Qualitative Analysis and Finite Element Approximation of a Class of Nonmonotone Nonlinear Dirichlet Problems

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Summary: Existence and convergence of finite element approximations of a class of nonlinear Dirichlet problems involving pseudomonotone operators are considered. A model problem, which is analyzed in some detail, is characterized by an operator which satisfies a generalized Gårding inequality and, thus, leads to nonlinear indefinite forms. Under assumptions on the local regularity of solutions, error estimates in the $W^{1,p}$-norm are obtained.

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1. Introduction

In this paper, we consider a class of nonlinear Dirichlet problems of the type

\[-\nabla \cdot (|\nabla u|^{p-2} \nabla u) + \alpha \frac{u}{\sqrt{1 + |\nabla u|^2}} = f \text{ in } \Omega \]
\[u = 0 \text{ on } \partial \Omega \]

(1.1)

with \(1 < p < \infty\)

where \(\Omega\) is an open bounded domain in \(\mathbb{R}^n\) with Lipschitzian boundary \(\partial \Omega\) and \(\alpha\) is a finite constant. The operator appearing in (1.1) is not necessarily monotone, and, consequently, multiple solutions may exist for fixed data \(f\).

Our aim is to investigate the qualitative behavior of solutions to (1.1), particularly conditions for their existence, uniqueness, and boundedness, and to study finite element-Galerkin approximations of these solutions. We also obtain a-priori error estimates for finite element approximations under assumptions on the regularity of solutions.

The special case \(\alpha = 0\) of (1.1) was studied in some detail by GLOWINSKI and MARROCO [6]. In this case, the operator appearing in (1.1) is strongly monotone from \(W^{1,p}_0(\Omega)\) into its dual \(W^{-1,p}'(\Omega)\) and unique solutions exist for any \(f\) in \(W^{-1,p}'(\Omega)\). When \(\alpha \neq 0\), we will show that the operator is of the Gårding type studied by ODEN [9] and, therefore, is pseudomonotone in the sense of BREZIS [3] and LIONS [7]. Analyses of finite element approximations of certain one-dimensional pseudo-
monotone problems were investigated by ODEN and NICOLAU del ROURE [10] and similar techniques were used in the study of certain nonlinear elliptic systems encountered in plane elasticity problems by ODEN and REDDY [11,12]. Additional references on finite element approximations of nonlinear strongly-monotone problems can be found in the work of GLOWINSKI and MARROCO [6] and in the survey article of BABUSKA [2]; see also ODEN and REDDY [11,12].

Following this introduction, we describe a variational formulation of problem (1.1) in Section 2 and we record a useful existence theorem for operators A of the type in (1.1) in Section 3 which establishes sufficient conditions for A to be pseudomonotone and surjective. Sections 4 and 5 are devoted to studies of properties of the specific operator A in our model problem (1.1), with Section 4 containing some preliminary inequalities in \( \mathbb{R}^n \). Conditions for the existence of weak solutions of (1.1) are laid down in Section 6 and finite element approximations of (1.1) are introduced in Section 7. There we discuss conditions for weak and strong convergence of sequences of approximations generated by regular refinements of a mesh containing \( C^0 \)-elements.

The lack of monotonicity of A leads to indefinite forms which resemble Gårding's inequality in linear elliptic theory; e.g.

\[
\langle A(u) - A(v), u - v \rangle \geq C \| u - v \|_{1,p}^p - \gamma \| u - v \|_{0,q}^p
\]

for positive constants C and \( \gamma \). The presence of the negative term presents some difficulties in the error analysis. One way of overcoming these difficulties is to first determine some "local" estimates for an appropriate linearized auxiliary problem. We analyze such a linearized
problem in Section 8 on the basis of assumptions on the global regularity of solutions to (1.1). Finally, in Section 9 of the paper, we obtain a-priori estimates of the error in the $W^{1,p}$-norm. In particular, we show that whenever a sequence of piecewise linear finite element approximations $u_h$ converges strongly in $W^{1,p}_0(\Omega)$ to a solution $u$ of our problem, then $\|u - u_h\|_{1,p}$ is of order $O(h^{1/(p-1)})$ as $h \to 0$.

2. Variational Formulation

We will consider a variational formulation of problem (1.1) which has meaning in the context of Sobolev spaces. Following standard notations, we denote by $W^{m,p}(\Omega)$ the Sobolev space of order $(m,p)$ consisting of equivalence classes of functions with generalized derivatives of order $\leq m$ in $L^p(\Omega)$, $m \geq 0$, $1 \leq p \leq \infty$. When equipped with the norm

$$
\|u\|_{m,p} = \left( \int_{\Omega} \left( \sum_{|\alpha| \leq m} |D^\alpha u|^p \right)^{1/p} \right)^{1/p},
$$

(2.1)

$$
\|u\|_{m,\infty} = \text{ess sup}_{x \in \Omega} \sum_{|\alpha| \leq m} |D^\alpha u(x)|
$$

the spaces $W^{m,p}(\Omega)$ are Banach spaces. Here $\Omega \subset \mathbb{R}^n$, $dx = dx_1 dx_2 \ldots dx_n$, $x = (x_1, x_2, \ldots, x_n) \in \Omega$, and $\alpha$ is a multi-index, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_i \geq 0$, $\alpha_i$ = integer, $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$. We are particularly interested in the space $W^{1,p}_0(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ in the $\| \cdot \|_{m,p}$-norm, and which is a reflexive Banach space equipped with the norm.
The dual of $W^{1,p}_0(\Omega)$ is the negative Sobolev space

$$\left(W^{1,p}_0(\Omega)\right)' = W^{-1,p'}(\Omega), \quad p' = p/(p-1)$$

(2.3)

equipped with the norm $\|\cdot\|_*$ where

$$\|f\|_* = \sup_{\vE \in W^{1,p}_0(\Omega)} \frac{\langle f, \vE \rangle}{\|\vE\|_{1,p}}, \quad \vE \neq 0$$

(2.4)

wherein $\langle \cdot, \cdot \rangle$ denotes the bilinear duality pairing on $W^{-1,p'}(\Omega) \times W^{1,p}_0(\Omega)$.

