow \partial \mathcal{G} \\
\ker \gamma = \mathcal{H}_0, \quad \ker \gamma^* = \mathcal{G}_0 \\
\mathcal{H}_0 \subset \mathcal{U} = \mathcal{U}' \subset \mathcal{H}_0, \quad \mathcal{G}_0 \subset \mathcal{V} = \mathcal{V}' \subset \mathcal{G}_0
\end{align*}
\] (1.19)

The inclusions \( \mathcal{H} \subset \mathcal{U}, \mathcal{G} \subset \mathcal{V}, \mathcal{H}_0 \subset \mathcal{U}, \mathcal{G}_0 \subset \mathcal{V} \), are dense and continuous.

Next, we introduce an operator \( B \) which maps pairs \((u, v)\), \( u \in \mathcal{H} \) and \( v \in \mathcal{G} \), linearly and continuously into real numbers:

\[
B : \mathcal{H} \times \mathcal{G} \rightarrow \mathbb{R}
\] (1.20)

We denote the values of \( B \) in \( \mathbb{R} \) by \( B(u, v) \), and we refer to \( B \) as a continuous bilinear form on \( \mathcal{H} \times \mathcal{G} \). That \( B \) is bilinear mean that

\[
\begin{align*}
B(\alpha u_1 + \beta u_2, v) &= \alpha B(u_1, v) + \beta B(u_2, v) \\
B(u, \alpha v_1 + \beta v_2) &= \alpha B(u, v_1) + \beta B(u, v_2)
\end{align*}
\] (1.21)

\( \forall u, u_1, u_2 \in \mathcal{H}, v, v_1, v_2 \in \mathcal{G}, \alpha, \beta \in \mathbb{R} \). That \( B \) is continuous means that it is bounded, i.e. there exists a positive constant \( M \) such that

\[
B(u, v) \leq M \|u\|_\mathcal{H} \|v\|_\mathcal{G} \quad \forall u \in \mathcal{H}, \quad \forall v \in \mathcal{G}
\] (1.22)

Now let \( u \) be fixed element of \( \mathcal{H} \) and let \( v \in \mathcal{G}_0 \). Then \( B(u, v) \) describes a continuous linear functional \( l_u \) on the space \( \mathcal{G}_0 \) for each choice of \( u \in \mathcal{H} : B(u, v) = l_u(v), \quad v \in \mathcal{G}_0 \). The linear functional \( l_u \) depends linearly and continuously on \( u \), and we describe this dependence in terms of a linear operator \( Au = l_u \). Thus

\[
B(u, v) = \langle Au, v \rangle_{\mathcal{G}_0}, \quad v \in \mathcal{G}_0
\] (1.23)

The operator \( A \) is called the formal operator associated with the bilinear form \( B \). Clearly

\[
A \in L(\mathcal{H}, \mathcal{G}_0)
\] (1.24)
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In a similar manner, if we fix \( v \in \mathcal{B} \), \( B(u, v) \) defines a continuous linear functional on \( \mathcal{H}_0 \), and we define a continuous linear operator \( A^* \) by

\[
B(u, v) = (A^*v, u)_{\mathcal{H}_0}, \quad u \in \mathcal{H}_0
\]

(1.25)

The operator \( A^* \) is known as the formal adjoint of \( A \), and

\[
A^* \in \mathcal{L}(\mathcal{B}, \mathcal{H}_0')
\]

(1.26)

The fact that \( \mathcal{H}_0 = \ker \gamma \) and \( \mathcal{B}_0 = \ker \gamma^* \) enables us to establish the following fundamental lemma.

Theorem 1.3.1 Let \( \mathcal{H} \) and \( \mathcal{B} \) denote the Hilbert spaces with the trace properties described above, and let \( B \) denote a continuous bilinear form from \( \mathcal{H} \times \mathcal{B} \rightarrow \mathbb{R} \) with formal association operators \( A \in \mathcal{L}(\mathcal{H}, \mathcal{B}_0) \) and \( A^* \in \mathcal{L}(\mathcal{B}, \mathcal{H}_0') \). Moreover, let \( \mathcal{H}_A \) and \( \mathcal{B}_{A^*} \) denote subspaces

\[
\mathcal{H}_A = \{ u \in \mathcal{H} : Au \in \mathcal{V} \} \quad (\mathcal{V} = \mathcal{V}' \subset \mathcal{H}_0)
\]

\[
\mathcal{B}_{A^*} = \{ v \in \mathcal{B} : A^*v \in \mathcal{U} \} \quad (\mathcal{U} = \mathcal{U}' \subset \mathcal{H}_0')
\]

(1.27)

Then there exists uniquely defined operators \( \delta \in \mathcal{L}(\mathcal{H}_A, \mathcal{B}) \) and \( \delta^* \in \mathcal{L}(\mathcal{B}_{A^*}, \mathcal{H}') \) such that the following formulae hold:

\[
B(u, v) = (v, Au)_\mathcal{V} + (\delta u, \gamma^* v)_{\mathcal{H}} \quad u \in \mathcal{H}_A, \quad v \in \mathcal{B}
\]

\[
B(u, v) = (A^* v, u)_{\mathcal{H}_0'} + (\delta^* v, \gamma^* u)_{\mathcal{H}_0} \quad u \in \mathcal{H}, \quad v \in \mathcal{B}_{A^*}
\]

(1.28)

Here \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \) and \( \langle \cdot, \cdot \rangle_{\mathcal{H}_0'} \) denote duality pairings on \( \partial \mathcal{H} \times \partial \mathcal{H} \) and \( \partial \mathcal{B} \times \partial \mathcal{B} \) respectively.

For a proof of the theorem, see Oden.\(^{22}\)

Let \( B : \mathcal{H} \times \mathcal{B} \rightarrow \mathbb{R} \) be a continuous bilinear form on Hilbert spaces \( \mathcal{H} \) and \( \mathcal{B} \) having the trace property. Let in addition, \( A \) be the formal operator associated with \( B(\cdot, \cdot) \) and let \( A^* \) be its formal adjoint. The operators \( \gamma \in \mathcal{L}(\mathcal{H}, \partial \mathcal{H}) \) and \( \delta \in \mathcal{L}(\mathcal{H}_A, \partial \mathcal{H}) \) described above are called the Dirichlet operator and the Neumann operator, respectively, corresponding to the operator \( A \). Likewise, the operators \( \gamma^* \in \mathcal{L}(\mathcal{B}, \partial \mathcal{B}) \) and \( \delta^* \in \mathcal{L}(\mathcal{B}_{A^*}, \partial \mathcal{B}') \) described above are called the Dirichlet and Neumann operators, respectively, corresponding to the operator \( A^* \).

1.3.4 Green's formulae

The relationships derived in Theorem 1.3.1. are called Green's formulae for the bilinear form \( B(\cdot, \cdot) \):

\[
B(u, v) = (v, Au)_\mathcal{V} + (\delta u, \gamma^* v)_{\mathcal{H}} \quad u \in \mathcal{H}_A, \quad v \in \mathcal{B}
\]

\[
B(u, v) = (A^* v, u)_{\mathcal{H}_0'} + (\delta^* v, \gamma^* u)_{\mathcal{H}_0} \quad u \in \mathcal{H}, \quad v \in \mathcal{B}_{A^*}
\]

(1.29)
If we take \( u \in \mathcal{H}_A \) and \( v \in \mathcal{G}_{A^*} \), we obtain the abstract Green's formula for the operator \( A \in \mathcal{L}(\mathcal{H}, \mathcal{G}) \cap \mathcal{L}(\mathcal{H}, \mathcal{V}) \):

\[
(A^*v, u)_{\mathcal{H}} = (v, Au)_{\mathcal{V}} + \langle \delta u, \gamma^*v \rangle_{\partial \Omega} - \langle \delta^*v, \gamma u \rangle_{\partial \Omega}, \quad u \in \mathcal{H}_A, \quad v \in \mathcal{G}_{A^*}.
\]

(1.30)

The collection of boundary terms,

\[
\Gamma(u, v) = \langle \delta u, \gamma^*v \rangle_{\partial \Omega} - \langle \delta^*v, \gamma u \rangle_{\partial \Omega}
\]

is called the bilinear concomitant of \( A; \Gamma: \mathcal{H}_A \times \mathcal{G}_{A^*} \rightarrow \mathbb{R} \).

**Example 1.3.1** Consider the case in which \( \Omega \) is a smooth open bounded subset of \( \mathbb{R}^2 \) with a smooth boundary \( \partial \Omega \) and

\[
\mathcal{H} = \mathcal{G} = H^1(\Omega) = \{ u: u, u_x, u_y \in L_2(\Omega) \}; \quad \mathcal{V} = \mathcal{V} = L_2(\Omega)
\]

Let \( a = a(x, y) \), \( b = b(x, y) \), and \( c = c(x, y) \) be sufficiently smooth functions of \( x \) and \( y \) (e.g. \( a, b, c \in C^1(\Omega) \)), and define the bilinear form \( B: H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{R} \) by

\[
B(u, v) = \int_\Omega (a \nabla u \cdot \nabla v + b u_x v + c v u_y) \, dx \, dy
\]

where \( \nabla u = \text{grad} \ u = (u_x, u_y) \). Thus,

\[
\partial \mathcal{H} = \partial \mathcal{G} = H^{1/2}(\partial \Omega); \quad \ker \gamma = H^1_0(\Omega) = \{ u \in H^1(\Omega); u = 0 \text{ on } \partial \Omega \}
\]

In this case,

\[
B(u, v) = \int_\Omega v(-\nabla \cdot (a \nabla u) + b u_x + c v u_y) \, dx \, dy + \oint_{\partial \Omega} a \frac{\partial u}{\partial n} \frac{\partial v}{\partial n} \, ds
\]

