RECENT DEVELOPMENTS IN THE THEORY OF
FINITE ELEMENT APPROXIMATIONS OF BOUNDARY VALUE
PROBLEMS IN NONLINEAR ELASTICITY

J.T. ODEN

The University of Texas at Austin, Austin, Texas, U.S.A.

1. Introduction

In this paper we address a number of theoretical questions, mostly mathematical in nature, that arise in the study of finite element approximations of boundary value problems in nonlinear elasticity. In particular, we consider:

1. sufficient conditions for the existence and weak convergence of finite element approximations of a class of elastostatics problems involving compressible hyperelastic materials.
2. additional conditions which guarantee that finite element approximations of the problems in item 1 above converge strongly (in some appropriate energy norm).
3. conditions for the existence of solutions and their finite element approximations for problems involving incompressible hyperelastic bodies.
4. the construction of a priori error estimates for cases in which the solutions are sufficiently regular.
5. the qualitative behavior of "small" solutions and their approximations in the neighborhood of global solutions.
6. conditions for the boundedness and convergence of approximations of hydrostatic pressures in the analysis of incompressible problems.

After some preliminaries and the introduction of various notations and conventions we outline briefly the formulation of finite element approximations of two classes of boundary value problems in elastostatics: (i) those involving compressible hyperelastic materials and (ii) those involving incompressible hyperelastic materials. In each case we assume that the material is isotropic and homogeneous in its reference configuration and that the strain energy function is such that the variational problems have meaning in an appropriate Sobolev space $W^{1,p}(\Omega) = (W^{1,p}(\Omega))^n$, $n = 1, 2, 3$.

In our study of item 1 we review the role of existence theory as a guide to the study of Galerkin approximations - first for monotone operators and then for a class of pseudomonotone operators satisfying a generalized Gårding inequality. We establish conditions for strongly continuous operators, satisfying Gårding-type inequalities, for which solutions exist and their finite element approximations exist, and we give sufficient conditions for weak and strong convergence of such approximations (item 2). For solutions sufficiently smooth to have second derivatives bounded in $W^{2,p}(\Omega)$ these results also apply to both classes of elastostatics problems considered here.

In both the compressible and incompressible problems an analysis of errors must be preceded by a local (linear) analysis due to the indefiniteness of forms (such as those in the generalized Gårding-
type inequality). Thus it is natural to investigate the behavior of local solutions (item 5) if global error estimates are desired. Sections 5, 6 and 7 contain an analysis of the compressible problem. Some results on existence theory for such problems are summarized in section 5, and details of a local analysis, including certain local error estimates, are recorded in section 6. A priori error estimates for finite element approximations of a class of compressible elastostatics problems are shown in section 7.

Incompressible elastostatics problems are taken up in sections 8 and 9. Some results on existence of solutions of this type are summarized in section 8, and preliminary results on a priori error estimates of finite element approximations are discussed in section 9. In the incompressible case some difficulties are encountered due to problems with the hydrostatic pressure. Existence of solutions for both the given problem and its approximations rest on the existence of a constant $C_2 > 0$ such that a generalization of the Babuska-Brezzi condition holds:

$$\sup_{V \in \mathcal{V}} \frac{\sum_{\Omega} \rho \text{adj} \nabla u : \nabla v \, dx}{\|v\|_{\mathcal{V}}} \geq C_2 \|p\|_{\mathcal{P}}.$$  \hfill (1.1)

where $\mathcal{V}$ is the space of admissible motions $u$, $\mathcal{P}$ is the space of the hydrostatic pressure $p$, $\text{adj}$ denotes the matrix of cofactors (see eq. (2.9)), and $\beta \geq 1$. When (1.1) holds, error estimates can be obtained, provided that the assumptions needed for a corresponding local analysis hold.

2. Notations and preliminaries

We consider two classes of nonlinear stationary boundary value problems:

I. Find the motion $u$ of a compressible elastic body $\Omega$ subjected to body forces $f$ and surface tractions $S$ such that the following principle of virtual work is satisfied:

$$\langle A(u), v \rangle = \langle F, v \rangle \quad \forall v \in K_0. \hfill (2.1)$$

II. Find a pair $(u, p)$, consisting of a motion $u$ and a hydrostatic pressure $p$ in an incompressible elastic body $\Omega$ such that the following principle of virtual work is satisfied:

$$\langle A(u), v \rangle + \langle C(u)p, v \rangle = \langle F, v \rangle \quad \forall v \in \mathcal{V}.$$  \hfill (2.2)

$$\langle J(u), q \rangle = \langle g, q \rangle \quad \forall q \in \mathcal{P}. \hfill (2.2)$$

In I and II we use the following conventions and notations:

$$\langle A(u), v \rangle = \int_{\Omega} \rho \frac{\partial \sigma(u)}{\partial \nabla u} : \nabla v \, dx. \hfill (2.3)$$

where
\( \rho \) = mass density in the reference configuration of \( \mathcal{B} \),
\( \Omega \) = domain of the particle positions \( x \in \Omega \subset \mathbb{R}^n \), \( n = 1, 2, 3 \),
\( \nabla u = (\partial u_i/\partial x_j) \), \( i \) \& \( j \) is the material gradient of \( u \).
\( \sigma \) = strain energy function, which is assumed to be objective and to be given as a function of the principal invariants \( I, II \) and \( J \) of the left Cauchy-Green tensor \( G = FF' \). \( F = \nabla u \):

\[
\begin{align*}
I & = \text{tr } G \quad (\text{tr } G = \text{trace of } G) \\
II & = \frac{1}{2} (\text{tr } G)^2 - \frac{1}{2} \text{tr}(G^2), \\
J & = \det F \quad (J = (\det G)^{1/2}).
\end{align*}
\]

(2.4)

\( \mathcal{U} \) = set of admissible motions in the sense that \( u \in \mathcal{U} \) implies that \( u(x) = x, x \in \partial \Omega \), where \( \partial \Omega \) is the boundary of \( \Omega \) \( (\partial \Omega = \partial \Omega_1 \cup \partial \Omega_2, \partial \Omega_1 \cap \partial \Omega_2 = \emptyset) \), and the energy \( \int_0^1 (A(u), u) \) is finite.

The manifold \( \mathcal{U} \) is a reflexive Banach space equipped with a norm \( \| \cdot \|_{\mathcal{U}} \) translated by a vector \( q, q(x) = x, x \in \partial \Omega \) in such a way as to satisfy boundary conditions, prescribing homogeneous displacements on \( \partial \Omega \). Its topological dual is denoted \( \mathcal{U}' \), and \( (\cdot, \cdot) \) denotes duality pairing on \( \mathcal{U}' \times \mathcal{U} : i.e. (\cdot, \cdot) : \mathcal{U}' \times \mathcal{U} \rightarrow \mathbb{R} \) is the bilinear canonical pairing of linear functionals \( F \) in \( \mathcal{U}' \) with element \( u \in \mathcal{U} \).

Likewise, \( \mathcal{V} \) is a Banach space of Lagrange multipliers in which the hydrostatic pressures \( p \) are defined. The norm on \( \mathcal{V} \) is denoted \( \| \cdot \|_{\mathcal{V}} \), and duality pairing on \( \mathcal{V}' \times \mathcal{V} \) is denoted \((\cdot, \cdot)\).