Returning to (1.1), we introduce an operator $A: W^{1,p}_0(\Omega) \to W^{-1,p'}(\Omega)$ defined by

$$\langle A(u), \vE \rangle = \int_\Omega \left( |\vE|^{p-2} \vE \cdot \vD + a(1 + |\vE|^2)^{-1/2} \vE \cdot \vD \right) dx$$

$$u, \vE \in W^{1,p}_0(\Omega)$$

(2.5)

Likewise, if $f$ is a given bounded linear functional in $W^{-1,p'}(\Omega)$, we write $f(\vE) = \langle f, \vE \rangle \quad \forall \vE \in W^{1,p}_0(\Omega)$.

We will consider the following variational boundary value problem:

Given $f \in W^{-1,p'}(\Omega)$, find $u \in W^{1,p}_0(\Omega)$ such that
\[ \langle A(u), v \rangle = \langle f, v \rangle \quad \forall \quad v \in W^{1,p}_0(\Omega) \] (2.6)

Since \( C^\infty_0(\Omega) \) is dense in \( W^{1,p}_0(\Omega) \), we take \( v \in C^\infty_0(\Omega) \) and conclude that \( A(u) \) in (2.6) is formally given by

\[ A(u) = -\nabla \cdot (|\nabla u|^{p-2} \nabla u) + \alpha \frac{u}{\sqrt{1 + |\nabla u|^2}} \] (2.7)

where \( A(u) \in \mathcal{D}'(\Omega) \); i.e., (2.7) is to be interpreted in the sense of distributions. With this interpretation, the equivalence of (1.1) and (2.6) is apparent.

3. An Existence Theorem for Gårding-Type Operators

The first question that arises in the study of problem (2.6) is that of existence of solutions. We will record here an existence theorem which will prove to be critical in the subsequent studies of approximations. We first review some definitions and preliminary results.

Let

- \( U \) and \( V \) be reflexive Banach spaces with the property that \( U \) is compactly embedded in \( V \),

\[ U \subset C \subset V \] (3.1)

- \( A : U \to U' \) is an operator mapping \( U \) into its strong topological dual \( U' \)
The operator \( A \) is bounded from \( U \) into \( U' \) if it maps (strongly) bounded sets in \( U \) into (strongly) bounded sets in \( U' \). \( A \) is hemicontinuous at a point \( u \in U \) if the real-valued function

\[
\phi(t) = \langle A(u + tv), w \rangle \quad t \in [0,1]
\]

is a continuous function of \( t \) for \( v, w \in U \). In (3.2), \( \langle \cdot, \cdot \rangle \) denotes duality pairing on \( U' \times U \). Moreover, \( A: U \rightarrow U' \) is coercive on \( U \) if

\[
\lim_{\| v \|_U \rightarrow \infty} \frac{\langle A(v), v \rangle}{\| v \|_U} = +\infty
\]

The following existence theorem was proved by ODEN [9]:

**Theorem 3.1.** Let conditions (3.1) hold and let \( A: U \rightarrow U' \) be an operator having the following properties:

(i) \( A \) is bounded

(ii) \( A \) is hemicontinuous

(iii) \( A \) is coercive

(iv) If \( B_\mu(0) \) is the open ball of radius \( \mu \) in \( U \), i.e.,

\[
B_\mu(0) = \{ v \in U : \| v \|_U < \mu \}
\]

then, \( \forall u, v \in B_\mu(0) \),

\[
\langle A(u) - A(v), u - v \rangle \geq -H(\mu, \| u - v \|_U)
\]

where \( H: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) (\( \mathbb{R}^+ = [0,\infty) \)) is a continuous real-valued function with the property that
\[
\lim_{\theta \to 0^+} \frac{1}{\theta} H(x, \theta y) = 0, \quad x, y \in \mathbb{R}^+
\] (3.6)

Then \( A \) is surjective, i.e., \( \forall f \in U' \) there exists at least one \( u \in U \) such that
\[
A(u) = f
\]

We remark that operators satisfying the conditions of Theorem 3.1 are monotone only in the special case in which \( H \equiv 0 \). Operators of this type were referred to as Gårding operators by ODEN [9] owing to the formal similarity of (3.5) to the Gårding inequality for linear elliptic operators. Note that \( H \) depends upon the norm \( \|u - v\|_V \) in the space \( V \) in which \( U \) is compactly embedded. This property is crucial in the study of the existence and approximation of solutions to abstract problems involving this class of nonmonotone operators. It effectively establishes that the operator \( A \) differs from a monotone operator by a component which is completely continuous from \( U \) into \( U' \) (whenever \( U \) is dense in \( V \)). It is not difficult to show that when the conditions of Theorem 3.1 hold (in particular, conditions (ii) and (iv)), the operator \( A \) is pseudomonotone in the sense of BREZIS [3] and LIONS [7]; i.e.,

If \( u_m \in U \) converges weakly to \( u \in U \) as \( m \to \infty \) and if
\[
\lim_{m \to \infty} \sup \langle A(u_m), u_m - u \rangle < 0,
\]

then
\[
\lim_{m \to \infty} \inf \langle A(u_m), u_m - v \rangle \geq \langle A(u), u - v \rangle \quad \forall v \in U
\]

(3.7)
4. Some Inequalities in $\mathbb{R}^n$

We will now establish several inequalities for vectors in $\mathbb{R}^n$ which will prove to be important in studying properties of the operator $A$ of (2.5). We use the following notation.