Thus, the formal operator corresponding to \( B(\cdot, \cdot) \) is

\[
Au = -\nabla \cdot (a \nabla u) + b \frac{\partial u}{\partial x} + c \frac{\partial u}{\partial y}
\]

if we take

\[
u \in \mathcal{H}_A = \{ u \in H^1(\Omega): Au \in L_2(\Omega) \}
\]

Also, we now have

\[
\delta u = a \left. \frac{\partial u}{\partial n} \right|_{\partial \Omega}, \quad \gamma^*v = v|_{\partial \Omega}
\]

Similarly, if \( v \in \mathcal{G}_{A^*} = \mathcal{H}_A \), we have

\[
B(u, v) = \int_\Omega u \left( -\nabla \cdot (a \nabla v) - \frac{\partial bv}{\partial x} - \frac{\partial cv}{\partial y} \right) \, dx \, dy + \oint_{\partial \Omega} \left( au \frac{\partial v}{\partial n} + b v u_x + c v u_y \right) \, ds
\]
where \( n_x, n_y \) are the components of the unit outward normal \( n \) to \( \partial \Omega \). Thus,
\[
\begin{align*}
A^* v &= -\nabla \cdot(a \nabla v) - \frac{\partial bv}{\partial x} - \frac{\partial cv}{\partial y} \\
\gamma u &= u|_{\partial \Omega} \\
\delta^* v &= \left[ a \frac{\partial v}{\partial n} + (b n_x + c n_y) v \right]|_{\partial \Omega}
\end{align*}
\]

The bilinear concomitant is then
\[
\Gamma(u, v) = \int_{\partial \Omega} \left[ a \frac{\partial u}{\partial n} - au \frac{\partial v}{\partial n} - (c n_x + b n_y) uv \right] ds
\]

**Example 1.3.2** Let \( \Omega \) be as in Example 1.3.1 and define
\[
B(u, v) = \int_{\Omega} \text{grad } u \cdot v \, dx \, dy
\]
as a bilinear form on \( \mathcal{H} \times \mathcal{G} \), where
\[
\mathcal{H} = H^1(\Omega) = \{ u : u, u_x, u_y \in L_2(\Omega) \}
\]
\[
\mathcal{G} = H^1(\Omega) = \{ \mathbf{v} = (v_1, v_2) : v_1, v_{1x}, v_{1y}, v_2, v_{2x}, v_{2y} \in L_2(\Omega) \}
\]
\( H^1(\Omega) \) is dense in \( \mathcal{U} = L_2(\Omega) \) and \( H^1(\Omega) \) is dense in \( \mathcal{V} = L_2(\Omega) = L_2(\Omega) \times L_2(\Omega) \). In this case, we may take \( \gamma^* v = v \cdot n|_{\partial \Omega} \) but \( \delta = 0 \). Indeed, if the formal operator associated with the given bilinear form is
\[
A u = \text{grad } u
\]
then \( \mathcal{H}_A = \mathcal{H} = H^1(\Omega) \) and \( C_A : \mathcal{H}_A \rightarrow \mathcal{G}_0^* \) is identically zero. We next note that
\[
B(u, v) = \int_{\Omega} u(-\text{div } v) \, dx \, dy + \oint_{\partial \Omega} v \cdot n u \, ds
\]
Thus, \( \mathcal{G}_A^* = H^1(\Omega) \) and
\[
A^* v = -\text{div } v; \quad \gamma u = u|_{\partial \Omega}; \quad \delta^* v = v \cdot n|_{\partial \Omega}
\]
The Green's formula is the classical relation,
\[
\oint_{\partial \Omega} (\text{grad } u \cdot v + u \text{ div } v) \, dx \, dy = \oint_{\partial \Omega} uv \cdot n \, ds
\]

**1.3.5 Mixed boundary condition**

An abstract Green's formula appropriate for operators with mixed boundary conditions can be obtained by introducing some additional operators and spaces.
Let
\[ \pi^*_1 = \text{a continuous linear projection defined on } \partial \mathcal{G} \text{ into itself} \]
\[ \pi^*_2 = I - \pi^*_1, \text{ } I \text{ being the identity map from } \partial \mathcal{G} \text{ onto itself} \]
\[ \gamma^*_1 = \pi^*_1 \gamma^*, \quad \gamma^*_2 = \pi^*_2 \gamma^* \quad \text{and} \quad \gamma^* = \gamma^*_1 + \gamma^*_2 \]
\[ \mathcal{F} = \ker \gamma^*_1 = \{ v \in \mathcal{G} : \pi^*_1 \gamma^* v = 0 \} \]

The space \( \mathcal{F} \) is a closed linear subspace of \( \mathcal{G} \) with the property
\[ \mathcal{F}_0 \subset \mathcal{F} \subset \mathcal{G} \]

The operators \( \gamma^*_1 \) and \( \gamma^*_2 \) effectively decompose \( \partial \mathcal{G} \) into two subspaces
\[ \partial \mathcal{G}_1 = \gamma^*_1(\mathcal{G}), \quad \partial \mathcal{G}_2 = \gamma^*_2(\mathcal{G}) \]
\[ \partial \mathcal{G} = \partial \mathcal{G}_1 + \partial \mathcal{G}_2 \]

A similar collection of operators can be introduced on \( \partial \mathcal{H} \):
\[ \pi_1 = \text{a continuous linear projection of } \partial \mathcal{H} \text{ into } \partial \mathcal{H} \]
\[ \pi_2 = I' - \pi_1 (I' : \partial \mathcal{H} \to \partial \mathcal{H}) \]
\[ \gamma_1 = \pi_1 \gamma, \quad \gamma_2 = \pi_2 \gamma, \quad \gamma = \gamma_1 + \gamma_2 \]
\[ \mathcal{F} = \ker \gamma_2 = \{ u \in \mathcal{H} : \pi_2 \gamma u = 0 \} \]
\[ \mathcal{H}_0 \subset \mathcal{F} \subset \mathcal{H} \]
\[ \partial \mathcal{H}_1 = \gamma_1(\mathcal{H}), \quad \partial \mathcal{H}_2 = \gamma_2(\mathcal{H}), \quad \partial \mathcal{H} = \partial \mathcal{H}_1 + \partial \mathcal{H}_2 \]

The Green formulae (1.28) now yield
\[ B(u, v) = (v, Au)_\gamma + (\delta u, \gamma_2^* v)_{\partial \mathcal{G}_2}, \quad u \in \mathcal{H}, \quad v \in \mathcal{F} \]
\[ B(u, v) = (A^* v, u)_\gamma + (\delta^* v, \gamma_1 u)_{\partial \mathcal{G}_1}, \quad v \in \mathcal{G}_\gamma, \quad u \in \mathcal{F} \]

We observe for \( v \in \mathcal{F} \),
\[ (\delta u, \gamma^* v)_{\partial \mathcal{G}} = (\delta u, \pi^*_1 \gamma^* v)_{\partial \mathcal{G}} + (\delta u, \pi^*_2 \gamma^* v)_{\partial \mathcal{G}} \]
\[ = (\delta u, \pi^*_2 \gamma^* v)_{\partial \mathcal{G}} \]
\[ = (\pi^*_2 \delta u, \gamma^* v)_{\partial \mathcal{G}} = (\delta_2 u, \gamma^* v)_{\partial \mathcal{G}}. \]

where
\[ \delta_2 = \pi^*_2 \delta, \quad \delta_2 \in \mathcal{L}(\mathcal{H}, \partial \mathcal{G}_2) \]

Finally, collecting (1.36) and (1.37), we arrive at the abstract Green's formula.
\[ (A^* v, u)_\gamma = (v, Au)_\gamma - (\delta^* v, \gamma_1 u)_{\partial \mathcal{G}_1} + (\delta_2 u, \gamma^* v)_{\partial \mathcal{G}_2} \]
\[ \forall u \in \mathcal{H} \cap \mathcal{F}, \quad v \in \mathcal{G}_\gamma \cap \mathcal{F} \]
1.4 ABSTRACT VARIATIONAL BOUNDARY-VALUE PROBLEMS

1.4.1 Some linear boundary-value problems

The Green's formulae and various properties of the bilinear forms $B(\cdot, \cdot)$ described in the previous section provide the basis for a theory of boundary-value problems involving linear operators on Hilbert spaces. We shall continue to use the notations and conventions of the previous section: $\mathcal{H}$ and $\mathcal{G}$ are Hilbert spaces with the trace property, densely embedded in pivot spaces $\mathcal{U}$ and $\mathcal{V}$, respectively, and $\gamma^* : \mathcal{G} \to \partial \mathcal{G}$, $\ker \gamma^* = \mathcal{G}_0$, $\gamma : \mathcal{H} \to \partial \mathcal{H}$, $\ker \gamma = \mathcal{H}_0$, etc.

Let $B : \mathcal{H} \times \mathcal{G} \to \mathbb{R}$ be a continuous bilinear form and let $A$ be the formal operator associated with $B(u, v)$. We consider three types of boundary-value problems associated with $A$.

(1) The Dirichlet problem for $A$. Given data $f \in \mathcal{V}$ and $g \in \partial \mathcal{H}$, the problem of finding $u \in \mathcal{H}_A$ such that
\[
\begin{align*}
Au &= f \\
\gamma u &= g
\end{align*}
\] (1.40)

is called the Dirichlet problem for the operator $A$.

(2) The Neumann problem for $A$. Given data $f \in \mathcal{V}$ and $s \in \partial \mathcal{G}$, the problem of finding $u \in \mathcal{H}_A$ such that
\[
\begin{align*}
Au &= f \\
\delta u &= s
\end{align*}
\] (1.41)

is called the Neumann problem for the operator $A$.

(3) The mixed problem for $A$. Given data $f \in \mathcal{V}$, $g \in \partial \mathcal{H}_1$, and $s \in \partial \mathcal{H}_2$, the problem of finding $u \in \mathcal{H}_A$ such that
\[
\begin{align*}
Au &= f \\
\gamma_1 u &= g \\
\delta_2 u &= s
\end{align*}
\] (1.42)

is called the mixed problem for the operator $A$.