Continuing,

\[
\langle F, v \rangle = \int_\Omega \rho f \cdot v \, dx + \int_{\partial \Omega_2} S \cdot v \, ds,
\]

(2.5)

\[
\langle C(u)p, v \rangle = \int_\Omega p \, \text{adj } \nabla u^t \cdot \nabla v \, dx,
\]

(2.6)

\[
(J(u), q) = \int_\Omega J(u)q \, dx,
\]

(2.7)

\[
(g, q) = \int_\Omega q \, dx,
\]

(2.8)

where \( \text{adj } \nabla u = \text{transpose of the matrix of cofactors of } \nabla u \), or, in cartesian components.

\[
\begin{bmatrix}
  u_{2,1}u_{3,2} - u_{2,3}u_{3,1} & u_{1,3}u_{3,2} - u_{1,2}u_{3,3} & u_{1,2}u_{2,3} - u_{1,3}u_{2,2} \\
  u_{2,1}u_{3,1} - u_{2,2}u_{3,3} & u_{1,1}u_{3,3} - u_{1,3}u_{3,1} & u_{1,3}u_{2,1} - u_{1,1}u_{2,3} \\
  u_{2,2}u_{3,1} - u_{2,3}u_{3,2} & u_{1,2}u_{3,3} - u_{1,1}u_{3,2} & u_{1,1}u_{2,2} - u_{1,2}u_{2,1}
\end{bmatrix}
\]

(2.9)
In the case of compressible materials (problem I) all motions are subject to the constraint

\[ J(u) > 0. \]  \hspace{1cm} (2.10)

Thus \( K_0 \) in (2.1) is the set

\[ K_0 = \{ v \in \mathcal{U} : J(v) > 0 \text{ a.e. in } \Omega \}. \]  \hspace{1cm} (2.11)

We frequently have reason to consider the related constraint set

\[ K_\nu = \{ v \in \mathcal{U} : J(v) \geq \nu > 0 \text{ a.e. in } \Omega \}. \]  \hspace{1cm} (2.12)

Then \( K_0 \subset K_\nu \). In the case of incompressible materials we have the incompressibility constraint

\[ J(v) = 1, \]  \hspace{1cm} (2.13)

which is accommodated for in formulation (2.2) by the introduction of a Lagrange multiplier \( \mu \).

We remark that both problems I and II are, in general, equivalent to problems of finding stationary values of certain energy functionals. In the case of problem (2.1) any minimizer of the potential energy

\[ II(v) = \int_{\Omega} \rho \sigma(v) dx - \langle F, v \rangle, \quad v \in K_0, \]  \hspace{1cm} (2.14)

satisfies (2.1), whereas any saddle point \((u, \mu) \in \mathcal{U} \times \mathcal{V}\) of the Lagrange functional

\[ L(v, \mu) = \int_{\Omega} \rho \sigma(v) dx - \langle F, v \rangle + \int_{\Omega} \mu (J(v) - 1) dx \]  \hspace{1cm} (2.15)

is a solution of (2.2).

We confine our attention to materials for which the form of the strain energy function is such that the space \( \mathcal{U} \) of the admissible motions can be identified with the Sobolev spaces \( W^{m,p}(\Omega) = (W^{m,p}(\Omega))^n \), \( 1 < p < \infty \), \( n = 1, 2 \) or 3, which are equipped with the norms

\[ \| u \|_{m,p} = \left( \sum_{i=1}^{n} \| u_i \|_{m,p}^p \right)^{1/p}, \quad n = 1, 2 \text{ or } 3, \quad u = \{u_i\}_{1 \leq i \leq n} \]  \hspace{1cm} (2.16)

\[ \| u_i \|_{m,p}^p = \int_{\Omega} \sum_{|\alpha| \leq m} |D^\alpha u_i|^p dx \]  \hspace{1cm} (2.17)
\[ \|u_i\|_{m,\infty} = \text{ess sup}_{x \in \Omega} \sum_{|\alpha| \leq m} |D^\alpha u_i(x)|. \]  
\hspace{1cm} (2.16)

Clearly,
\[ W^{0,p}(\Omega) = L^p(\Omega) = (L^p(\Omega))'. \]

Since \( \sigma \) involves functions of \( \nabla u \), we generally associate \( \mathcal{U} \) with \( W^{1,p}(\Omega) \). We also make use of the seminorms.

\[ |u|^p_{m,p} = \sum_{i=1}^n \int_\Omega \sum_{|\alpha| = m} |D^\alpha u_i|^p \, dx \]
and use the notation
\[ |u|^p_{1,p} = \int_\Omega |\nabla u|^p \, dx, \quad |\nabla u|^p = \sum_{i,j=1}^n \left| \frac{\partial u_i}{\partial x_j} \right|^p. \]  
\hspace{1cm} (2.18)

We also make reference to the spaces
\[ W_0^{m,p}(\Omega) = C_0^\infty(\Omega) = \{ v \in W^{m,p}(\Omega) : D^\alpha u_i = 0 \text{ on } \partial \Omega, \, |\alpha| \leq m-1 \}. \]  
\hspace{1cm} (2.19)
\[ W_\ast^{m,p}(\Omega) = \{ v \in W^{m,p}(\Omega) : D^\alpha u_i = 0 \text{ on } \partial \Omega_1, \, |\alpha| \leq m-1 \}. \]  
\hspace{1cm} (2.20)
and their duals \((p' = p/(p - 1))\)
\[ W^{-m,p'}(\Omega) = (W_0^{-m,p}(\Omega))', \]  
\hspace{1cm} (2.21)
\[ W^{-m,p'}(\Omega) = (W_\ast^{-m,p}(\Omega))'. \]

Throughout our analysis we assume that \( \Omega \) is a smooth, simply connected, open, bounded domain in \( \mathbb{R}^n \) with a smooth (e.g. Lipschitzian) boundary \( \partial \Omega \).

3. Finite element approximations

Finite element approximations of problems I and II are constructed in the usual way: We introduce a partition \( Q_h \) of \( \Omega \) into a collection of \( C^0 \)-elements \( G \) whose maximum diameter is \( h \) and whose local approximations are given containing complete polynomials of degree \( \leq k \) for the representation of \( u = \{ u_1, \ldots, u_n \} \) and of degree \( \leq t \) for representations of \( p \). We consider regular refinements of the finite element mesh, and thereby produce families of subspaces \( \{ \mathcal{U}_h \}_{0 < h < 1} \), \( \{ \mathcal{V}_h \}_{0 < h < 1} \) of the spaces \( \mathcal{U} \) and \( \mathcal{V} \), respectively, where
\[ \mathcal{U}_h = \{ \psi_h \in \mathcal{U} : (\psi_h)_G \in \mathcal{P}_k(G); 1 \leq i \leq n, G \in \mathcal{Q}_h \}, \]

\[ \mathcal{V}_h = \{ q_h \in \mathcal{V} : q_h|_G \in \mathcal{P}_t(G); G \in \mathcal{Q}_h \}. \]

(3.1)

and \( \mathcal{P}_k(G) \) and \( \mathcal{P}_t(G) \) are the spaces of polynomials of degree \( \leq k \) and \( \leq t \) over element \( G \), respectively.