$x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$

$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$ for $x, y \in \mathbb{R}^n$ \hspace{1cm} (4.1)

and

$|x| = \langle x, x \rangle^{1/2}$ \hspace{1cm} (4.2)

Lemma 4.1. Let $x, y$ be any two vectors in $\mathbb{R}^n$. Then,

(i) for $p \geq 2$,

$$(|x|^{p-2} x - |y|^{p-2} y, x - y) \geq \left(\frac{1}{2}\right)^{p-1} |x - y|^p$$ \hspace{1cm} (4.3)

(ii) for $1 < p \leq 2$

$$(|x| + |y|)^{2-p}(|x|^{p-2} x - |y|^{p-2} y, x - y) \geq (p-1)|x - y|^2$$ \hspace{1cm} (4.4)

Proof:

(i) For $p = 2$, the inequality is obvious. Also the equality occurs for $x = y$. So we consider $p > 2$ and $x \neq y$. By direct expansion
\[ \phi_1(x, y) = (|x|^{p-2} - |y|^{p-2}, x - y) = |x|^p + |y|^p - (|x|^{p-2} + |y|^{p-2})(x, y) \]

Noting that \((x, y) = \frac{1}{2} (|x|^2 + |y|^2 - |x-y|^2)\), we get

\[ \phi_1(x, y) = \frac{1}{2} \left[ (|x|^{p-2} + |y|^{p-2})|x-y|^2 + (|x|^2 - |y|^2)(|x|^{p-2} - |y|^{p-2}) \right] \]

The second term on the right hand side is positive for \(p > 2\). Therefore,

\[ \phi_1(x, y) \geq \frac{1}{2} \frac{|x|^{p-2} + |y|^{p-2}}{|x-y|^{p-2}} |x-y|^p \geq \frac{1}{2} \frac{|x|^{p-2} + |y|^{p-2}}{(|x| + |y|)^{p-2}} |x-y|^p \]

For \(a, b, r > 0\), we have \(a^r + b^r \geq [\max(a, b)]^r \geq \left(\frac{a+b}{2}\right)^r\). Hence,

\[ \phi_1(x, y) \geq \left(\frac{1}{2}\right)^{p-1} |x-y|^p \]

(ii) For \(p = 2\) or \(x = y\), the above inequality obviously reduces to an equality. We therefore consider \(1 < p < 2\). When \(|x| = |y|\) we get

\[ (|x| + |y|)^{2-p} (|x|^{p-2} x - |y|^{p-2} y, x - y) = 2^{2-p} |x-y|^2 \]

where we note that \(2^{2-p} > p-1\) for \(1 < p < 2\). Therefore we need only consider the case \(|x| \neq |y|\). In this case, let
\[ \phi_2(x, y) = \frac{(|x| + |y|)^{2-p} (|x|^{p-2} x - |y|^{p-2} y, x - y)}{|x - y|^2} \]

\[ = \frac{1}{2} \frac{(|x| + |y|)^{2-p}}{|x - y|^2} \left[ (|x|^{p-2} + |y|^{p-2}) |x - y|^2 \right. \]

\[ + (|x|^2 - |y|^2)(|x|^{p-2} - |y|^{p-2}) \]

In the above expression we made use of the expansion of \( 2(x, y) = |x|^2 + |y|^2 - |x - y|^2 \). Denoting \( r = 2 - p \) so that \( 0 < r < 1 \),

\[ \phi_2(x, y) = \frac{1}{2} \left( \frac{|x| + |y|}{|x| + |y|} \right)^r \left[ |x|^r + |y|^r - \frac{(|x|^2 - |y|^2)(|x|^r - |y|^r)}{|x - y|^2} \right] \]

\[ = \frac{1}{2} \left( \frac{|x| + |y|}{|x| + |y|} \right)^r \left[ |x|^r + |y|^r - \frac{|x|^2 - |y|^2}{|x - y|^2} \right. \]

\[ \left. - \frac{|x|^r - |y|^r}{|x - y|^2} \right] \]

\[ \geq \frac{1}{2} \left( \frac{|x| + |y|}{|x| + |y|} \right)^r \left[ |x|^r + |y|^r - \frac{(|x| + |y|)}{|x|^r - |y|^r} \right. \]

\[ \left. - \frac{|x|}{|y|} \right] \]

Without loss in generality, let \( |x| > |y| \) and denote

\[ t = \frac{|x|}{|y|} > 1 \]
Then

\[ \phi_2(x, y) \geq \frac{1}{2} \left( 1 + \frac{1}{t} \right)^r \left[ 1 + t^r - (t+1) \frac{t^r - 1}{t-1} \right] \]

\[ = \frac{1}{2} \left( 1 + \frac{1}{t} \right)^r \left[ \frac{2(t-t^r)}{t-1} \right] \geq 1 - \frac{t^r - 1}{t-1} \]

\[ \geq 1 - r = p - 1 . \]

Here we used the fact that \( \frac{t^r - 1}{t-1} < r \) for \( t > 1 \), \( 0 < r < 1 \).

Lemma 4.2. Let \( x, y \) be arbitrary vectors in \( \mathbb{R}^n \). Then

(i) for \( p \geq 2 \),

\[ | |x|^{p-2} x - |y|^{p-2} y | \leq \beta_3 |x - y| (|x| + |y|)^{p-2} \quad (4.5) \]

where

\[ \beta_3 = \begin{cases} \sqrt{p+1} & \text{for } 2 \leq p \leq 3 \\ p - 1 & \text{for } p \geq 3 \end{cases} \quad (4.6) \]

(ii) for \( 1 < p \leq 2 \),

\[ | |x|^{p-2} x - |y|^{p-2} y | \leq \sqrt{5} |x - y|^{p-1} \quad (4.7) \]
Proof:

(1) Cases $p = 2$ or $x = y$ or $|x| = |y|$ are obvious. We will therefore consider $p > 2$ and $|x| \neq |y|$.