Now the bilinear form $B : \mathcal{H} \times \mathcal{G} \to \mathbb{R}$ described in the previous section can be used to construct variational boundary-value problems analogous to those for $A$. We shall consider the following variational problems:

(1) The variational Dirichlet problem for $A$. Given data $f \in \mathcal{V}$ and $g \in \partial \mathcal{H}$, find $w \in \mathcal{H}_0 = \ker \gamma$ such that
\[
B(w, v) = (f, v) + B(\gamma^{-1} g, v) \forall v \in \mathcal{G}_0
\] (1.43)

where $\gamma^{-1}$ is a right inverse of $\gamma$. 

(2) The variational Neumann problem for \( A \). Given data \( f \in \mathcal{V} \) and \( s \in \partial \mathcal{G} \), find \( u \in \mathcal{H} \) such that

\[
B(u, v) = (f, v)_\mathcal{V} + (s, \gamma^* v)_{\partial \mathcal{G}} \quad \forall v \in \mathcal{G} \tag{1.44}
\]

(3) The variational mixed problem for \( A \). Given data \( f \in \mathcal{V} \), \( g \in \partial \mathcal{H}_1 \), and \( s \in \partial \mathcal{G}_2 \), find \( w \in \ker \gamma_1 = \ker \pi_1 \gamma \) such that

\[
B(w, v) = (f, v)_\mathcal{V} - B(\gamma_1^{-1} g, v) + (s, \gamma^* v)_{\partial \mathcal{G}_2} \quad \forall v \in \mathcal{G} \tag{1.45}
\]

where \( \gamma_1^{-1} \) is a right inverse of \( \gamma_1 \) and \( \mathcal{G} = \ker \gamma_1^* = \ker \pi_1 \gamma^* \).

**Remark.** There are, of course, several other abstract boundary-value problems for the operator \( A \) that could be mentioned. For example, a second bilinear form \( b : \mathcal{H} \times \mathcal{H} \to \mathbb{R} \) could be introduced at this point which would permit the construction of oblique boundary conditions and which would lead to a formulation more general than (1.42). However, such technical generalizations obscure the simple structure of the ideas we wish to clarify here, and so they are omitted.

**Theorem 1.4.1** The Dirichlet problem (1.40) for the operator \( A \) and the variational Dirichlet problem (1.43) are equivalent in the following sense. Let \( \gamma^{-1} \) be the inverse map of \( \gamma \); if \( u \) is a solution of (1.40) then \( w = u - \gamma^{-1} g \) is a solution of (1.43). Conversely, if \( w \) is a solution of (1.43), then \( u = w + \gamma^{-1} g \) is a solution of (1.40). Moreover, the Neumann problem (1.41) for the operator \( A \) is equivalent to the variational Neumann problem (1.44) in the sense that any solution of (1.41) is also a solution of (1.44) and, conversely, any solution of (1.44) is a solution of (1.41). Likewise, problems (1.42) and (1.45) are equivalent in a similar sense.

For a proof see Reference 22.

**Example 1.4.1** Let \( \mathcal{H} = \mathcal{G} = H^1(\Omega) \), \( \mathcal{V} = \mathcal{V} = L^2(\Omega) \), \( \mathcal{H}_0 = \mathcal{G}_0 = H^1_0(\Omega) \), and \( \partial \mathcal{H} = \partial \mathcal{G} = H^{1/2}(\partial \Omega) \), where \( \Omega \) is a smooth open bounded domain in \( \mathbb{R}^2 \). The bilinear form

\[
B(u, v) = \int_{\Omega} (\nabla u \cdot \nabla v + \alpha uv) \, dx \, dy
\]

where \( \alpha \) is a non-negative constant, is a continuous bilinear form from \( H^1(\Omega) \times H^1(\Omega) \) into \( \mathbb{R} \). The formal operator associated with \( B(\cdot, \cdot) \) is \( A = -\Delta + \alpha \), where \( \Delta = \nabla \cdot \nabla = \nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 \) is the Laplacian operator. We denote

\[
\mathcal{H}_A = \mathcal{G}_A^* = H^1(A, \Omega) = \{ u \in H^1(\Omega) : -\Delta u + \alpha u \in L^2(\Omega) \}.
\]
The Dirichlet problem for \( A \) is to find \( u \in H^1(\Omega) \) such that

\[
-\Delta u + \alpha u = f \quad \text{in } \Omega, \quad f \in L^2(\Omega)
\]

\[
u = g \quad \text{on } \partial \Omega, \quad g \in H^{1/2}(\partial \Omega)
\]

In view of Theorem 1.4.1, this problem is equivalent to the problem of finding \( w \in H^1_0(\Omega) \) such that

\[
\int_{\Omega} (\nabla w \cdot \nabla v + \alpha w v) \, dx \, dy = \int_{\Omega} fv \, dx \, dy \quad - \int_{\Omega} (\nabla w_0 \cdot \nabla v + \alpha w_0 v) \, dx \, dy \quad \forall v \in H^1_0(\Omega)
\]

where \( w_0 \) is any function in \( H^1(\Omega) \) such that \( w_0 = g \) on \( \partial \Omega \). Then \( u = w + w_0 \) is the solution of the Dirichlet problem for \( A \).

The Neumann problem for \( A \) is to find \( u \in H^1(\Omega, \Omega) \) such that

\[
-\Delta u + \alpha u = f \quad \text{in } \Omega, \quad f \in L^2(\Omega)
\]

\[
\frac{\partial u}{\partial n} = s \quad \text{on } \partial \Omega, \quad s \in (H^{1/2}(\partial \Omega))'
\]

and it is equivalent to seeking \( u \in H^1(\Omega) \) such that

\[
\int_{\Omega} (\nabla u \cdot \nabla v + \alpha uv) \, dx \, dy = \int_{\Omega} fv \, dx \, dy + \oint_{\partial \Omega} sv \, ds \quad \forall v \in H^1(\Omega)
\]

where the contour integral \( \oint_{\partial \Omega} \) denotes duality pairing on \( (H^{1/2}(\partial \Omega))' \times H^{1/2}(\partial \Omega) \).

We observe that boundary conditions enter the statement of a variational boundary-value problem in two distinct ways. The essential or stable boundary conditions enter by simply defining the spaces \( \mathcal{H}_0 \) and \( \mathcal{G}_0 \) on which the problem is posed. The natural or unstable boundary conditions are introduced in the definition of the bilinear form \( B(u, v) \) and are defined on the spaces \( \mathcal{H}_A \) and \( \mathcal{G}_A \).

### 1.4.2 Compatibility of the data

We shall discuss briefly here the issue of the compatibility of the data \( f, g, s \) with various boundary-value problems. The ideas are derived from the classical theorem (recall Theorem 1.2.1).

**Theorem 1.4.2** Let \( A \) be a bounded linear operator from a Hilbert space \( \mathcal{H} \) into a Hilbert space \( \mathcal{G} \). Let \( A^* \) be its adjoint. Let \( \mathcal{N}(A), \mathcal{N}(A^*) \) denote the null
spaces of \( A \) and \( A^* \) respectively and \( \mathcal{R}(A) \) and \( \mathcal{R}(A^*) \) denote the ranges of \( A \) and \( A^* \) respectively. Then

\[
\begin{align*}
\mathcal{R}(A)^\perp &= \mathcal{N}(A^*), & \overline{\mathcal{R}(A)} &= \mathcal{N}(A^*)^\perp \cr
\mathcal{R}(A^*)^\perp &= \mathcal{N}(A), & \overline{\mathcal{R}(A^*)} &= \mathcal{N}(A)^\perp \end{align*}
\]

(1.46)

where \( \perp \) denotes the orthogonal complement of the spaces indicated and an overbar indicates the closure.

**Proof:** For a proof see, for example, Taylor.21

We shall first address the compatibility question in connection with the Dirichlet problem for the operator \( A \): find \( u \in \mathcal{H}_A \) such that

\[
Au = f, \quad f \in \mathcal{V}
\]

\[
\gamma u = g, \quad g \in \partial \mathcal{H}
\]

Let \( A^* \in \mathcal{L}(\mathcal{G}_{A^*}, \mathcal{H}_0) \) denote the formal adjoint of \( A \) and let \( \gamma^* \in \mathcal{L}(\mathcal{G}, \partial \mathcal{G}) \), \( \ker \gamma^* = \mathcal{G}_0 \). Then the adjoint Dirichlet problem corresponding to \( A \) is the problem of finding \( v \in \mathcal{G}_{A^*} \) such that

\[
A^*v = f^*, \quad f^* \in \mathcal{U}
\]

\[
\gamma^*v = g^*, \quad g^* \in \partial \mathcal{G}
\]

We also introduce the null spaces,

\[
\mathcal{N}(A, \gamma) = \{ u \in \mathcal{H}_A : Au = 0, \gamma u = 0 \}
\]

\[
\mathcal{N}(A^*, \gamma^*) = \{ v \in \mathcal{G}_{A^*} : A^*v = 0, \gamma^*v = 0 \}
\]

We shall assume that these spaces are finite-dimensional.

Now if \( \mathcal{N}(A, \gamma) \) is finite-dimensional, it is closed in \( \mathcal{U} \) and we have

\[
\mathcal{U} = \mathcal{N}(A, \gamma) \oplus \mathcal{N}(A, \gamma)^\perp
\]

where

\[
\mathcal{N}(A, \gamma)^\perp = \{ u \in \mathcal{U} : (u, v)_\mathcal{U} = 0 \forall v \in \mathcal{N}(A, \gamma) \}
\]

Then \( A \) can be regarded as continuous linear operator from \( \mathcal{N}(A, \gamma)^\perp \) onto its range \( \mathcal{R}(A) \subset \mathcal{V} \). By the Banach theorem, a continuous inverse \( A^{-1} \) exists from \( \mathcal{R}(A) \) onto \( \mathcal{N}(A, \gamma)^\perp \), and \( \mathcal{R}(A) \) is closed in \( \mathcal{V} \). Thus \( \mathcal{V} = \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp \) where \( \mathcal{R}(A)^\perp = \{ v \in \mathcal{V} : (v, w)_{\mathcal{V}} = 0 \forall w \in \mathcal{R}(A) \} \). The Dirichlet problem for \( A \) clearly has at least one solution whenever the data \( f \in \mathcal{R}(A) \) and \( g \in \mathcal{R}(\gamma) \). Data satisfying these requirements are said to be compatible with the operators \( (A, \gamma) \).