It is known (see for example Ciarlet [2]) that spaces of this type have the following interpolation properties: If \( \omega \in \mathcal{W}^{m,p}(\Omega) \) and \( r \in \mathcal{W}^{h,p}(\Omega) \), the constants \( C_0 = C_0(\Omega, m, k, p) \) and \( C_1 = C_1(\Omega, s, t, p) \) and elements \( \tilde{\omega}_h \in \mathcal{U}_h \) and \( \tilde{r}_h \in \mathcal{V}_h \) exist such that for \( h \) sufficiently small

\[ \| \omega - \tilde{\omega}_h \|_{s,p} \leq C_0 h^\alpha \| \omega \|_{m,p}, \]

(3.2a)

\[ \| r - \tilde{r}_h \|_{l,p} \leq C_1 h^\gamma \| r \|_{l,p}, \]

(3.2b)

where

\[ \alpha = \min(k + 1 - s, m - s), \quad 0 \leq s \leq m - 1, \]

(3.3a)

\[ \gamma = \min(t + 1 - j, l - j). \quad 0 \leq j \leq l - 1. \]

(3.3b)

The finite element approximation of (2.1) consists of seeking \( u_h \in K_0^h \) such that

\[ \langle A(u_h), \psi_h \rangle = \langle F, \psi_h \rangle \quad \forall \psi_h \in K_0^h, \]

(3.4)

where, for example, \( K_0^h = K_0 \cap \mathcal{U}_h \), and the finite element approximation of (2.2) consists of seeking \( (u_h, p_h) \in \mathcal{U}_h \times \mathcal{V}_h \) such that

\[ \langle A(u_h), \psi_h \rangle + \langle C(u_h)p_h, \psi_h \rangle = \langle F, \psi_h \rangle \quad \forall \psi_h \in \mathcal{U}_h, \]

(3.5)

\[ \langle J(u_h), q_h \rangle = \langle g, q_h \rangle \quad \forall q_h \in \mathcal{V}_h. \]

We are interested here in the qualitative analysis of problems (3.4) and (3.5): (i) Under what conditions are we guaranteed that solutions exist? (ii) Are they unique? (iii) Do the finite element approximations converge to solutions of I or II? (iv) If so, in what sense is convergence obtained? (v) Can the error be estimated? (vi) If so, what is the rate of convergence? We address these questions in subsequent sections.

4. Existence theory as a guide to approximation

With rare exceptions, a constructive proof of the existence of solutions to variational boundary value problems provides most of the tools needed to construct an approximation theory and a priori error estimates. For example, suppose that we wish to determine if solutions exist to the abstract problem
\[ A(u) = f \quad \text{or} \quad \langle A(u), v \rangle = \langle f, v \rangle \quad \forall v \in \mathcal{U}, \]  

(4.1)

where \( \mathcal{U} \) is a reflexive Banach space and \( A : \mathcal{U} \to \mathcal{U}' \) is an operator, generally nonlinear, satisfying the following conditions:

\begin{enumerate}
\item[(4.i)] Boundedness:
\[ \|u\|_{\mathcal{U}} \leq M_1 \Rightarrow \|A(u)\|_{\mathcal{U}'} \leq M_2. \quad M_1 \text{ and } M_2 \text{ are constants}, \]

\item[(4.ii)] Coerciveness:
\[ \frac{\langle A(v), v \rangle}{\|v\|_{\mathcal{U}}} \to +\infty \text{ as } \|v\|_{\mathcal{U}} \to \infty, \]

\item[(4.iii)] Monotonicity:
\[ \langle A(u) - A(v), u - v \rangle \geq 0 \quad \forall u, v \in \mathcal{U}. \]

\item[(4.iv)] Hemicontinuity:
\[ \phi(t) = \langle A(u + tw), w \rangle, \quad t \in [0, 1], \text{ is a continuous function of } t. \]
\[ \phi(t) \to \langle A(u), w \rangle \text{ as } t \to 0. \]
\end{enumerate}

A standard way of proving the existence of solutions to (4.1) is to use Galerkin methods and so-called compactness arguments which make use of the fact that in reflexive Banach spaces every bounded sequence has a weakly convergent subsequence. The steps in the proof of existence when conditions (4.i)–(4.iv) hold are as follows:

1. Construct a family \( \{ \mathcal{U}_h \} \) of finite-dimensional subspaces of \( \mathcal{U} \). with \( \bigcup_h \mathcal{U}_h \) everywhere dense in \( \mathcal{U} \). and construct a Galerkin approximation of (4.1) as follows: Find \( u_h \in \mathcal{U}_h \) such that
\[ \langle A(u_h), v_h \rangle = \langle f, v_h \rangle \quad \forall v_h \in \mathcal{U}_h. \]

(4.2)

Conditions (4.i)–(4.iv) are sufficient to guarantee the existence of a solution \( u_h \) to the approximate problem (4.2).

2. Let \( h \to 0 \). Since
\[ \langle A(u_h), u_h \rangle = \langle f, u_h \rangle \leq \|f\|_{\mathcal{U}'} \|u_h\|_{\mathcal{U}'}, \]

the coerciveness condition (4.ii) guarantees that the sequence of Galerkin approximations \( \{u_h\} \) is bounded in \( \mathcal{U} \). Hence, there must exist a subsequence, also denoted \( \{u_h\} \) for simplicity, that converges weakly to an element \( u \in \mathcal{U} \).
3. Since $A$ is bounded (condition (4.1)), there exists an $X \in \mathcal{U}'$ such that $A(u_h) \to X$ weakly in $\mathcal{U}'$.

4. Monotonicity and hemicontinuity are used in this last step to show that $X = A(u)$ and, therefore, to complete the proof of existence of solutions to (4.1). Indeed, if $v$ is arbitrary and $u$ is the weak limit of $u_h$, then

$$0 \leq \lim_{h \to 0} \langle A(u_h) - A(u), u_h - u \rangle = \langle X - A(u), u - v \rangle.$$ 

Set $v = u + tw$ and divide by $t$ to get $\langle X - A(u + tw), w \rangle \geq 0$ for every $w \in \mathcal{U}$. Then, by the hemi-continuity of $A$ (condition (4.iv)) we can take the limit as $t \to 0$ to get

$$\langle A(u) - X, w \rangle \geq 0 \quad \forall w \in \mathcal{U},$$

which can only hold if $A(u) = X$.

Steps 1–3 are quite general and need not depend upon the monotonicity of $A$. If $A$ is strictly monotone, the solution to (4.1) for given $f \in \mathcal{U}'$ is unique. For example, if $A: W^{1,p}_0(\Omega) \to W^{-1,p}(\Omega)$ is given formally by

$$A(u) = -\nabla \cdot (|\nabla u|^{p-2} \nabla u), \quad 1 \leq p < \infty,$$

then it can be shown that

$$\langle A(u) - A(v), u - v \rangle \geq C\|u - v\|_{1,p}^p, \quad C > 0.$$  

(4.4)

i.e. $A$ is strongly monotone and there exists a unique solution to $A(u) = f$, $u|_{\partial \Omega} = 0$. Also, it can easily be shown that $A$ of (4.3) is continuous and bounded; coerciveness follows from (4.4).

These properties are preserved on finite-dimensional subspaces $\mathcal{U}_h$, and so the Galerkin approximations (4.2) are also uniquely solvable for any $h > 0$.