Let

$$
\phi_3(x, y) = \frac{|x|^{p-2}x - |y|^{p-2}y}{|x - y|(|x| + |y|)^{p-2}}
$$

Then by direct expansion and using $2(x, y) = |x|^2 + |y|^2 - |x - y|^2$, we get

$$
(\phi_3(x, y))^2 = \left(\frac{|x| |y|}{(|x| + |y|)^2}\right)^{p-2} + \frac{|x|^p - |y|^p}{|x-y|^2(|x| + |y|)^{p-2}} \cdot \frac{|x|^{p-2} - |y|^{p-2}}{(|x| + |y|)^{p-2}}
$$

\[< 1 + \frac{(|x| + |y|)^2}{(|x| - |y|)^2} \cdot \frac{|x|^p - |y|^p}{(|x| + |y|)^p} \cdot \frac{|x|^{p-2} - |y|^{p-2}}{(|x| + |y|)^{p-2}}\]

Without loss in generality, we take $|x| > |y|$ and denote $s = \frac{|y|}{|x|} < 1$. Then,

$$
\phi_3^2(x, y) \leq 1 + \left(\frac{1+s}{1-s}\right)^2 \frac{1-s^p}{(1+s)^p} \frac{1-s^{p-2}}{(1+s)^{p-2}}
$$

\[= 1 + \frac{1}{(1+s)^{2(p-2)}} \frac{1-s^p}{1-s} \frac{1-s^{p-2}}{1-s}
$$

\[< 1 + \frac{1-s^p}{1-s} \frac{1-s^{p-2}}{1-s}.\]
for $0 < s < 1$, \( \frac{1-s}{1-s} < \begin{cases} r & \text{when } r \geq 1 \\ 1 & \text{when } r < 1 \end{cases} \).

Therefore
\[
\phi_3^2 \leq \begin{cases} 1 + p(p-2) = (p-1)^2 & p \geq 3 \\ 1 + p & 2 < p \leq 3 \end{cases}.
\]

The result follows immediately.

(ii) When $p = 2$, the inequality is obvious. When $|\overline{x}| = |\overline{y}|$ we observe that
\[
| |\overline{x}|^{p-2} \overline{x} - |\overline{y}|^{p-2} \overline{y}| = |\overline{x}|^{p-2} |\overline{x} - \overline{y}| = |\overline{x} - \overline{y}|^{p-1} \left( \frac{|\overline{x} - \overline{y}|}{|\overline{x}|} \right)^{2-p} \leq 2^{2-p} |\overline{x} - \overline{y}|^{p-1} \leq \sqrt{5} |\overline{x} - \overline{y}|^{p-1}
\]

We now consider $1 < p < 2$ and $|\overline{x}| \neq |\overline{y}|$. Let $r = p - 1$ so that $0 < r < 1$. Let
\[
\phi_4(\overline{x}, \overline{y}) = \frac{| |\overline{x}|^{p-2} \overline{x} - |\overline{y}|^{p-2} \overline{y}|}{|\overline{x} - \overline{y}|^{p-1}} = \frac{| |\overline{x}|^{r-1} \overline{x} - |\overline{y}|^{r-1} \overline{y}|}{|\overline{x} - \overline{y}|^{r}}
\]

Also we note that $\cos \theta = \frac{(\overline{x}, \overline{y})}{|\overline{x}| |\overline{y}|}$, where $\theta \leq \pi$ represents the angle between vectors $\overline{x}$ and $\overline{y}$. 
\[
\phi_4(x, y) = \frac{\sqrt{|x|^2 + |y|^2 - 2|x| |y| \cos \theta}}{(\sqrt{|x|^2 + |y|^2 - 2|x| |y| \cos \theta})^\theta}
\]

Without loss in generality, let \(|x| > |y|\) and let \(t = \frac{|y|}{|x|} < 1\).

Then,

\[
\phi^2_4(x, y) = \frac{1 + t^2 - 2t \cos \theta}{(1 + t^2 - 2t \cos \theta)^\theta} = \frac{(1 - t^\theta)^2 + 4t \sin^2(\frac{\theta}{2})}{[(1 - t^\theta)^2 + 4t \sin^2(\frac{\theta}{2})]^\theta}
\]

\[
= \frac{(1 - t^\theta)^2}{[(1 - t^\theta)^2 + 4t \sin^2(\frac{\theta}{2})]^\theta} + \frac{4t \sin^2(\frac{\theta}{2})}{[(1 - t^\theta)^2 + 4t \sin^2(\frac{\theta}{2})]^\theta}
\]

\[
\leq \left(\frac{1 - t^\theta}{1 - t}\right) + \frac{4t \sin^2(\frac{\theta}{2})}{4t \sin^2\theta} \leq 1 + 4^{1-r}\left(\sin^2(\frac{\theta}{2})\right)^{1-r}
\]

\[
\leq 1 + 4^{1-r} \leq 5
\]

\[
\phi_4(x, y) \leq \sqrt{5} \quad \square
\]

GLOWINSKI and MARROCO [6] proved the inequalities of Lemmas 4.1 and 4.2 for vectors in \(\mathbb{R}^2\). In addition to extending these results to \(\mathbb{R}^n\), we have also obtained here sharper estimates on the constants appearing in these inequalities.

As a final lemma, we establish a useful elementary inequality:
Lemma 4.3. Let \( a, b \in \mathbb{R} \). Then,

\[
|\sqrt{1+a^2} - \sqrt{1+b^2}| \leq |a-b|
\]  

(4.8)

Proof: We multiply and divide the left hand side by \( \sqrt{1+a^2} + \sqrt{1+b^2} \).