A convenient test for compatibility of the data is given in the following theorem.
Theorem 1.4.3 Let \( f \in \mathcal{V} \) and \( g \in \mathcal{K} \) be data in a Dirichlet problem for the operator \( A \). Then a necessary and sufficient condition that there exists a solution of the Dirichlet problem is that
\[
(f, v)_{\mathcal{V}} - \langle \delta^* v, g \rangle_{\mathcal{K}} = 0 \quad \forall v \in \mathcal{N}(A^*, \gamma^*)
\] (1.47)

For the Neumann problems
\[
A u = f, \quad A^* v = f^*
\]
\[
\delta u = s, \quad \delta^* v = s^*
\]
we define
\[
\mathcal{N}(A, \delta) = \{ u \in \mathcal{H}_A : A u = 0, \delta u = 0 \}
\]
\[
\mathcal{N}(A^*, \delta^*) = \{ v \in \mathcal{H}_{A^*}^* : A^* v = 0, \delta^* v = 0 \}
\]
and, analogously, have

Theorem 1.4.4 A necessary and sufficient condition for the existence of at least one solution of the Neumann problem for the operator \( A \) is that the data \( (f, s) \) satisfy
\[
(f, v)_{\mathcal{V}} + \langle s, \gamma^* v \rangle_{\mathcal{K}} = 0 \quad \forall v \in \mathcal{N}(A^*, \delta^*)
\] (1.48)

A similar compatibility condition can be developed for mixed boundary-value problems:

Theorem 1.4.5 A necessary and sufficient condition for the existence of a solution to the mixed boundary-value problem for \( A \) is that the data \( (f, g, s) \) satisfy
\[
(f, v)_{\mathcal{V}} - \langle \delta^* v, g \rangle_{\mathcal{K}} + \langle s, \gamma^* v \rangle_{\mathcal{K}} = 0 \quad \forall v \in \mathcal{N}(A^*, \gamma_1^*, \delta_2^*)
\] (1.49)

where
\[
\mathcal{N}(A^*, \gamma_1^*, \delta_2^*) = \{ v \in \mathcal{H}_{A^*}^* : A^* v = 0, \gamma_1^* v = 0, \delta_2^* v = 0 \}
\] (1.50)

Whenever the compatibility conditions hold, a solution to the Dirichlet, Neumann, or mixed problems may exist, but it will not necessarily be unique. The solution \( u \) is unique, of course, whenever \( \mathcal{N}(A, \gamma) = \{ 0 \} \) for the Dirichlet problem, whenever \( \mathcal{N}(A, \delta) = \{ 0 \} \) for the Neumann problem, and when \( \mathcal{N}(A, \gamma, \delta) = \{ 0 \} \) for the mixed problem. However, these conditions often do not hold. We can, however, force the solutions to either class of problems to be unique by imposing an additional condition.

Theorem 1.4.6 Let \( u \in \mathcal{H}_A \) be a solution to the Dirichlet problem for the operator \( A \). Then \( u \) is the only solution to this problem if
\[
(u, w)_u = 0 \quad \forall w \in \mathcal{N}(A, \gamma)
\] (1.51)
Likewise, a solution $u$ of the Neumann (mixed) problem for $A$ is unique if $(u, w)_{\mathcal{H}} = 0 \forall w \in \mathcal{N}(A, \delta) \forall w \in \mathcal{N}(A, \gamma, \delta)$, respectively.

### 1.4.3 Existence theory

The theory presented thus far suggests the following general setting for linear variational boundary-value problems.

—Let $\mathcal{H}$ and $\mathcal{G}$ be arbitrary Hilbert spaces (now $\mathcal{H}$ and $\mathcal{G}$ are not necessarily the spaces appearing earlier in this section—they do not necessarily have the trace property).

—$B : \mathcal{H} \times \mathcal{G} \to \mathbb{R}$ is a bilinear form. Then find an element $u \in \mathcal{H}$ such that

$$B(u, v) = f(v) \quad \forall v \in \mathcal{G}$$

where $f \in \mathcal{G}'$.

The essential question here is: What conditions can be imposed so that we are guaranteed that a unique solution exists which depends continuously on the choice of data $f$? This question was originally resolved for certain choices of $B$ by Lax and Milgram. A more general form of their classic theorem made popular by Babuška (see also Nečas [17] and Oden and Reddy [19]) is given as follows.

**Theorem 1.4.7** Let $B : \mathcal{H} \times \mathcal{G} \to \mathbb{R}$ be a bilinear functional on $\mathcal{H} \times \mathcal{G}$, $\mathcal{H}$ and $\mathcal{G}$ being Hilbert spaces, which has the following three properties.

(i) There exists a constant $M > 0$ such that

$$B(u, v) \leq M \|u\|_{\mathcal{H}} \|v\|_{\mathcal{G}} \quad \forall u \in \mathcal{H}, \quad \forall v \in \mathcal{G}$$

where $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{G}}$ denote the norms on $\mathcal{H}$ and $\mathcal{G}$, respectively.

(ii) There exists a constant $\gamma > 0$ such that

$$\inf_{u \in \mathcal{H}} \sup_{v \in \mathcal{G}} |B(u, v)| \geq \gamma > 0$$

where $\inf_{\|u\|_{\mathcal{H}} = 1} \|u\|_{\mathcal{G}} \leq 1$.

(iii) We have

$$\sup_{u \in \mathcal{H}} |B(u, v)| > 0, \quad v \neq 0$$

Then there exists a unique solution to the problem of finding $u \in \mathcal{H}$ such that

$$B(u, v) = f(v) \quad \forall v \in \mathcal{G}, \quad f \in \mathcal{G}'$$

Moreover, the solution $u_0$ depends continuously on the data; in fact,

$$\|u_0\|_{\mathcal{H}} \leq \frac{1}{\gamma} \|f\|_{\mathcal{G}'}.$$
Property (i) $B(\cdot, \cdot)$ is, of course, a continuity requirement; $B(\cdot, \cdot)$ is assumed to be a bounded linear functional on $\mathcal{H}$ and on $\mathcal{G}$. Properties (ii) and (iii) serve to establish the existence of a continuous inverse of the associated operator $A$.

Corollary 1.4.8 Let $B: \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ be a bilinear form on a real Hilbert space $\mathcal{H}$. Let there exist positive constants $M$ and $\gamma$ such that

$$B(u, v) \leq M \|u\|_\mathcal{H} \|v\|_\mathcal{H} \quad \forall u, v \in \mathcal{H}$$

$$B(u, u) \geq \gamma \|u\|_\mathcal{H}^2$$

Then there exists a unique $u \in \mathcal{H}$ such that

$$B(u, v) = f(v) \quad \forall v \in \mathcal{H}, \quad f \in \mathcal{H}'$$

and

$$\|u\|_\mathcal{H} \leq \frac{1}{\gamma} \|f\|_\mathcal{H}'$$

1.5 CONSTRUCTION OF VARIATIONAL PRINCIPLES

Let $W$ be a real Banach space and $W'$ its topological dual. If $\mathcal{P}$ is an operator from $W$ into $W'$, not necessarily linear, we may consider the abstract problem of finding $u \in W$ such that

$$\mathcal{P}(u) = 0, \quad 0 \in W'$$

(1.52)

Now it is well known that in many cases an alternative problem can be formulated, equivalent to (1.52), which involves seeking a $u \in W$ such that

$$K'(u) = 0$$

where $K$ is an appropriate functional defined on $W$ and $K'(u)$ is the Gâteaux derivative of $K$ at $u$, i.e.

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (K(u + \varepsilon \eta) - K(u)) = \langle K'(u), \eta \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes duality pairing on $W' \times W$. Thus, if there exists a Gâteaux differentiable functional $K: W \rightarrow \mathbb{R}$ such that

$$\mathcal{P} = K'$$

(1.53)

then (1.52) is equivalent to the classical variational problem of finding elements $u \in W$ which are critical points of the functional $K$. We say that $\mathcal{P}$ is the gradient of $K$ and we sometimes write $\mathcal{P} = \text{grad} K$.

Any operator $\mathcal{P}: W \rightarrow W'$ for which there exists a functional $K: W \rightarrow \mathbb{R}$ such that $\mathcal{P} = \text{grad} K$ is called a potential operator. It is well known (see, for example, Vainberg or Nashed) that if a continuous Gâteaux differential
Given a potential operator $\mathcal{P}: W \to W'$, the problem of determining a functional $K$ such that (1.53) holds is called the inverse problem of the calculus of variations. Its solution is provided by the following theorem, the proof of which can be found in the monograph of Vainberg.

Theorem 1.5.1 Let $\mathcal{P}: W \to W'$ be a potential operator on the Banach space $W$. Then $\mathcal{P}$ is the gradient of the functional $K: W \to \mathbb{R}$ given by

$$K(u) = \int_0^1 \langle \mathcal{P}(u_0 + s(u - u_0)), u_0 \rangle \, ds + K_0$$

where $u_0$ is a fixed point in $W$, $K_0 = K(u_0)$, and $s \in [0, 1]$. ■

By an appropriate identification of the space $W$ and the duality $\langle \cdot, \cdot \rangle$, all of the classical variational principles of mathematical physics can be constructed using (1.54). A lengthy list of applications of (1.54) to this end can be found in Chapter 5 of Oden and Reddy.