Error estimates are also straightforward. Use of Hölder’s inequality reveals that $\forall u, v, w \in \mathcal{U} = W^{1,p}_0(\Omega), p \geq 2$:

$$\langle A(u) - A(v), w \rangle \leq g(u, v)\|u - v\|_{1,p} \|w\|_{1,p},$$

(4.5)

and

$$C\|e_h\|_{1,p}^p = \langle A(u) - A(u_h), e_h \rangle \quad (\text{from (4.4)}).$$

(4.6a)

$$= \langle A(u) - A(u_h), u - \tilde{u}_h \rangle \quad (\text{from orthogonality}),$$

(4.6b)

$$\leq g(u, u_h)\|e_h\|_{1,p} \|u - \tilde{u}_h\|_{1,p} \quad (\text{from (4.5)}).$$

(4.6c)

$$\leq g(u, u_h)\|e_h\|_{1,p} C_0 h^a \|u\|_{m,p} \quad (\text{from (3.2a)}).$$

(4.6d)
i.e.,

$$|e_h|_{1,p} \leq g(u)h^{\alpha(p-1)}||u||_{m,p} \quad \text{as } h \to 0.$$  

(4.7)

where \( g(u) \geq (C_0/C)g(u, u_h) \) is a constant depending on \( u \). In (4.6b) "orthogonality" refers to the fact that upon setting \( v = v_h \) in (4.1) and subtracting (4.2) we get

$$\langle A(u) - A(u_h), v_h \rangle = 0 \quad \forall v_h \in \mathcal{U}_h.$$  

(4.8)

The quantity \( \tilde{u}_h \) in (4.6) is the interpolant of \( u \) in the interpolation estimate (3.2a).

**Nonmonotone operators.** Unfortunately, the rather elegant analysis described above is not applicable to either of the general finite elasticity problems I or II described in section 2 for the following reasons:

1. In general, unique solutions to problems I and II do not exist. Else the governing equations would not be sufficiently general to encompass the theory of elastic stability.
2. The operators of finite elasticity are not monotone.
3. The boundary value problems I and II are posed on sets \( K_0 \) and \( K_1 \) due to the constraints \( J > 0 \) or \( J = 1 \) and cannot be correctly described by operators on linear spaces.

These factors represent formidable difficulties in developing a reasonable theory for finite element approximations in general nonlinear elasticity. To attempt to overcome these obstacles, let us examine some possible generalizations of the methods outlined above.

First, it is natural to attempt to extend the notion of monotone operators since the Galerkin method makes use of monotonicity in only the final step. One extension, which has been proposed recently in [7], is to replace condition (4.iii) by

(4.iii)' Generalized Garding property: For \( u, v \) in a ball \( B_\mu \) of radius \( \mu \) in \( \mathcal{U} \)

$$\langle A(u) - A(v), u - v \rangle \geq F(||u - v||_{\mathcal{U}}) - H(\mu, ||u - v||_{\mathcal{U}}),$$  

(4.9)

where \( F(\cdot) \) and \( H(\cdot, \cdot) \) are continuous, nonnegative-valued functions of their arguments, \( F(0) = 0 \), and \( H \) is homogeneous of degree \( > 1 \) in its second argument, i.e.

$$F(x) \geq 0, \quad F(x) = 0 \Rightarrow x = 0, \quad \lim_{b \to 0^+} \frac{1}{b} H(x, b y) = 0.$$

In (4.9), \( F \) may be identically zero. Also, \( \mathcal{V} \) is a reflexive Banach space in which the "solution" space \( \mathcal{U} \) is compactly embedded; e.g. \( \mathcal{U} \subseteq \mathcal{C}_0 \mathcal{V} \) and if \( u_m \to u \) weakly in \( \mathcal{U} \), then \( u_m \to u \) strongly in \( \mathcal{V} \).

It is shown in [7] that conditions (4.i), (4.ii), (4.iii)' and (4.iv) are sufficient to guarantee the existence of solutions to (4.1) and its Galerkin approximation (4.2). However, multiple solutions to each may exist since the right-hand side of (4.9) can be negative. Inequality (4.9) represents an abstraction of "generalized Garding" inequalities, for example

$$\langle A(u) - A(v), u - v \rangle \geq C_1||u - v||_{1,p} - C_2(\mu)||u - v||_{0,p}', \quad 2 \leq p < \infty, \quad p' = p/(p - 1),$$  

(4.10)

where \( C_1 > 0, C_2(\mu) > 0 \), and \( ||u||_{1,p}, ||v||_{1,p} < \mu \).
On the surface, it would seem that this generalization of (4.iii) overcomes difficulties 1 and 2 mentioned above. Another generalization which extends these methods is that when (4.i), (4.ii), (4.iii') and (4.iv) hold for \( A : K \rightarrow \mathcal{U}' \), where \( K \) is any nonempty closed convex set in \( \mathcal{U} \); then there exist \( u \in \mathcal{U} \) such that

\[
\langle A(u) - f, v - u \rangle \geq 0 \quad \forall v \in K,
\]  

(4.11)

and (4.11) reduces to an equality whenever \( u \in \text{interior} (K) \). Indeed, the following general approximation results can be shown to hold (for a complete proof, see Oden and Reddy [10]):

**THEOREM 4.1.** Let \( \mathcal{U} \) and \( \mathcal{V} \) be reflexive Banach spaces, \( \mathcal{U} \) being compactly embedded in \( \mathcal{V} \), and let \( K \) be a nonempty closed convex subset of \( \mathcal{U} \). Let \( A : K \rightarrow \mathcal{U}' \) satisfy conditions (4.i), (4.ii), (4.iii') and (4.iv) on \( K \). Then there exists at least one solution \( u \in K \) to (4.11) for any \( f \in \mathcal{U}' \). Moreover, if \( u \in \text{interior}(K) \), then

\[
\langle A(u), v \rangle = \langle f, v \rangle \quad \forall v \in K.
\]  

(4.12)

Continuing, let \( \{ \mathcal{U}_h \}_{0 < h < 1} \) be a family of subspaces such that \( \cup_h \mathcal{U}_h \) is everywhere dense in \( \mathcal{U} \), \( \{ K_h \}_{0 < h < 1} \) is a family of nonempty, closed convex subsets of \( K \) such that \( K_h \subset K \cap \mathcal{U}_h \forall h \), and \( A \) satisfies the above four conditions. Then there exists at least one solution \( u_h \) to the Galerkin approximation of (4.11):

\[
\langle A(u_h) - f, v_h - u_h \rangle \geq 0 \quad \forall v_h \in K_h.
\]  

(4.13)

If \( u_h \in \text{interior}(K_h) \), then

\[
\langle A(u_h), v_h \rangle = \langle f, v_h \rangle \quad \forall v_h \in K_h.
\]  

(4.14)

Finally, let \( \{ u_h \} \) be a sequence of solutions of the Galerkin approximations (4.13). Then

(i) If \( F = 0 \) in (4.9), there exists a subsequence \( \{ u_{h'} \} \) which converges weakly to a solution \( u \) of (4.11).

(ii) If \( F \neq 0 \) in (4.9), there exists a subsequence \( \{ u_{h''} \} \) which converges strongly to a solution \( u \) of (4.12).