Thus,

\[
|\sqrt{1+a^2} - \sqrt{1+b^2}| = \frac{|(\sqrt{1+a^2} - \sqrt{1+b^2})(\sqrt{1+a^2} + \sqrt{1+b^2})|}{\sqrt{1+a^2} + \sqrt{1+b^2}}
\]

\[
= \frac{|a^2 - b^2|}{\sqrt{1+a^2} + \sqrt{1+b^2}}
\]

\[
= \frac{|a+b|}{(\sqrt{1+a^2} + \sqrt{1+b^2})} \frac{|a-b|}{|a+b|} |a-b|
\]

We obviously have \( |a+b|/(\sqrt{1+a^2} + \sqrt{1+b^2}) \leq 1 \) and thus inequality (4.8) follows immediately. \( \Box \)

5. Some Properties of the Operator \( A \)

We now return to the operator \( A \) defined in (2.5). In this section, we establish a number of properties of \( A \) that are crucial in proving the existence of solutions to (2.6) and to subsequent studies of finite element approximations.
Theorem 5.1. Let the operator \( A : W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega) \), be as defined in (2.5), and let

\[
p > \frac{3n}{n+2}
\]

(5.1)

where \( n \) is the dimension of \( \Omega \). Then, for \( u, v, w \in W_0^{1,p}(\Omega) \), we have

(i) \( p > 2 \)

\[
| \langle A(u) - A(v), w \rangle | \leq g_1(u,v) \| u - v \|_{1,p} \| w \|_{1,p}
\]

(5.2)

where

\[
g_1(u,v) = \beta_3 (\| u \|_{1,p} + \| v \|_{1,p})^{p-2} + C_1 |\alpha| (1 + \| v \|_{1,p})
\]

(5.3)

where \( \beta_3 \) is defined by (4.6) and \( C_1 \) is a positive constant.

(ii) \( 1 < p < 2 \)

\[
| \langle A(u) - A(v), w \rangle | \leq g_2(u,v) \| u - v \|_{1,p}^{p-1} \| w \|_{1,p}
\]

(5.4)

\[
g_2(u,v) = \sqrt{5} + C_1 |\alpha| (1 + \| v \|_{1,p}) \| u - v \|_{1,p}^{2-p}
\]

(5.5)

Proof: It is convenient to decompose \( A \) into two parts,

\[
A = A_p + A_o
\]

(5.6)

where, formally,
\[ A_p(u) = -\nabla \cdot (|\nabla u|^{p-2} \nabla u) \quad \forall u \in \dot{W}^{1,p} (\Omega) \]  \hfill (5.7)

\[ A_0(u) = \alpha \frac{u}{\sqrt{1 + |\nabla u|^2}} \quad \forall u \in \dot{W}^{1,p} (\Omega) \]  \hfill (5.8)

Then,

\[ \langle A_p(u) - A_p(v), w \rangle = \int_{\Omega} (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \cdot \nabla w \, dx \]  \hfill (5.9)

For \( p \geq 2 \), we make use of the inequality (4.5) of Lemma 4.2 to obtain

\[ |\langle A_p(u) - A_p(v), w \rangle| \leq \beta_3 \int_{\Omega} (|\nabla u| + |\nabla v|)^{p-2} |\nabla u - \nabla v| |\nabla w| \, dx \]

where \( \beta_3 \) is defined in (4.6). Now using Hölder's inequality, we obtain

\[ |\langle A_p(u) - A_p(v), w \rangle| \leq \beta_3 \left( \|\nabla u\|_{L^p(\Omega)} + \|\nabla v\|_{L^p(\Omega)} \right)^{p-2} \|\nabla u - \nabla v\|_{L^1(\Omega)} \|\nabla w\|_{L^p(\Omega)} \]

\[ \leq \beta_3 \left( \|u\|_{1,p} + \|v\|_{1,p} \right)^{p-2} \|u - v\|_{1,p} \|w\|_{1,p} \]  \hfill (5.10)

For \( 1 < p < 2 \), we make use of the inequality (4.7) of Lemma 4.2 to obtain

\[ |\langle A_p(u) - A_p(v), w \rangle| \leq \sqrt{5} \int_{\Omega} |\nabla (u - v)| \|\nabla w\|^{p-1} \, dx \]

\[ \leq \sqrt{5} \|u - v\|_{1,p}^{p-1} \|w\|_{1,p} \]  \hfill (5.11)
We now consider operator $A_0$:

$$\langle A_0(u) - A_0(v), w \rangle = \alpha \int_{\Omega} \left( \frac{u}{\sqrt{1 + |u|^2}} - \frac{v}{\sqrt{1 + |v|^2}} \right) w \, dx \quad (5.12)$$

Clearly,

$$|\langle A_0(u) - A_0(v), w \rangle| = |\alpha \int_{\Omega} \left( \frac{(u-v)w}{\sqrt{1 + |u|^2}} + \left( \frac{1}{\sqrt{1 + |u|^2}} - \frac{1}{\sqrt{1 + |v|^2}} \right) vw \right) \, dx|$$

$$\leq |\alpha| \left[ \int_{\Omega} |u-v| |w| \, dx + \int_{\Omega} \left| \frac{1}{\sqrt{1 + |u|^2}} - \frac{1}{\sqrt{1 + |v|^2}} \right| |v| |w| \, dx \right]$$

By the inequality (4.8) of Lemma (4.3), we have

$$|\langle A_0(u) - A_0(v), w \rangle| \leq |\alpha| \left[ \int_{\Omega} |u-v| |w| \, dx + \int_{\Omega} |\nabla (u-v)| |v| |w| \, dx \right]$$

(5.13)

We recall here the Sobolev embedding theorem (see, for example, ADAMS [1]):

Suppose $z \in W_o^{1,p}(\Omega), \Omega \subset \mathbb{R}^n$, then

(a) for $p > n$, $u \in C^0(\Omega)$ and

$$\sup_{x \in \Omega} |z(x)| \leq \text{const.} \left\| \nabla z \right\|_{L^p(\Omega)}$$

(5.14)
where the constant depends on $\text{mes } \Omega$ but not on $z$.