1.6 THE CLASSICAL VARIATIONAL PRINCIPLES

1.6.1 A general class of boundary-value problems

We shall now describe a general class of abstract boundary-value problems that is encountered with remarkable frequency in linear problems of mathematical physics and mechanics. Continuing to use the notation of the previous section, we introduce a linear, symmetric, operator

$$E: \mathcal{V}' \to \mathcal{V}$$

which effects a continuous, isometric isomorphism of the dual of the pivot space $\mathcal{V}'$ onto $\mathcal{V}$, which has a continuous inverse, $E^{-1}: \mathcal{V} \to \mathcal{V}'$. If $\mathbf{v} \in \mathcal{V}'$, we shall denote the elements in $E(\mathcal{V}')$ by

$$\mathbf{v} = Ev$$

The Green's formula (1.30) can now be written

$$(A^* \mathbf{v}, u)_\mathcal{H} = (Au, \mathbf{v})_\mathcal{V} + \langle \delta^* \mathbf{v}, \gamma_1 u \rangle_{\mathcal{X}} - \langle \delta u, \gamma_2^* (\mathbf{v}) \rangle_{\mathcal{X}}_\mathcal{A}, \quad \forall u \in \mathcal{X}_\mathcal{H}, \quad \forall \mathbf{v} \in \mathcal{X}_\mathcal{A}$$

(1.57)
Now let us consider the following abstract problem. Given \( f \in \mathcal{U} \), \( g \in \partial \mathcal{H}_1 \), and \( s \in \partial \mathcal{H}_2 \), find \( u \in \mathcal{H}_A \) such that

\[
\begin{align*}
A^*EAu &= f \\
\gamma_1 u &= g \\
\gamma_2^*EAu &= s
\end{align*}
\]  
(1.58)

where we have used the notations of (1.32) and (1.35). We shall exploit the fact that this problem can be rewritten in the following canonical form: find

\[
(u, v, \sigma) \in \mathcal{H}_A \times \mathcal{Y}' \times \mathcal{G}_A^*
\]

such that

\[
\begin{align*}
Au = v & \quad \gamma_1 u = g \\
Ev = \sigma & \quad A^*\sigma = f \\
A^*u &= s
\end{align*}
\]  
(1.59)

We would now like to construct a variational principle (i.e. a potential functional) corresponding to (1.60). Towards this end, we introduce the matrix operator

\[
\begin{bmatrix}
0 & -1 & A & 0 & 0 \\
-1 & E & 0 & 0 & 0 \\
A^* & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \pi_1 & 0 \\
0 & 0 & 0 & \pi_2^* & 0
\end{bmatrix}
\begin{bmatrix}
\sigma \\
v \\
u \\
\gamma^*\sigma \\
\gamma u
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
f \\
g \\
s
\end{bmatrix}
\]  
(1.61)

or

\[
\mathcal{P}(\lambda) = f
\]  
(1.62)

where \( \mathcal{P} \) is the coefficient matrix of operators in (1.61).

\[
\lambda^T = (\sigma, v, u, \gamma^*\sigma, \gamma u) \in \mathcal{G}_A^* \times \mathcal{Y}' \times \mathcal{H}_A \times \partial \mathcal{H} \times \partial \mathcal{H}
\]
\[
f^T = (0, 0, f, g, s) \in \mathcal{Y}' \times \mathcal{U} \times \partial \mathcal{H}_1 \times \partial \mathcal{H}_2
\]

If

\[
\mathcal{W} = \mathcal{Y}' \times \mathcal{Y} \times \mathcal{U} \times \partial \mathcal{H}_1 \times \partial \mathcal{H}_2
\]

and

\[
\langle \cdot, \cdot \rangle_{\mathcal{W}} = \langle \cdot, \cdot \rangle_{\mathcal{Y}'} + \langle \cdot, \cdot \rangle_{\mathcal{Y}} + \langle \cdot, \cdot \rangle_{\mathcal{U}} + \langle \cdot, \cdot \rangle_{\partial \mathcal{H}_1} + \langle \cdot, \cdot \rangle_{\partial \mathcal{H}_2}
\]

We compute easily the functional.

\[
L(u, v, \sigma) = \int_0^1 \langle \mathcal{P}(s\lambda) - f, \lambda \rangle_{\mathcal{W}} \, ds
\]

\[
= \frac{1}{2} \langle \mathcal{P}(\lambda), \lambda \rangle_{\mathcal{W}} - \langle f, \lambda \rangle_{\mathcal{W}}
\]

\[
= \frac{1}{2} \langle -v + Au, \sigma \rangle_{\mathcal{Y}'} + \frac{1}{2} \langle -\sigma + Ev, v \rangle_{\mathcal{Y}} + \frac{1}{2} \langle A^*\sigma, u \rangle_{\mathcal{U}}
\]

\[
+ \frac{1}{2} \langle \pi_1 \gamma u, \delta^*\sigma \rangle_{\partial \mathcal{H}_1} + \frac{1}{2} \langle \pi_2^* \gamma^*\sigma, \delta u \rangle_{\partial \mathcal{H}_2} - \langle f, u \rangle_{\mathcal{U}} - \langle g, \delta^*\sigma \rangle_{\partial \mathcal{H}_1} - \langle s, \delta u \rangle_{\partial \mathcal{H}_2}
\]
wherein
\[ \hat{\lambda}^T = (\sigma, v, u, \delta^*\sigma, \delta u) \in \mathcal{V} \times \mathcal{V}^* \times \partial \mathcal{X} \times \partial \mathcal{G} \]

Applying Green's formula to the term \( \frac{1}{2} \langle A^*\sigma, u \rangle_u \) gives
\[ L(u, v, \sigma) = \frac{1}{2} \langle Ev, v \rangle_{\mathcal{V}} + \langle Au - v, \sigma \rangle_{\mathcal{V}} - \langle f, u \rangle_u \]
\[ + \langle \gamma_1 u - g, \delta^*\sigma \rangle_{\partial \mathcal{X}} - \langle \delta u, \delta \sigma \rangle_{\partial \mathcal{G}} \]  \hspace{1cm} (1.63)

The Euler equations are (1.60). Indeed,
\[ \delta L(u, v, \sigma; \tilde{u}, \tilde{v}, \tilde{\sigma}) = \langle Ev, \tilde{v} \rangle_{\mathcal{V}} + \langle Au - v, \tilde{\sigma} \rangle_{\mathcal{V}} - \langle f, \tilde{u} \rangle_u + \langle \gamma_1 u - g, \delta^*\tilde{\sigma} \rangle_{\partial \mathcal{X}} \]
\[ - \langle \delta \tilde{u}, \delta \sigma \rangle_{\partial \mathcal{G}} + \langle A\tilde{u} - \tilde{v}, \sigma \rangle_{\mathcal{V}} + \langle \gamma_1 \tilde{u}, \delta^*\sigma \rangle_{\partial \mathcal{X}} \]
\[ = \langle Ev - \sigma, \tilde{v} \rangle_{\mathcal{V}} + \langle Au - v, \tilde{\sigma} \rangle_{\mathcal{V}} + \langle A^*\sigma - f, \tilde{u} \rangle_u \]
\[ + \langle \gamma_1 u - g, \delta^*\tilde{\sigma} \rangle_{\partial \mathcal{X}} + \langle \gamma_2^*\sigma - s, \delta \tilde{u} \rangle_{\partial \mathcal{G}} \]

where we have applied Green's formula into \( \langle A\tilde{u}, \sigma \rangle_{\mathcal{V}} \).

Next, we list a number of additional principles that can be derived directly from the functional \( L \):

II. Constraint:
\[ \sigma = E^{-1}\sigma \]  \hspace{1cm} (1.64)

\[ R_{\sigma}(u, \sigma) = L(u, v(\sigma), \sigma) \]
\[ = -\frac{1}{2} \langle E^{-1}\sigma, \sigma \rangle_{\mathcal{V}} + \langle Au, \sigma \rangle_{\mathcal{V}} - \langle f, u \rangle_u \]
\[ + \langle \delta^*\sigma, \gamma_1 u - g \rangle_{\partial \mathcal{X}} - \langle \delta u, s \rangle_{\partial \mathcal{G}} \]  \hspace{1cm} (1.65)

Euler equations:
\[ Au = E^{-1}\sigma \hspace{1cm} \gamma_1 u = g \]
\[ A^*\sigma = f \hspace{1cm} \gamma_2^*\sigma = s \]  \hspace{1cm} (1.66)

Constraint:
\[ \tilde{\sigma} = Ev \]  \hspace{1cm} (1.67)

\[ R_{v}(u, v) = L(u, v, \sigma(v)) \]
\[ = -\frac{1}{2} \langle v, Ev \rangle_{\mathcal{V}} + \langle Au, Ev \rangle_{\mathcal{V}} - \langle f, u \rangle_u \]
\[ + \langle \delta^*\sigma, \gamma_1 u - g \rangle_{\partial \mathcal{X}} - \langle \delta u, s \rangle_{\partial \mathcal{G}} \]  \hspace{1cm} (1.68)

Euler equations:
\[ Au = v \hspace{1cm} \gamma_1 u = g \]
\[ A^*Ev = f \hspace{1cm} \gamma_2^*Ev = s \]  \hspace{1cm} (1.69)