These results do represent substantial improvements over the monotone theory outlined earlier. However, there are still some shortcomings:

S.1. Gårding inequalities of the type (4.9) and (4.11) hold for the operators of finite elasticity when motions \( u \) and \( v \) have bounded second partial derivatives in (for example) \( L^P(\Omega) \). This is not a serious shortcoming from the point of view of approximation theory, but it invalidates the existence theory just described for it requires that we establish the regularity of solutions whose existence has not been proved.

S.2. The sets \( K_0 \) and \( K_1 \) (recall (2.11) and (2.12)) encountered in nonlinear elasticity problems are not convex.

S.3. Owing to the presence of the negative term in (4.9) (or (4.10)), the error analysis outlined in (4.6) breaks down. This is not surprising. Since neither the solutions \( u \) nor their approximations
5. Results for problem \( I \) — compressible materials

The shortcomings S.1 and S.2 in the previous section have been recently overcome in a paper by Oden and Kikuchi [8] using minimization arguments. A summary of these results follows:

1. The basic idea is to prove the existence of minimizers \( u \) of the potential energy \( \Pi \) of (2.14) and to then show that these minimizers are characterized as solutions of the equilibrium equations (2.1). In other words, we seek \( u \in K_0 \) such that

\[
\inf_{v \in K_0} \Pi(v) = \Pi(u). \tag{5.1}
\]

2. Sufficient conditions for (5.1) to be solvable are that \( K_0 \) be weakly sequentially closed (i.e. \( u_m \in K_0 \) and \( u_m \rightharpoonup u \) weakly imply \( u \in K_0 \)) and that \( \Pi: K \to \mathbb{R} \) be proper and weakly lower semicontinuous (i.e. \( \Pi \not\equiv +\infty \) and \( u_m \rightharpoonup u \) weakly imply \( \lim_{m} \inf \Pi(u_m) \geq \Pi(u) \)) and coercive (\( \Pi(u) \to +\infty \) as \( \|u\|_V \to \infty \)). The set \( K_0 \) is not weakly sequentially closed; so we attempt to first solve (5.1) on the sets \( K_{\nu} \) of (2.12) which are.

3. Suppose \( \Pi \) is given by (2.14) and the strain energy is of the form

\[
\sigma = \sigma_1(I, II) + \sigma_2(J), \tag{5.2}
\]

where \( I, II, J \) are the invariants in (2.4), \( \sigma_1 \) is a differentiable polynomial in \( I \) and \( II \) convex in \( u \), and \( \sigma_2 \) is a differentiable convex function of \( J \) such that \( \sigma_2 \to +\infty \) as \( J \to 0 \). The form of \( \sigma_1 \) implies that the appropriate space of admissible functions is \( W^{1,p}_{\nu}(\Omega) \) for some \( p \). \( 2 \leq p \leq \infty \). The coefficients in \( \sigma_1 \) can be selected so that \( \sigma_1 \) is convex on all of \( W^{1,p}_{\nu}(\Omega) \).

4. It can be shown (e.g. [8]) that the function \( J = \det \nabla u : (W^{1,p}_{\nu}(\Omega))^{n} \to L^{p/n}(\Omega) \) is weakly sequentially continuous, \( p/n > 1 \). This fact and the form of \( \sigma \) assumed in step 3 can be used to verify that \( \Pi \) is weakly lower semicontinuous on \( K_{\nu}, \nu > 0 \).

5. The elastic constants can be selected so that \( \Pi \) is coercive on \( K_{\nu} \), and this property is independent of \( \nu \). Hence, by 1, there exists a minimizer of \( \Pi \) on \( K_{\nu} \) for each \( \nu > 0 \).

6. Let \( \{u_{\nu}\} \) be a sequence of minimizers, one representing a minimizer of \( \Pi \) on \( K_{\nu} \) for each \( \nu > 0 \). Coerciveness of \( \Pi \) shows that this sequence is uniformly bounded in \( \nu \). Hence, a subsequence \( \{u_{\nu'}\} \) exists which converges weakly to an element \( u \) in \( W^{1,p}_{\nu}(\Omega) \). By the weak lower semicontinuity of \( \Pi \), it follows that \( \lim_{\nu'} \inf \Pi(u_{\nu'}) \geq \Pi(u) \). Moreover, \( \Pi(u) \) is bounded. Hence, by the property \( \sigma_2 \to +\infty \) as \( J \to 0 \), \( u \) is a minimizer of \( \Pi \) in \( K_0 \).

7. \( K_0 \) of (2.11) is open and nonempty, and \( \Pi \) can be differentiated in the sense of Gâteaux at a
minimizer $u$:

$$
\langle D\Pi(u), v \rangle = \lim_{t \to 0^+} \frac{\partial}{\partial t} \Pi(u + tv) = 0,
$$

where $\langle \cdot, \cdot \rangle$ denotes duality pairing on $(W^{1,p}_*(\Omega))' \times W^{1,p}_*(\Omega)$. Equation (5.3) is precisely the weak form of the equilibrium equations (2.1). Thus the existence of solutions to (2.1) under the above assumptions is guaranteed.

In addition to the conditions outlined above, if we assume that the data and domain $\Omega$ are such that motions $u$ and $v$ are bounded in $W^{2,p}_*(\Omega)$, then one can impose conditions on the elastic constants such that a generalized Gårding inequality of the following type holds:

$$
\langle A(u) - A(v), u - v \rangle \geq C|u|_p - |v|_p - \gamma|u - v|_p.
$$

Here $A$ is defined in (2.3), $C$ is a positive constant, and $\gamma$ is a positive constant depending upon the data ($\rho_0$ and $\sigma$), the elastic constants, and the bound $\mu$ of $u$ and $v$ in $W^{2,p}_*(\Omega)$. Also,

$$
\langle A(u) - A(v), w \rangle \leq g(u, v)|u - v|_p|w|_p.
$$

for any $u, v, w \in K_0 \cap W^{1,p}_*(\Omega)$, where $g(u, v)$ is a continuous function of $|u|_p$ and $|v|_p$.

Thus, difficulties S.1 and S.2 listed in the previous section can be resolved for an important class of boundary value problems in finite elastostatics. Problem S.3 requires some special considerations, and we examine these in section 7.

We remark that there are elastostatics problems which cannot be resolved using the method outlined in this section. For example, if the strain energy function contains terms such as $I/J$, a proof that $\Pi$ is weakly lower semicontinuous is not known. Knowles and Sternberg (e.g. [3, 4]) have shown that in materials of this type the equations of elasticity may become hyperbolic, and "elastostatic" shocks (surfaces of discontinuity in first derivatives of $u$) can be developed. No existence theory or approximation theory or even any numerical experiments for problems of this type are known.

6. Local analysis

The key to the resolution of difficulty S.3 of section 4 – error analysis in the presence of multiple solutions – lies in the study of "solutions in the small", i.e. linear analysis of perturbations in solutions about global solutions of the boundary value problem. Once again, these studies also provide guidelines to both approximation theories and computational methods; importantly, they also provide a necessary tool for studying bifurcations and the multiplicity of solutions.