(b) for $p \leq n$ and $(n-p)q \leq np$, $z \in L^q_\Omega$ and

$$
\|z\|_{L^q_\Omega} \leq \text{const. } \|v\|_{L^p_\Omega}, \quad q < \infty
$$

(5.15)

where the constant depends on $\text{mes } \Omega$, $q$, $p$, $n$, but not on $z$.

Now, returning to (5.13), we observe that when $V(u-v) \in L^p_\Omega$, we need $v, w \in L^q_\Omega$, where $q \geq 2p' = 2p/(p-1)$. Then, according to the Sobolev embedding results given above, we need the condition (5.1). When (5.1) holds, we use Hölder's inequality and the inequalities (5.14) or (5.15) to obtain

$$
|\langle A_0(u) - A_0(v), w \rangle| \leq |a| C_1(1+\|v\|_{L^p_\Omega}) \|u-v\|_{L^p_\Omega} \|w\|_{L^p_\Omega}
$$

(5.16)

where $C_1$ is a constant dependent on $\text{mes } \Omega$.

Combining results (5.10), (5.11), and (5.16) gives (5.2) and (5.4).

**Theorem 5.2.** Let the conditions of Theorem 5.1 hold. Then the operator $A$ of (2.5) is strongly continuous and bounded from $W^{1,p}_0(\Omega)$ into $W^{-1,p'}(\Omega)$ for any $p$, $1 < p < \infty$, satisfying (5.1).

**Proof:** Let $\{u_k\}$ be a sequence in $W^{1,p}_0(\Omega)$ converging strongly to $u \in W^{1,p}_0(\Omega)$ with $p$ satisfying (5.1). Then $\|u_k\|_{L^p_\Omega}$ and $\|u\|_{L^p_\Omega}$ are bounded. Hence
\[ ||A(u) - A(u_k)||_{L^p} = \sup_{v \in W_o^{1,p}(\Omega)} \frac{\langle A(u) - A(u_k), v \rangle}{||v||_{L^p}} \]

\[ \leq M \begin{cases} ||u - u_k||_{L^p} & p \geq 2 \\ ||u - u_k||_{L^{p-1}} & 1 < p < 2 \end{cases} \]

where M is a constant depending on the functions \( g_1 \) and \( g_2 \) of (5.3) and (5.5) and on the bound of \( ||u_k||_{L^p} \). Hence \( A(u_k) \rightarrow A(u) \) strongly as \( k \rightarrow \infty \). Boundedness of A also follows; indeed, since \( A(0) = 0 \), we have

\[ ||A(v)||_{L^p} \leq \begin{cases} g_1(v) ||v||_{L^p} & p \geq 2 \\ g_2(v) ||v||_{L^{p-1}}^{p-1} & 1 < p < 2 \end{cases} \]

(5.17)

where

\[ g_1(v) = \beta_3 2^{p-2} ||v||_{L^p}^{p-2} + C_1 |a| (1 + ||v||_{L^p}) \]

\[ g_2(v) = \sqrt{5} + C_1 |a| (1 + ||v||_{L^p}) ||v||_{L^p}^{2-p} \]

and \( C_1 \) is the constant appearing in (5.3) and (5.5). \( \square \)

We note that the continuity of A implies that A is also hemi-continuous at every \( v \in W_o^{1,p}(\Omega) \).
Theorem 5.3. Let $A$ be the operator defined in (2.5). Then $A$ is coercive from $W_{0}^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$ whenever the following conditions hold:

(i) $\alpha > 0$, $1 \leq p \leq \infty$

(ii) $\alpha < 0$, $p > 2$

(iii) $\alpha < 0$, $p = 2$, and $1 - |\alpha| C_{2} > 0$

$$C_{2} = \operatorname{mes}(\Omega)^{1-2/p} (d(\Omega))^{2/2}$$

where $d(\Omega) = \max \operatorname{dist}(x, y)$ is the diameter of $\Omega$.

Proof: Clearly

$$\langle A(v), v \rangle = \|v\|_{1,p}^{p} + \alpha \int_{\Omega} \frac{v^{2}}{\sqrt{1 + |\nabla v|^{2}}} \, dx$$

(i) If $\alpha > 0$, then

$$\langle A(v), v \rangle \geq \|v\|_{1,p}^{p}$$

Hence $\langle A(v), v \rangle \rightarrow +\infty$ as $\|v\|_{1,p} \rightarrow \infty$ for any $p$, $1 \leq p \leq \infty$.

(ii) If $\alpha < 0$, $p > 2$. Then

$$\langle A(v), v \rangle \geq \|v\|_{1,p}^{p} - |\alpha| \|v\|_{0,2}^{2}$$
For \( p \geq 2 \), \( W^{1,p}_0(\Omega) \hookrightarrow W^{1,2}_0(\Omega) \) and
\[
\|v\|_{1,2} \leq \frac{1}{\text{mes}(\Omega)^2} \frac{1}{p} \|v\|_{1,p}
\]

Also, from Poincaré's inequality,
\[
\|v\|_{0,2} \leq \frac{d(\Omega)}{\sqrt{2}} \|\nabla v\|_{0,2}
\]

where \( d(\Omega) \) is the diameter of \( \Omega \). Hence
\[
\langle A(v), v \rangle \geq \|v\|_{1,p}^p - |\alpha| \text{mes}(\Omega)^{1-\frac{2}{p}} \frac{(d(\Omega))^2}{2} \|v\|_{1,p}^2
\]

so that the positive term dominates the growth of \( \langle A(v), v \rangle \) as
\[
\|v\|_{1,p} \to \infty \text{ if } p > 2.
\]

(iii) \( \alpha < 0 \), \( p = 2 \). In this case
\[
\langle A(v), v \rangle \geq \left[ 1 - |\alpha| \text{mes}(\Omega)^{1-\frac{2}{p}} \frac{1}{2}(d(\Omega))^2 \right] \|v\|_{1,2}^2
\]

so that coerciveness is obtained whenever the coefficient on the right side of this inequality is positive.