III. Constraints:
\[ v = Au \hspace{1cm} \gamma_1 u = g \hspace{1cm} \sigma = EAu \]  \hspace{1cm} (1.70)
\[ I(u) = L(u, v(u), \sigma(u)) = \frac{1}{2} \langle Au, EAu \rangle_{\mathcal{V}} - \langle f, u \rangle_u - \langle \delta u, s \rangle_{\partial \mathcal{G}} \]  \hspace{1cm} (1.71)
Euler equations:

\[ A^*E_2 = f \quad \gamma_2^*E_2 = s \]  (1.72)

\[ \delta I(u, \bar{u}) = \langle A\bar{u}, E_2u \rangle_v - \langle f, \bar{u}\rangle_u - \langle \delta \bar{u}, s \rangle_{\theta_2} \]

\[ = \langle A^*E_2u, \bar{u} \rangle_u - \langle \delta^*E_2u, \gamma_1u \rangle_{\theta_2} + \langle \delta \bar{u}, \gamma_2^*E_2u \rangle_{\theta_2} \]

\[ - \langle f, \bar{u} \rangle_u - \langle \delta \bar{u}, s \rangle_{\theta_2} \]

\[ = \langle A^*E_2u - f, \bar{u} \rangle_u + \langle \delta \bar{u}, \gamma_2^*E_2u - s \rangle_{\theta_2}; \quad \bar{u} \in \ker \gamma_1 \]

IV. Constraints:

\[ v = E^{-1}\sigma \quad A^*\sigma = f \quad \gamma_2^*\sigma = s \]  (1.73)

\[ J(\sigma) = L(u(\sigma), v(\sigma), \sigma) \]

\[ = -\frac{1}{2}\langle E^{-1}\sigma, \sigma \rangle_v - \langle \delta^*\sigma, g \rangle_{\theta_2} \]  (1.74)

Euler equations:

\[ Au = E^{-1}\sigma \quad \gamma_1u = g \]  (1.75)

\[ \delta J(\sigma, \tilde{\sigma}) = -\langle E^{-1}\sigma, \tilde{\sigma} \rangle_v - \langle \delta^*\tilde{\sigma}, g \rangle_{\theta_2} \]

\[ + \langle Au, \tilde{\sigma} \rangle_v + \langle \delta^*\tilde{\sigma}, \gamma_1u \rangle_{\theta_2} - \langle \delta u, \gamma_2^*\tilde{\sigma} \rangle_{\theta_2} - \langle A^*\tilde{\sigma}, u \rangle_u \]

\[ = \langle Au - E^{-1}\sigma, \tilde{\sigma} \rangle_v + \langle \delta^*\tilde{\sigma}, \gamma_1u - g \rangle_{\theta_2}; \quad \tilde{\sigma} \in \ker A^* \cap \ker \gamma_2^* \]

V. Constraints:

\[ Au = v \quad \gamma_1u = g \]  (1.76)

\[ A^*\sigma = f \quad \gamma_2^*\sigma = s \]

\[ K(v) = \frac{1}{2}\langle v, Ev \rangle_v - \langle v, \tilde{\sigma} \rangle_v, \quad \tilde{\sigma} \in \{\sigma : A^*\sigma = f, \gamma_2^*\sigma = s\} \]  (1.77)

Euler equation:

\[ Ev = \sigma \]  (1.78)

VI. Constraints:

\[ Au = v \quad \gamma_1u = g \]  (1.79)

\[ M(u, \sigma) = \langle Au, E_2u \rangle_v + \frac{1}{2}\langle E^{-1}\sigma, \sigma \rangle_v - \langle f, u \rangle_u \]

\[ - \langle Au, \sigma \rangle_v - \langle \delta u, s \rangle_{\theta_2} \]  (1.80)

Euler equations:

\[ Au = E^{-1}\sigma \]

\[ A^*(2E_2u - \sigma) = f \quad \gamma_2^*(2E_2u - \sigma) = s \]  (1.81)
VII. Constraints:

\[ A^* \sigma = f \quad \gamma_2^* \sigma = s \]  

\[ N(v, \sigma) = \frac{1}{2} \langle v, E \nu \rangle - \langle v, \sigma \rangle + \langle \delta^* \sigma, g \rangle_{\mathcal{A}^*} \]  

Euler equations:

\[ v = Au \quad \gamma_1 u = g \quad \sigma = Ev \]  

We summarize these results in Table 1.1.

<table>
<thead>
<tr>
<th>Functional</th>
<th>Definition</th>
<th>Constraints</th>
<th>Euler equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>( L(u, v, \sigma) )</td>
<td>( f )</td>
<td>( A^* \sigma = f ); ( \gamma_2^* \sigma = s )</td>
</tr>
<tr>
<td>II</td>
<td>( R_u(u, \sigma) )</td>
<td>( v = E^{-1} \sigma )</td>
<td>( Au = v; \gamma_1 u = g )</td>
</tr>
<tr>
<td>III</td>
<td>( I(u) )</td>
<td>( v = Au; \gamma_1 u = g )</td>
<td>( A^* \sigma = f; \gamma_2^* \sigma = s )</td>
</tr>
<tr>
<td>IV</td>
<td>( J(\sigma) )</td>
<td>( v = E^{-1} \sigma )</td>
<td>( Au = E^{-1} \sigma; \gamma_1 u = g )</td>
</tr>
<tr>
<td>V</td>
<td>( K(\nu) )</td>
<td>( v = Au; \gamma_1 u = g )</td>
<td>( \gamma_2^* \sigma = s )</td>
</tr>
<tr>
<td>VI</td>
<td>( M(u, \sigma) )</td>
<td>( Au = v; \gamma_1 u = g )</td>
<td>( A^* \sigma = f; \gamma_2^* \sigma = s )</td>
</tr>
<tr>
<td>VII</td>
<td>( N(v, \sigma) )</td>
<td>( A^* \sigma = f; \gamma_2^* \sigma = s )</td>
<td>( v = Au; \gamma_1 u = g ); ( \sigma = Ev )</td>
</tr>
</tbody>
</table>

1.7 APPLICATIONS IN ELASTICITY

We now apply the theory in the previous section to the construction of seven variational principles of linear elastostatics.
The Lame-Navier equations of linear elasticity are
\[
\begin{align*}
(A^* E A u)^i &= -(E^{i rs} u_{rs})_i = f_i \quad \text{in } \Omega \\
(\gamma_1 u)_i &= u_i = \hat{u}_i \quad \text{on } \partial \Omega_1 \\
(A^2 E A u)^i &= n_i (E^{i rs} u_{rs}) = \hat{T}_i \quad \text{on } \partial \Omega_2
\end{align*}
\] (1.85)
which correspond to equations (1.58). The canonical equations (1.60) are the strain-displacement equations, kinematical boundary conditions, constitutive equations, and equilibrium equations:
\[
\begin{align*}
(A u)^i &= \frac{1}{2} (u_{ij} + u_{ji}) = \epsilon_{ij} \quad \text{in } \Omega; \quad (\gamma_1 u)_i &= u_i = \hat{u}_i \quad \text{on } \partial \Omega_1 \\
(E e)^{ij} &= E^{i rs} e_{rs} = \sigma^{ij} \quad \text{in } \Omega \\
(A^* \sigma)^i &= -\sigma^{ij}_i = f_i \quad \text{in } \Omega; \quad (\gamma_2 \sigma)^i &= n_i \sigma^{ij} = \hat{T}_i \quad \text{on } \partial \Omega_2
\end{align*}
\] (1.86)

In this case,
\[
\mathcal{U} = L_2(\Omega) = \{u : u_i \in L_2(\Omega), i = 1, 2, 3\} \\
\mathcal{V} = L_2(\Omega) = \{\sigma : \sigma_{ij} \in L_2(\Omega), i, j = 1, 2, 3\}
\] (1.87)
and
\[
\mathcal{H} = H^1(\Omega) = \{u : u_i \in H^1(\Omega), i = 1, 2, 3\} \\
\mathcal{G} = H^1(\Omega) = \{\sigma : \sigma_{ij} \in H^1(\Omega), i, j = 1, 2, 3\}
\] (1.88)

Hence, the canonical problem of the elasticity can be expressed as follows: given
\[
(f, \hat{u}, \hat{T}) \in \mathcal{U} \times \partial \mathcal{H}_1 \times \partial \mathcal{G}_2 = (L_2(\Omega))' \times H^{1/2}(\partial \Omega_1) \times H^{-1/2}(\partial \Omega_2)
\] (1.89)
find
\[
(u, \epsilon, \sigma) \in \mathcal{H}_A \times \mathcal{V}' \times \mathcal{G}_{A^*} = H^1(A, \Omega) \times (L_2(\Omega))' \times H^1(A^*, \Omega)
\] (1.90)
such that equations (1.86) are satisfied. Here
\[
\mathcal{H}_A = H^1(A, \Omega) = \{u \in H^1(\Omega) : Au \in L_2(\Omega)\} \\
\mathcal{G}_{A^*} = H^1(A^*, \Omega) = \{\sigma \in H^1(\Omega) : A^* \sigma \in L_2(\Omega)\}
\] (1.91)

The Green's formula (1.57) becomes, for this problem,
\[
\int_{\Omega} -\sigma^{ij}_i u_j \, dx = \int_{\Omega} u_j \sigma_{ij} \, d\Omega + \int_{\partial \Omega_1} -n_i \sigma^{ij}_i u_j \, ds - \int_{\partial \Omega_2} u_i n_i \sigma^{ij}_i \, ds \\
\forall u \in H^1(A, \Omega), \quad \forall \sigma \in H^1(A^*, \Omega)
\] (1.92)
from which we can identify the operators \( \delta \in \mathcal{L}(H^1(A, \Omega), H^{-1/2}(\partial \Omega)) \) and
The Classical Variational Principles of Mechanics

\[ \delta^* \in \mathcal{L}(H^1(A^*, \Omega), H^{-1/2}(\partial \Omega)):\]

\[
\begin{align*}
(\delta u)_i &= u_i \quad \text{on} \quad \partial \Omega_2 \\
(\delta^* \sigma)^{ij} &= -\eta_i \sigma^{ij} \quad \text{on} \quad \partial \Omega_1
\end{align*}
\]  

(1.93)

Now we are able to construct the variational theory for linear elastostatics. According to Table 1.1, we obtain:

I. The Hu–Washizu principle

\[ L(u, \varepsilon, \sigma) = \int_{\Omega} \left\{ \frac{1}{2} \varepsilon_{ij} E^{ijrs} \varepsilon_{rs} + \left[ \frac{1}{2}(u_{i,j} + u_{j,i}) - \varepsilon_{ij} \right] \sigma^{ij} - f^i u_i \right\} \, dx \]

\[- \int_{\partial \Omega_1} n_i \sigma^{ij} (u_i - \hat{u}_i) \, ds - \int_{\partial \Omega_2} u_i \hat{T}_i \, ds \]  

(1.94)

Euler equations:

\[
\begin{align*}
\frac{1}{2}(u_{i,j} + u_{j,i}) &= \varepsilon_{ij} \quad \text{in} \quad \Omega; \quad u_i = \hat{u}_i \quad \text{on} \quad \partial \Omega_1 \\
E^{ijrs} \varepsilon_{rs} &= \sigma^{ij} \quad \text{in} \quad \Omega \\
-\sigma^{ij}_{,i} &= f^i \quad \text{in} \quad \Omega; \quad n_i \sigma^{ij} = \hat{T}_i \quad \text{on} \quad \partial \Omega_2
\end{align*}
\]  

(1.95)

II. The Hellinger–Reissner principle

(i) Constraints:

\[ \varepsilon_{rs} = C_{ijrs} \sigma^{ij}(C_{ijrs} = (E^{ijrs})^{-1}) \]  

(1.96)

\[ R_\varepsilon(u, \sigma) = \int_{\Omega} \left[ -\frac{1}{2} C_{ijrs} \sigma^{ij} \varepsilon_{rs} + \frac{1}{2}(u_{i,j} + u_{j,i}) \sigma^{ij} - f^i u_i \right] \, dx \]

\[- \int_{\partial \Omega_1} n_i \sigma^{ij} (u_i - \hat{u}_i) \, ds - \int_{\partial \Omega_2} u_i \hat{T}_i \, ds \]  

(1.97)

Euler equations:

\[
\begin{align*}
\frac{1}{2}(u_{i,j} + u_{j,i}) &= C_{rsij} \sigma^{rs} \quad \text{in} \quad \Omega; \quad u_i = \hat{u}_i \quad \text{on} \quad \partial \Omega_2 \\
-\sigma^{ij}_{,i} &= f^i \quad \text{in} \quad \Omega; \quad n_i \sigma^{ij} = \hat{T}_i \quad \text{on} \quad \partial \Omega_2
\end{align*}
\]  

(1.98)

(ii) Constraints:

\[ \sigma^{ij} = E^{ijrs} \varepsilon_{rs} \quad \text{in} \quad \Omega \]  

(1.99)

\[ R_\sigma(u, \varepsilon) = \int_{\Omega} \left[ -\frac{1}{2} \varepsilon_{ij} E^{ijrs} \varepsilon_{rs} + \frac{1}{2}(u_{i,j} + u_{j,i}) E^{ijrs} \varepsilon_{rs} - f^i u_i \right] \, dx \]

\[- \int_{\partial \Omega_1} n_i \sigma^{ij} (u_i - \hat{u}_i) \, ds - \int_{\partial \Omega_2} u_i \hat{T}_i \, ds \]  

(1.100)
Euler equations:

\[
\frac{1}{2}(u_{i,j} + u_{j,i}) = \varepsilon_{ij} \quad \text{in} \quad \Omega; \quad u_i = \hat{u}_i \quad \text{on} \quad \partial\Omega_1
\]

\[-(E^{iir} \varepsilon_r) = f^i \quad \text{in} \quad \Omega; \quad n_i E^{iir} \varepsilon_r = \hat{T}_i \quad \text{on} \quad \partial\Omega_2
\]  \hspace{1cm} (1.101)

III. Potential energy principle

Constraints:

\[
\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{in} \quad \Omega; \quad u_i = \hat{u}_i \quad \text{on} \quad \partial\Omega_1
\]

\[
\sigma_{ij} = \frac{1}{2}E^{iir}(u_{r,s} + u_{s,r}) \quad \text{in} \quad \Omega
\]  \hspace{1cm} (1.102)

\[
I(u) = \int_\Omega \left( \frac{1}{2}u_{i,j}E^{iir}u_{r,s} - f^i u_i \right) \, dx - \int_{\partial\Omega_2} u_i \hat{T}_i \, ds
\]  \hspace{1cm} (1.103)

Euler equations:

\[-(E^{iir} u_{r,s})_i = f^i \quad \text{in} \quad \Omega; \quad n_i E^{iir} u_{r,s} = \hat{T}_i \quad \text{on} \quad \partial\Omega_2
\]  \hspace{1cm} (1.104)

IV. Complementary energy principle

Constraints:

\[
\varepsilon_{rs} = C_{iir} \sigma_{ij} \quad \text{in} \quad \Omega
\]

\[-\sigma_{ij}^{rs} = f^i \quad \text{in} \quad \Omega; \quad n_i \sigma_{ij}^{rs} = \hat{T}_i \quad \text{on} \quad \partial\Omega_2
\]  \hspace{1cm} (1.105)

\[
J(\sigma) = -\int_\Omega \frac{1}{2}C_{iirj} \sigma_{ij}^{rs} \sigma_{rs} \, dx + \int_{\partial\Omega_2} n_i \sigma_{ij}^{rs} \hat{u}_i \, ds
\]  \hspace{1cm} (1.106)

Euler equations:

\[
\frac{1}{2}(u_{r,s} + u_{s,r}) = C_{iir} \sigma_{ij} \quad \text{in} \quad \Omega; \quad u_i = \hat{u}_i \quad \text{on} \quad \partial\Omega_1
\]  \hspace{1cm} (1.107)

V. A constitutive variational principle

Constraints:

\[
\frac{1}{2}(u_{i,j} + u_{j,i}) = \varepsilon_{ij} \quad \text{in} \quad \Omega; \quad u_i = \hat{u}_i \quad \text{on} \quad \partial\Omega_1
\]

\[-\sigma_{ij}^{rs} = f^i \quad \text{in} \quad \Omega; \quad n_i \sigma_{ij}^{rs} = \hat{T}_i \quad \text{on} \quad \partial\Omega_2
\]  \hspace{1cm} (1.108)

\[
K(\varepsilon) = \int_\Omega \left( \frac{1}{2}E^{iir} \varepsilon_r \varepsilon_r - \varepsilon_{ij} \hat{\sigma}_{ij} \right) \, dx
\]  \hspace{1cm} (1.109)

\[
\hat{\sigma}_{ij} \in \{ \sigma^{ij} : -\sigma^{ij} = f^i \quad \text{in} \quad \Omega; \quad n_i \sigma^{ij} = \hat{T}_i \quad \text{on} \quad \partial\Omega_2 \}
\]

Euler equations:

\[
E^{iir} \varepsilon_r = \sigma^{ij} \quad \text{in} \quad \Omega
\]  \hspace{1cm} (1.110)
VI. A constitutive-potential energy principle

Constraints:
\[ \frac{1}{2}(u_{i,j} + u_{j,i}) = \varepsilon_{ij} \quad \text{in } \Omega; \quad u_i = \hat{u}_i \quad \text{on } \partial \Omega_1 \]  
(1.111)

\[ M(u, \sigma) = \int_{\Omega} \left[ u_{i,j}E^{i\alpha}u_{\alpha,j} + \frac{1}{2}C_{ij\alpha}\sigma^{ij}\sigma^{\alpha} \right] \, dx - \int_{\partial \Omega_1} u_i \hat{T}^i \, ds \]  
(1.112)

Euler equations:
\[ \frac{1}{2}(u_{r,s} + u_{s,r}) = C_{ij\alpha}\sigma^{ij} \quad \text{in } \Omega \]
\[ -\{E^{i\alpha}(u_{r,s} + u_{s,r}) - \sigma^{ij}\} = f^i \quad \text{in } \Omega \]  
(1.113)

\[ n^i\{E^{i\alpha}(u_{r,s} + u_{s,r}) - \sigma^{ij}\} = \hat{T}^i \quad \text{on } \partial \Omega_2 \]

VII. A compatibility-constitutive variational principle

Constraints:
\[ -\sigma_{ij} = f^i \quad \text{in } \Omega; \quad n_i\sigma^{ij} = \hat{T}^i \quad \text{on } \partial \Omega_2 \]  
(1.114)

\[ N(\varepsilon, \sigma) = \int_{\Omega} \left( \frac{1}{2}\varepsilon_{ij}E^{i\alpha}\varepsilon_{\alpha} - \varepsilon_{ij}\sigma^{ij} \right) \, dx - \int_{\partial \Omega_1} n_i\sigma^{ij}\hat{u}_i \, ds \]  
(1.115)

Euler equations:
\[ \varepsilon_{ij} = \frac{1}{2}(i_{ij} + u_{j,i}) \quad \text{in } \Omega; \quad u_i = \hat{u}_i \quad \text{on } \partial \Omega_1 \]
\[ \sigma^{ij} = E^{i\alpha}\varepsilon_{\alpha} \quad \text{in } \Omega \]  
(1.116)

1.8 DUAL PRINCIPLES

1.8.1 The dual problem

It was pointed out by Oden and Reddy\textsuperscript{15} (see also Reference 14) that a parallel collection of so-called dual variational principles can be constructed in one-to-one correspondence with the variational principles described in the previous section. While we shall not elaborate on the detailed features of these dual functionals, we shall outline briefly the essential concepts for the sake of completeness.