We again list the principal ideas:

1. Suppose $\mathcal{U} = W^{1,p}_0(\Omega), p \geq 2$, and that we wish to solve a nonlinear variational boundary value problem of the form
\[ \langle A(u), v \rangle = \langle F, v \rangle \quad \forall v \in \mathcal{U}. \]  
\hspace{1cm} (6.1)

with \( \langle \cdot, \cdot \rangle : \mathcal{U}' \times \mathcal{U} \to \mathbb{R} \) and \( F \) given in \( \mathcal{U}' \). We assume that \( F \) is given as a \( C^1 \)-function of a real parameter \( s \) and that \( A \) is Gâteaux-differentiable in some ball \( B_{\rho_0}(u) \) about a solution \( u \in \mathcal{U} \). Let \( (u(s_0), F(s_0)) \) be a solution for \( F \) evaluated at \( s = s_0 \), and let \( (u_0(s_0 + s), F(s_0 + s)) \) be another solution for \( F(s_0 + s) \). Then the increments \( dw/ds \) satisfy the linear boundary value problem

\[ \langle DA(u(s_0)) \cdot \dot{w}, v \rangle = \langle F'(s_0), v \rangle \quad \forall v \in \mathcal{U}, \]  
\hspace{1cm} (6.2)

(possibly in a neighbourhood of \( u(s_0) \)).

2. Under appropriate conditions on \( DA(u_0) \) \((u_0 = u(s_0))\) problem (6.2) represents a variational boundary value problem on \( W_0^{1,2}(\Omega) \); \( DA(u_0) : W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega) \), and there exist positive functionals \( M : m, q : \mathcal{U} \to \mathbb{R}^+ \) such that for any \( \eta, \xi \in W_0^{1,2}(\Omega) \) and \( w \in \mathcal{U} \),

\[ \langle DA(w)\eta, \xi \rangle \leq M(w)\|\eta\|_{1,2} \|\xi\|_{1,2}, \]  
\hspace{1cm} (6.3a)

\[ \langle DA(w)\eta, \eta \rangle \geq m(w)\|\eta\|_{1,2}^2 - q(w)\|w\|_{0,2}^2. \]  
\hspace{1cm} (6.3b)

Inequality (6.3b) is a Gårding inequality for the linearized problem (6.2). According to Poincaré's inequality, a constant \( C_p > 0 \) exists such that

\[ \|\eta\|_{0,2} \leq C_p\|\eta\|_{1,2}. \]

Hence, (6.3b) can be written

\[ \langle DA(w)\eta, \eta \rangle \geq (m(w) - C_p^2(w))\|\eta\|_{1,2}^2. \]  
\hspace{1cm} (6.4)

3. When (6.3) holds, (6.2) is solvable, but multiple solutions may exist because the right-hand side of (6.3b) (or (6.4)) can be negative. Whenever \( M(w) > 0 \) and

\[ m(w) - C_p^2(w) > 0, \]  
\hspace{1cm} (6.5)

unique solutions exist to (6.2) by the Lax-Milgram Theorem (see for example [6]).

4. Let

(a) (6.5) hold in a ball \( B_{\varepsilon}(u_0) \) of radius \( \varepsilon \) about \( u_0 \) (i.e. \( \|w - u_0\|_{\mathcal{U}} < \varepsilon \Rightarrow w \) satisfies (6.5)).

(b) \( \{u_h\} \) be a sequence of Galerkin approximations converging strongly to a solution \( u_0 \) of (6.1).

Then there exists an \( h_\varepsilon \) such that \( u_h \in B_{\varepsilon}(u_0) \) for all \( h < h_\varepsilon \). Then, for any \( \theta \in [0, 1] \),

\[ \theta u_0 + (1 - \theta)u_h \in B_{\varepsilon}(u_0). \]  
\hspace{1cm} (6.6)

5. The following mean-value theorem holds for all \( w \in \mathcal{U} \):

\[ \langle A(u_0) - A(u_h), w \rangle = \langle DA(\theta u_0 + (1 - \theta)u_h) \cdot (u_h - u_0), w \rangle \]  
\hspace{1cm} (6.7)
for some \( \theta \in [0, 1] \). We assume that (6.6) holds and that, from what was established earlier, the problem

\[
\langle DA(z_{\theta h} \mathbf{m}, \zeta) \rangle = \langle F, v \rangle \quad \forall v \in W^{1,2}_0(\Omega).
\]

(6.8)

\[ z_{\theta h} = \theta u_0 + (1 - \theta)u_h \]

is uniquely solvable for \( \zeta \) and any \( \psi \in L^p(\Omega), p \geq 2 \). Moreover,

\[ \| \zeta \|_{2,2} \leq \hat{C} \| \psi \|_{0,2}. \]

6. Suppose that (3.2), (4.5), (4.8) and the above conditions hold. Then the approximation error \( e_h = u_0 - u_h \) satisfies

\[
\| e_h \|_{0,p} = \sup_{\psi} \frac{\langle \psi, e_h \rangle}{\| \psi \|_{0,p}}.
\]

\[ = c \sup_{\psi} \frac{\langle DA(\theta u_0 + (1 - \theta)u_h), (u_h - u), \zeta \rangle}{\| \psi \|_{0,p}} \]

\[ = c \sup_{\psi} \frac{\langle A(u_0) - A(u_h), \zeta - \tilde{\zeta}_h \rangle}{\| \psi \|_{0,p}} \]

\[ \leq g(u_0, u_h)Ch_\alpha \hat{C} |e_h|_{1,p} \]

where \( \tilde{\zeta}_h \) is the finite element interpolant of \( \zeta \), and \( c, C, \hat{C} \) are positive constants. Since \( g(\cdot, \cdot) \) is continuous and \( u_0 \) and \( u_h \) are bounded, there exists a number \( g(u_0) > 0 \), depending on \( u_0 \), such that for \( h \) sufficiently small

\[ \| e_h \|_{0,p} \leq g(u_0)h_\alpha |e_h|_{1,p}, \]

(6.9)

with \( \alpha \) given by (3.3a).

The procedure leading to (6.9) is essentially a version of the well-known Aubin-Nitsche method (cf. [11, 51]). Inequalities of the type (6.9) are crucial in completing an error analysis for the indefinite forms of the type often encountered in nonmonotone problems (e.g. recall (5.4)).

7. A priori error estimates — problem I

When results of the type (6.9) are available from a local analysis, it is possible to obtain a priori estimates of the approximation error \( e_h \) to problem I. Let the operator \( A \) in (2.3) satisfy the following conditions for \( u, v, w \in K \subset W^{1,p}_*(\Omega) \):
\[ (A(u) - A(v), w) \leq g(u, v)|u - v|_{1,p}^p |w|_{1,p}^p. \]  
\[ (A(u) - A(v), u - v) \geq C_1 |u - v|_{1,p}^p - \gamma(\mu) |u - v|_0^p. \]  
\( (7.1) \)

Here \( g(\cdot, \cdot) \) is a continuous function of \( |u|_{1,p}, |v|_{1,p}, C \) and \( \gamma \) are constants with \( \gamma \) depending on a bound \( \mu \) of \( u \) and \( v \) in \( W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \) and \( \infty > p \gg 2. \)

Further, suppose that

(a) \( u \) is a solution of \((2.1)\) and \( \{u_h\} \) is a sequence of finite element approximations of \((2.1)\) obtained using piecewise linear elements \( (\alpha = \min(1, l - 1)) \) such that \( u_h \to u \) strongly as \( h \to 0. \)

(b) Interpolation properties \((3.2)\) hold.