Theorem 5.4. Let \( A \) be the operator defined in (2.5). Let \( B_\mu(0) \) be the ball of radius \( \mu \) in \( W^{1,p}_0(\Omega) \), i.e.,
\[
B_\mu(0) = \{ w \in W^{1,p}_0(\Omega) : \|w\|_{1,p} \leq \mu \}
\]
For $\alpha \neq 0$, let condition (5.1) of Theorem 5.1 hold. Then for every $u, v \in B_\mu(0)$ and arbitrary $\epsilon > 0$, there exist positive constants $\gamma_1(\epsilon, \mu)$ and $\gamma_2(\epsilon, \mu)$, dependent on $\epsilon$ and $\mu$, such that

(i) for $p \geq 2$

$$\langle A(u) - A(v), u - v \rangle \geq \left[ \frac{1}{2} \right] \| u - v \|_{1,p}^p - |\alpha| C_\alpha \| u - v \|_{0,2}^2 - \gamma_1(\epsilon, \mu) \| u - v \|_{0,2p}^{p'}$$

where

$$C_\alpha = \begin{cases} 0 & \text{if } \alpha > 0 \\ 1 & \text{if } \alpha < 0 \end{cases}$$

and

$$\gamma_1(\epsilon, \mu) = \frac{1}{p'(\epsilon p)^{p'/p}} (|\alpha| C \mu)^{p'}$$

In (5.24), $C$ is the constant appearing in (5.15) and $p' = p/(p-1)$.

(ii) for $1 < p < 2$,

$$\langle A(u) - A(v), u - v \rangle \geq \left[ (p-1)(2\mu)^{p-2} - \epsilon \right] \| u - v \|_{1,p}^2$$

where

$$\gamma_2(\epsilon, \mu) = \frac{1}{4\epsilon} (|\alpha| C \mu)^2$$
Proof: We make use of the decomposition given in (5.6) for \( u, v \in W^{1,p}_{0}(\Omega) \) and observe that

\[
\langle A(u) - A(v), u - v \rangle = \langle A_\alpha(u) - A_\alpha(v), u - v \rangle + \langle A_\alpha(u) - A_\alpha(v), u - v \rangle
\]

\[ p \geq 2 \]

(5.27)

We consider the two cases: \( p \geq 2 \) and \( 1 < p < 2 \).

**Case (i).** \( p \geq 2 \). Application of the inequality (4.3) of Lemma 4.1 yields

\[
\langle A_\alpha(u) - A_\alpha(v), u - v \rangle \geq \left(\frac{1}{2}\right)^{p-1} \|u - v\|^p_{1,p}
\]

(5.28)

For the operator \( A_\alpha \), we have

\[
\langle A_\alpha(u) - A_\alpha(v), u - v \rangle = \alpha \int_{\Omega} \left[ \frac{u}{\sqrt{1 + |\nabla u|^2}} - \frac{v}{\sqrt{1 + |\nabla v|^2}} \right] (u - v) \, dx
\]

\[
= \alpha \int_{\Omega} \left[ \frac{(u - v)^2}{\sqrt{1 + |\nabla u|^2}} + \left( \frac{1}{\sqrt{1 + |\nabla u|^2}} - \frac{1}{\sqrt{1 + |\nabla v|^2}} \right) v(u - v) \right] \, dx
\]

(5.29)

We now apply the inequality (4.8) and observe that the first term need not be considered when \( \alpha > 0 \). Moreover, for \( p \geq 2 \),

\( z \in W^{1,p}(\Omega) \implies z \in L^2(\Omega) \).

Thus,
\[
\langle A_0(u) - A_0(v), u - v \rangle \geq - |\alpha| \left( C_\alpha \|u - v\|_{0,2}^2 \right) + \int_\Omega |v(u-v)| |v| \|u-v\| \, dx \]  

(5.30)

where \( C_\alpha = 0 \) if \( \alpha \geq 0 \) and \( C_\alpha = 1 \) if \( \alpha < 0 \).

For \( p > \frac{3n}{n+2} \), \( W^{1,p}(\Omega) \) is embedded in \( L^{2p'}(\Omega) \), \( p' = p/(p-1) \).

Applying the Hölder inequality to (5.30), we get

\[
\langle A_0(u) - A_0(v), u - v \rangle \geq - |\alpha| \left( C_\alpha \|u - v\|_{0,2}^2 \right) + \|v(u-v)\|_{0,p} \|v\|_{0,2p'} \|u-v\|_{0,2p'}
\]  

(5.31)

By the imbedding theorems (recall (5.15)), there exists a constant

\( C \) such that

\[
\|v\|_{0,2p'} \leq C \|v\|_{1,p}
\]  

(5.32)

For \( u, v \in B_u(0) \), we then have

\[
\langle A_0(u) - A_0(v), u - v \rangle \geq - |\alpha| \left( C_\alpha \|u - v\|_{0,2}^2 \right) - |\alpha| \left( C_u \|u - v\|_{1,p} \|u-v\|_{0,2p'} \right)
\]  

(5.33)

We now apply the following form of Young's inequality: For \( a, b \in \mathbb{R}^+ \), and arbitrary \( \varepsilon > 0 \),
\[ ab \leq \varepsilon a^p + \frac{1}{p'(\varepsilon p)^{p'/p}} b^{p'} , \quad \frac{1}{p} + \frac{1}{p'} = 1 \]  

(5.34)

Hence

\[
\langle A_\alpha(u) - A_\alpha(v), u - v \rangle \geq -|\alpha| C_\alpha \|u - v\|_{0,2}^2 
- \varepsilon \|u - v\|_{1,p}^p - \gamma_1(\varepsilon, \mu) \|u - v\|_{0,2, p'}^p
\]  

(5.35)

where \( \gamma_1(\varepsilon, \mu) \) is precisely (5.24).