To construct the dual principles, we consider a new Hilbert space $\mathcal{S}$ and a linear operator
\[ C : \mathcal{S} \rightarrow \mathcal{G} \]  
(1.117)
such that

\[
\mathcal{R}(C) = \mathcal{N}(A^*) \\
\mathcal{N}(C^*) \subset \mathcal{R}(A)
\]  

(1.118)

Next, we consider the dual problem of finding \( \varphi \in \mathcal{S} \) such that

\[
\begin{aligned}
C^*E^{-1}C\varphi &= \eta \\
\gamma_1^{(C)}\varphi &= p \\
\gamma_2^{(C)*}E^{-1}C\varphi &= q
\end{aligned}
\]  

(1.119)

where

\[
\eta \in \mathcal{S}', \quad p \in \partial \mathcal{S}_1', \quad q \in \partial \mathcal{S}_2'
\]  

(1.120)

Owing to the similarity of (1.119) and (1.58), we can immediately construct the following dual functionals:

I'.

\[
\mathcal{L}(\varphi, \sigma, v) = \frac{1}{2}\langle E^{-1}\sigma, \sigma \rangle_\nu - \langle \psi, C\varphi - \sigma \rangle_\nu - \langle \eta, \varphi \rangle_\nu \\
+ \langle \delta^{(C)}v, \gamma_1^{(C)}\varphi - p \rangle_\nu - \langle q, \delta^{(C)}\varphi \rangle_\nu
\]  

(1.121)

II'.

\[
\mathcal{R}_v(\varphi, v) = -\frac{1}{2}\langle v, Ev \rangle_\nu + \langle v, C\varphi \rangle_\nu - \langle \eta, \varphi \rangle_\nu \\
+ \langle \delta^{(C)}v, \gamma_1^{(C)}\varphi - p \rangle_\nu - \langle q, \delta^{(C)}\varphi \rangle_\nu
\]  

(1.122)

or

\[
\mathcal{R}_v(\varphi, \sigma) = -\frac{1}{2}\langle E^{-1}\sigma, \sigma \rangle_\nu - \langle E^{-1}C\varphi, C\varphi \rangle_\nu - \langle \eta, \varphi \rangle_\nu \\
+ \langle \delta^{(C)}E^{-1}\sigma, \gamma_1^{(C)}\varphi - p \rangle_\nu - \langle q, \delta^{(C)}\varphi \rangle_\nu
\]  

(1.123)

III'.

\[
\mathcal{J}(\varphi) = \frac{1}{2}\langle E^{-1}C\varphi, C\varphi \rangle_\nu - \langle \eta, \varphi \rangle_\nu \\
- \langle v, C\varphi \rangle_\nu - \langle q, \delta^{(C)}\varphi \rangle_\nu
\]  

(1.124)

IV'.

\[
\mathcal{G}(v) = -\frac{1}{2}\langle v, Ev \rangle_\nu - \langle \delta^{(C)}v, v \rangle_\nu
\]  

(1.125)

V'.

\[
\mathcal{K}(\sigma) = \frac{1}{2}\langle E^{-1}\sigma, \sigma \rangle_\nu - \langle v, \sigma \rangle_\nu
\]  

(1.126)

VI'.

\[
\mathcal{M}(v, \varphi) = \frac{1}{2}\langle E^{-1}C\varphi, C\varphi \rangle_\nu + \langle v, Ev \rangle_\nu - \langle \eta, \varphi \rangle_\nu \\
- \langle v, C\varphi \rangle_\nu - \langle q, \delta^{(C)}\varphi \rangle_\nu
\]  

(1.127)

VII'.

\[
\mathcal{N}(\sigma, v) = \frac{1}{2}\langle E^{-1}\sigma, \sigma \rangle_\nu - \langle v, \sigma \rangle_\nu - \langle \delta^{(C)}v, v \rangle_\nu
\]  

(1.128)

Clearly, a table similar to Table 1.1 can also be constructed directly from (1.121)–(1.128) by using the following correspondences:

\[
\begin{aligned}
\mathcal{S}' &\sim \mathcal{S}, & \mathcal{F} &\sim \mathcal{H}, & \varphi &\sim \mu, & C &\sim A \ \\
C^* &\sim A^*, & E^{-1} &\sim E, & \sigma &\sim v, & v &\sim \sigma \\
\gamma_1^{(C)} &\sim \gamma_2^{*}, & \gamma_2^{(C)*} &\sim \gamma_1, & q &\sim g, & p &\sim s
\end{aligned}
\]  

(1.129)
1.8.2 Application to elastostatics

In the case of linear elasticity, we set

\[ \varphi = \varphi_{ij} = \text{tensor of stress functions} \]
\[ \sigma \sim \sigma^{ij} = \text{stress tensor} \]
\[ \nu \sim \epsilon_{ij} = \text{strain tensor} \]
\[ \eta \sim \eta_{ij} = \text{dislocation tensor} \]

Then (1.121)–(1.128) assume the following specific forms:

\[ \mathcal{L}(\varphi_{ij}, \sigma^{ij}, \epsilon_{ij}) = \int_{\Omega} \left[ \frac{1}{2} \sigma^{ij} C_{ijrs} \sigma_{rs} - \epsilon_{ij}(\sigma^{ij} - \epsilon^{imn} \epsilon^{ikr} \varphi_{m,km}) \right. \]
\[ \left. - \eta^{ij} \varphi_{ij} \right] \, dx + \langle \delta^{*} \epsilon_{ij}, \gamma(\varphi_{ij}) - p \rangle_{\partial \Omega_{1}} \]
\[ - \langle q, \delta(\varphi_{ij}) \rangle_{\partial \Omega_{2}} \]  

(1.130)

where, for compactness in notation, we have denoted

\[ \langle \delta^{*} \epsilon_{ij}, \gamma(\varphi_{ij}) - p \rangle_{\partial \Omega_{1}} = \int_{\partial \Omega_{1}} \left[ (F^{imjnpqu} n_{p} \varphi_{iu} - p^{imjnp}_{1}) e_{im,jps} \right. \]
\[ - (F^{imjnpqu} n_{p} \varphi_{iu,m} - p^{imjnp}_{2}) e_{ij,eq} \] \, ds

\[ \langle q, \delta(\varphi_{ij}) \rangle_{\partial \Omega_{2}} = \int_{\partial \Omega_{2}} (q^{ijkl} \varphi_{ijkl} - q^{ijkl}_{2} \varphi_{ij}) \, ds \]

wherein

\[ F^{imjnpqu} = \epsilon^{imr} \epsilon^{ins} C_{rsk} e^{kpe} e^{kqu} \quad i, j, k, l, m, n, p, q, r, s, t, u = 1, 2, 3 \]

\[ F^{imjnpqu} n_{p} \varphi_{iu} = p^{imjnp}_{1} \] on \( \partial \Omega_{1} \)
\[ F^{imjnpqu} n_{p} \varphi_{iu,m} = p^{imjnp}_{2} \] on \( \partial \Omega_{2} \)

\[ \mathcal{R}_{e}(\varphi_{ij}, \epsilon_{ij}) = \int_{\Omega} \left[ -\frac{1}{2} \epsilon_{ij} E^{ijrs} \epsilon_{rs} + \epsilon_{ij} \epsilon^{imn} \epsilon^{ikr} \varphi_{r,km} - \eta^{ij} \varphi_{ij} \right] \, dx \]
\[ + \langle \delta^{*} \epsilon_{ij}, \gamma(\varphi_{ij}) - p \rangle_{\partial \Omega_{1}} - \langle q, \delta(\varphi_{ij}) \rangle_{\partial \Omega_{2}} \]

(1.131)

\[ \mathcal{R}_{\sigma}(\varphi_{ij}, \sigma^{ij}) = \int_{\Omega} \left[ -\frac{1}{2} \sigma^{ij} C_{ijrs} \sigma_{rs} + C_{ijrs} \sigma^{ij} \epsilon^{imn} \epsilon^{ikr} \varphi_{pq,mk} \right. \]
\[ \left. - \eta^{ij} \varphi_{ij} \right] \, dx + \langle \delta^{*} (C_{ijrs} \sigma^{rs}), \gamma(\varphi_{ij}) - p \rangle_{\partial \Omega_{1}} \]
\[ - \langle q, \delta(\varphi_{ij}) \rangle_{\partial \Omega_{2}} \]  

(1.132)
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\[ \mathcal{F}(\varphi_{ij}) = \int_\Omega \left[ \frac{1}{2} C_{ijklm} \varepsilon^{imn} \varepsilon^{rps} \varepsilon^{tuh} \varphi_{mn,pq} y_{ak,th} - \eta^{ij} \varphi_{ij} \right] dx \]
\[-(q, \delta(\varphi_{ij}))_{\partial\Omega}, \tag{1.133} \]

\[ \mathcal{G}(\epsilon_{ij}) = \int_\Omega \left( -\frac{1}{2} \epsilon^{ir} \epsilon_{ij} \epsilon_{rs} \right) dx - (\delta^* \epsilon_{ij}, p)_{\partial\Omega} \tag{1.134} \]

\[ \mathcal{H}(\sigma^{ij}) = \int_\Omega \left( \frac{1}{2} C_{ijklm} \sigma^{ir} \sigma_{ij} - \sigma^{ij} \sigma_{ij} \right) dx \tag{1.135} \]

\[ \mathcal{M}(\varphi_{ij}, \epsilon_{ij}) = \int_\Omega \left[ \frac{1}{2} C_{ijklm} \varepsilon^{imn} \varepsilon^{rps} \varepsilon^{tuh} \varphi_{mn,pq} y_{ak,th} \right. \]
\[ + E^{ir} \epsilon_{ij} \epsilon_{rs} - 2 \epsilon_{ij} \epsilon^{ir} \epsilon^{mk} \varphi_{sk,rm} - \eta^{ij} \varphi_{ij} \left. \right] dx \]
\[-(q, \delta(\varphi_{ij}))_{\partial\Omega}, \tag{1.136} \]

\[ \mathcal{N}(\sigma^{ij}, \epsilon_{ij}) = \int_\Omega -\sigma^{ij} \epsilon_{ij} dx + \int_{\partial\Omega} \epsilon_{ij} \varphi^{ij} ds - (\delta^* \epsilon_{ij}, p)_{\partial\Omega} \tag{1.137} \]

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