(c) \( u \in K_0 \cap W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega). \) \( (7.2) \)

(d) For \( h \) sufficiently small \((6.9)\) holds, i.e.

\[ \|e_h\|_{0,p} \leq g(u)h\|e_h\|_{1,p}. \]

where \( e_h = u - u_h \) is the approximation error.

Then, if \( \tilde{u}_h \) is the finite element interpolant of \( u \) (cf. \((3.2))\), \((7.1)\) yields

\[ C_1 \|e_h\|_{1,p}^p \leq (A(u) - A(u_h), u - \tilde{u}_h + u_h - \tilde{u}_h) + \gamma(\mu)\|e_h\|_0^p \]
\[ \leq g(u, u_h)\|e_h\|_{1,p} C_0 h\|u\|_{2,p} + \gamma(\mu)g(u)^p h^p \|e_h\|_{1,p}^p. \]

Since \( \|e_h\|_{1,p} \to 0 \) as \( h \to 0 \) (recall theorem 4.1 and \( (a) \) of \((7.2))\), there is a constant \( L \) such that

\[ \|e_h\|_{1,p}^p \leq G(u)h^{l/(p-1)} + H(u)h^p/(p-1), \quad \text{as } h \to 0. \]

\( (7.3) \)

wherein \( G(u) \) and \( H(u) \) are positive constants depending on \( u, C_0, C_1, \gamma(\mu), L \) and \( p \) but not on \( h. \)

Inequality \((7.3)\) holds as an asymptotic global a priori estimate of the error in finite element approximations of \((2.1)\) whenever conditions \((7.1)\) and \((7.3)\) hold.

8. Results for problem II — incompressible materials

The analysis of the elastostatics problem for incompressible materials has some complications which are quite different from those for compressible materials. Following the plan used in section 5, we summarize some results on existence theory for such problems obtained recently by Oden, Kikuchi and Sheu [13]:

1. The basic idea is to prove the existence of saddle points of the Lagrangian \( L \) of \((2.15)\) and to then show that such saddle points can be characterized as solutions of \((2.2)\). In other words, we seek pairs \( (u, p) \in \mathcal{U} \times \mathcal{V} \) such that

\[ L(u, q) \leq L(u, p) \leq L(v, p) \quad \forall v \in \mathcal{U}, \forall q \in \mathcal{V}. \]

\( (8.1) \)
2. Sufficient conditions for the existence of solutions to (8.1) are that
   (a) \( \forall v \in \mathcal{V}, v \to L(v, q) \) is weakly lower semicontinuous.
   (b) \( \forall v \in \mathcal{U}, q \to L(v, q) \) is concave and upper semicontinuous.
   (c) \( \exists q_0 \in \mathcal{V}, v \to L(v, q_0) \) is coercive in the sense that
   \[ L(v, q_0) \to +\infty \quad \text{as} \quad \|v\|_{\mathcal{U}} \to \infty. \]
   (d) \( q \to L(v, q) \) is coercive in the sense that
   \[ \lim \left( \inf_{v \in \mathcal{U}} L(v, q) \right) = -\infty \quad \text{as} \quad \|q\|_{\mathcal{V}} \to \infty. \]

3. Suppose that the strain energy function \( \sigma = \sigma(I, II) \) is a polynomial in the invariants \( I \) and \( II \)
   and is a convex function of \( u \) which, with proper choices of the elastic constants, is such that
   \( f_{\Omega} \rho \sigma \, dx \) is coercive on \( W^{1,p}_c(\Omega), 2 \leq p < \infty \) with \( \mathcal{V} = L^p(\Omega). \) Then (a), (b) and (c) above are satisfied. However, (d) is not. To overcome this shortcoming, we can introduce the perturbed
   Lagrangian
   \[ L_\varepsilon(v, q) = L(v, q) - \varepsilon \|q\|_{\mathcal{V}}^\alpha \]  
   for \( \varepsilon > 0 \) and \( \alpha > 1. \) Then (a)-(d) hold for \( L_\varepsilon. \) Thus there exists a saddle point \( (u_\varepsilon, p_\varepsilon) \) of \( L_\varepsilon \) for any \( \varepsilon > 0. \)

4. Condition (c) holds for \( L_\varepsilon. \) Hence \( \{u_\varepsilon\} \) is bounded and therefore has a subsequence, also denoted \( \{u_\varepsilon\} \), which converges weakly to an element \( u \in W^{1,p}_c(\Omega). \) However, \( p_\varepsilon \) may not converge in any sense unless \( \|p_\varepsilon\|_{\mathcal{V}} \) is bounded for all \( \varepsilon > 0. \)

5. Suppose that
   (a) \( \forall q \in \mathcal{V}, v \to L(v, q) \) is Gâteaux-differentiable with respect to \( v: \)
   \[ \left( \frac{\partial L(v, q)}{\partial v}, w \right) = \lim_{t \to 0^+} \frac{\partial}{\partial t} L(v + tw, q), \]
   (b) \( \forall v \in \mathcal{U}, q \to L(v, q) \) is Gâteaux-differentiable with respect to \( q: \)
   \[ \left( \frac{\partial L(v, q)}{\partial q}, r \right) = \lim_{t \to 0^+} \frac{\partial}{\partial t} L(v, q + tr). \]

Then solutions \((u, p)\) to (8.1) are characterized by
\[ \left( \frac{\partial L(u, p)}{\partial v}, w \right) = 0 \quad \forall w \in \mathcal{U}, \]
\[ \left( \frac{\partial L(u, p)}{\partial q}, r \right) = 0 \quad \forall r \in \mathcal{V}, \]  
which for \( L \) given by (2.15) is precisely (2.2).
6. It is clear that the existence of solutions to (2.2) is proved under the conditions laid down in steps 1–5 above if we can show that the pressures \( p_e \) in the perturbed problem (8.2) are uniformly bounded in \( \mathcal{V} (= L^p(\Omega)) \) independent of \( e \). This is a difficult question. To resolve it, note that under the assumptions in step 5, equations (8.3) for \( L_e \) reduce to

\[
\begin{align*}
\langle A(u_e), v \rangle + \langle C(u_e) p_e, v \rangle &= \langle F, v \rangle \quad \forall v \in \mathcal{U}, \\
\langle J(u_e), q \rangle &= \alpha E(p_e, q) = \langle g, q \rangle \quad \forall q \in \mathcal{V},
\end{align*}
\]

(8.4)

where \( E(p_e, q) \) is the Gâteaux differential of \( \|p_e\|_{\mathcal{V}}^\beta \). From these equations it can be shown that the sequence \( \{p_e\} \) is bounded in \( \mathcal{V} \) whenever the following condition holds: there exist constants \( m_0 > 0, \beta > 1 \) independent of \( e \) such that

\[
\sup_{v \in \mathcal{U}} \frac{|\langle C(u_e) p_e, v \rangle|}{\|v\|_{\mathcal{U}}} \geq m_0 \|p_e\|_{\mathcal{V}}^\beta
\]

(8.5)
i.e. for the class of problems described here

\[
\sup_{v \in W^{1,p}_0(\Omega)} \frac{|\int_\Omega p_e \text{adj} \nabla u^1_e : \nabla v \, dx|}{\|v\|_{1,p}} \geq m_0 \|p_e\|_{\mathcal{V}}^\beta
\]

(8.6)

When assumptions listed in steps 1–5 and condition (8.5) (or (8.6)) hold, we are guaranteed the existence of at least one solution of (2.2).

9. A priori error estimates – problem II

The analysis of errors in finite element approximations of the incompressible case is somewhat more complicated. We summarize here some recent results of Sheu [14]. Let us assume that a solution \((u, p)\) of (2.2) exists in \( W^{2,p}(\Omega) \times W^{1,p}(\Omega) \) and that a corresponding finite element approximation \((u_h, p_h)\) exists which satisfies (3.5). i.e.

\[
\begin{align*}
\langle A(u), v \rangle + \langle C(u)p, v \rangle &= \langle F, v \rangle \quad \forall v \in \mathcal{U}, \\
\langle J(u), v \rangle &= \langle g, q \rangle \quad \forall q \in \mathcal{V}
\end{align*}
\]

(9.1)

and

\[
\begin{align*}
\langle A(u_h), v_h \rangle + \langle C(u_h)p_h, v_h \rangle &= \langle F, v_h \rangle \quad \forall v_h \in \mathcal{U}_h, \\
\langle J(u_h), q_h \rangle &= \langle g, q_h \rangle \quad \forall q_h \in \mathcal{V}_h.
\end{align*}
\]

(9.2)

We also assume that
\[ \langle A(u) - A(v), u - v \rangle + \langle C(u)p - C(v)q, u - v \rangle \]
\[ \geq C_0\|u - v\|_{1,p} - \gamma_0(\mu, \nu)\|u - v\|_{0,p} - \gamma_1(\mu)\|p - q\|_{0,p} \quad (9.3) \]

and

\[ \langle A(u) - A(v), z \rangle + \langle C(u)p - C(v)q, z \rangle \]
\[ \leq G_1(\mu)\|u - v\|_{1,p}\|z\|_{1,p} + G_2(\mu, \nu)\|u - v\|_{1,p}\|z\|_{1,p} + G_3(\mu, \nu)\|p - q\|_{0,p}\|z\|_{1,p}, \]
\[ (J(u) - J(v), q) \leq G_4(\mu)\|u - v\|_{1,p}\|q\|_{0,p} \quad (9.4) \]

for all \( u, v, z \) in a ball of radius \( \mu \) in \( W^{2,p}(\Omega) \) and for all \( p, q \) in a ball of radius \( \nu \) in \( L^p(\Omega) \). In (9.3) \( \gamma_0 \) and \( \gamma_1 \) are positive constants depending continuously on \( \mu \) and \( \nu \). In (9.4) \( G_i, i = 1, 2, 3, 4 \), are positive constants depending on \( \mu \) and \( \nu \). It can be shown that inequalities of the type (9.3) and (9.4) hold for a large class of nonlinear incompressible materials.

We also have from (9.1) and (9.2) the orthogonality condition

\[ \langle A(u) - A(u_h), v_h \rangle + \langle C(u)p - C(u_h)p_h, v_h \rangle = 0 \quad \forall v_h \in \mathcal{V}_h. \quad (9.5) \]

Finally, if \( u_h \to u \) and \( p_h \to p \), let us assume that a local analysis of the type described in section 6 yields (under appropriate assumptions on \( u \) and its regularity)

\[ \|e_u\|_{0,p} + \|e_p\|_{0,0} \leq C(u, p)h(\|e_u\|_{0,p} + \|e_p\|_{0,p}). \quad (9.6) \]

where \( C \) is a positive constant depending on \( (u, p) \), and that (9.6) holds in a sufficiently small neighborhood of \((u, p)\) when \( h \) is sufficiently small. In (9.6) \( e_u \) and \( e_p \) are the approximation errors.

\[ e_u = u - u_h, \quad e_p = p - p_h. \quad (9.7) \]

Clearly, for \( h \) small enough

\[ \|e_p\|_{0,p} \leq \frac{C(\mu, \nu)h}{1 - C(\mu, \nu)h} \|e_u\|_{1,p} \]

and

\[ \|e_p\|_{0,p} \leq C(\mu, \nu)h\|e_u\|_{1,p}, \quad \text{as } h \to 0. \quad (9.8) \]

According to (9.3), (9.4) and (9.5) we have
\[ C_0 \| e_u \|_{1,p} - \gamma_0 \| e_u \|_{0,p} - \gamma_1 \| e_p \|_{0,p} \leq G_{12}(\mu, \nu) \| e_u \|_{1,p} \| u - v_h \|_{1,p} + G_3(\mu) \| e_p \|_{0,p} \| u - v_h \|_{1,p} \quad \forall v_h \in \mathcal{V}_h. \]  

(9.9)

where \( G_{12} = G_1 + G_2 \). Assuming (8.6) holds as \( h \to 0 \) and setting \( v_h = \bar{u}_h \), we get

\[ \| e_u \|_{1,p} \leq M_0(\mu, \nu) \| e_u \|_{1,p} h^\alpha + M_1(\mu, \nu) h \| e_u \|_{1,p} h^\alpha + \max(\gamma_0, \gamma_1) \| e_p \|_{0,p} + \| e_p \|_{0,p}. \]

where \( M_0(\mu, \nu) \) and \( M_1(\mu, \nu) \) are positive numbers depending on \( u \) but not \( h \). Introducing the local estimates (9.6) and (9.8) yields

\[ \| e_u \|_{1,p} \leq M_0(\mu, \nu) \| e_u \|_{1,p} h^\alpha + \max(\gamma_0, \gamma_1)(C(\mu, \nu)h)^p \| e_u \|_{1,p} + (C(\mu, \nu)h)^p \| e_u \|_{1,p}. \]

By Young's inequality, \( \forall \varepsilon > 0 \).

\[ (C(\mu, \nu)h)^p \| e_u \|_{1,p} + \frac{1}{r} (C(\mu, \nu)h)^{p-1}(p-2), \quad r = \frac{(p-1)^2}{p(p-2)}. \]

Thus, as \( h \to 0 \), we have the estimate by setting \( \varepsilon \) sufficiently small

\[ \| e_u \|_{1,p} \leq M_0(\mu, \nu) h^{\alpha/(p-1)} + M_2(\mu, \nu) h^{1/(p-2)}. \]  

(9.10)

An estimate for the error in hydrostatic pressures follows from (9.8):

\[ \| e_p \|_{0,p} \leq M_3(\mu, \nu) h^{\alpha/(p-1)} + h^{1/(p-2)}. \]  

(9.11)

Additional details are given in [14].

We remark that a condition for numerical stability of such finite element approximations arises naturally from the theory in section 8. The finite element equations (9.2) have solutions whenever (8.5) (or (8.6)) holds for the spaces \( \mathcal{U}_h \) and \( \mathcal{V}_h \), i.e., there exists a constant \( C_h > 0 \) such that

\[ C_h \leq \inf_{q_h \in \mathcal{V}_h} \sup_{v_h \in \mathcal{U}_h} \| (C(u_h)q_h, v_h) \|. \]  

(9.12)

We must choose the approximations of \( v_h \) and \( q_h \) such that

\[ C_h \geq C_0 > 0 \quad \text{as} \quad h \to 0, \]

where \( C_0 \) is a constant independent of \( h \). This condition has not yet been thoroughly investigated, but we conjecture that it implies that we should take

\[ t \leq k - 1. \]

where \( t \) and \( k \) are the degrees of the respective polynomials defined in (3.1).
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