We obtain (5.22) by combining (5.28) and (5.35).

**Case (ii).** \( 1 < p < 2 \). By making use of inequality (4.4), following the steps used by Glowinski and Marroco [ ], we have

\[
(p-1) \|u - v\|_{1,p}^2 \leq \langle A_p(u) - A_p(v), u - v \rangle \left( \|u\|_{1,p} + \|v\|_{1,p} \right)^{2-p}
\]  

(5.36)

For \( u, v \in B_{\mu}(0) \), we have

\[
\langle A_p(u) - A_p(v), u - v \rangle \geq (p-1) (2\mu)^{p-2} \|u - v\|_{1,p}^2
\]  

(5.37)

The arguments for \( A_\alpha \) follow those given in case (i). For \( p \geq \frac{3n}{n+2} \), we get

\[
\langle A_\alpha(u) - A_\alpha(v), u - v \rangle \geq -|\alpha| C_\alpha \|u - v\|_{0,2}^2 
- \varepsilon \|u - v\|_{1,p}^2 - \gamma_2(\varepsilon, \mu) \|u - v\|_{0,2, p'}^2
\]  

(5.38)

where \( C_\alpha \) is given by (5.23) and \( \gamma_2(\varepsilon, \mu) \) is given by (5.26). (5.25)
follows by combining (5.37) and (5.38). □

An examination of the last steps in the above proof lead to the following useful result.

**Corollary 5.4.1.** Let the conditions of Theorem 5.4 hold and in addition let \( v \) be bounded in \( W^{0,q}(\Omega) \). Then \( \| \cdot \|_{0,2p'} \) in (5.22) and (5.25) can be replaced by \( \| \cdot \|_{0,r} \), where

\[
\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1
\]

(5.39) □

**Corollary 5.4.2.** If the condition

\[
p > \frac{3n}{n+2}
\]

(5.40)

holds instead of (5.1), then the operator \( A \) is of the Gårding type. If (5.40) holds, \( W_{o}^{1,p}(\Omega) \) is compactly imbedded in \( L^{2p'}(\Omega) \). □

6. Existence Theorem

We can now collect the results of Section 5 and state conditions under which problem (2.6) has a solution:

**Theorem 6.1.** Let \( A : W_{o}^{1,p}(\Omega) \to W^{-1,p'}(\Omega) \) be the operator defined in (2.5) and let \( \alpha \) and \( p \) satisfy the following conditions.

(i) \( p > \frac{3n}{n+2} \)

(ii) \( \alpha \geq 0 \), \( 1 < p < \infty \)

(iii) \( \alpha < 0 \), \( p > 2 \)

(iv) \( \alpha < 0 \), \( p = 2 \), \( 1 - \frac{1}{2} |\alpha| (\text{mes } \Omega) \frac{1-2}{p} (d(\Omega))^{2} > 0 \)


where \( n \) is the dimension of \( \Omega \) and \( d(\Omega) \) is the diameter of \( \Omega \).

Then for every \( f \in W^{-1,p'}(\Omega) \) there exists a solution \( u \) of the problem (2.6).

**Proof:** By the application of condition (i) and Theorems 5.1 and 5.2, we show that \( A \) is strongly continuous. \( A \) is therefore bounded and hemicon- 
tinuous. Coercivity of \( A \) follows from the conditions (ii) - (iv) and 

Theorem 5.3. By Theorem 5.4, we have a Gårding inequality with 

\( W^{1,p}_0(\Omega) \) compactly embedded in \( L^{2p'}(\Omega) \). Thus by Theorem 3.1, for every 

\( f \in W^{-1,p'}(\Omega) \) there exists at least one \( u \) such that (2.6) holds. \( \square \)

We emphasize that the condition (i) \( p > 3n/(n+2) \) is needed in 

order to assure that \( W^{1,p}(\Omega) \) is compact in \( L^{2p'}(\Omega) \), so that application 

of Hölder's inequality in (5.30) yields the Gårding inequality (5.22) or 

(5.25).

7. Finite Element Galerkin Approximations

Let us first consider a general approximation theorem for problems 

of the type covered by Theorem 3.1. Again, let (3.1) hold and consider the 

abstract problem of finding \( u \in \mathcal{U} \) such that 

\[
\langle A(u), v \rangle = \langle f, v \rangle \quad \forall \ v \in \mathcal{U} \tag{7.1}
\]

where \( f \) is given in \( \mathcal{U}' \). Let \( h \) be a real parameter, \( 0 < h \leq 1 \), and 
suppose that \( \{\mathcal{U}_h\} \) is a family of finite dimensional subspaces of \( \mathcal{U} \) with
the property that

\[ \bigcup_{h} U_h \text{ is everywhere dense in } U \]  \hspace{1cm} (7.2)

For any particular \( h \), we can consider a Galerkin approximation of (7.1) by seeking \( u_h \in U_h \) such that

\[ \langle A(u_h), v_h \rangle = \langle f, v_h \rangle \quad \forall \ v_h \in U_h \]  \hspace{1cm} (7.3)

Assuming (7.3) is solvable for each \( h \), there exists a sequence of Galerkin approximations \( \{ u_h \} \) which, we hope, converges in some sense to a solution of (7.1). Sufficient conditions for convergence of such approximations are listed in the following theorem due to ODEN and REDDY [12].

**Theorem 7.1.** Let conditions (3.1) hold and let \( A \) be a bounded, hemicontinuous, coercive operator from \( U \) into \( U' \). In addition, suppose that

\[ \langle A(u) - A(v), u - v \rangle \geq F(\|u - v\|_U) - H(\mu, \|u - v\|_U) \]

\[ \forall \ u, v \in B_\mu(0) \subset U \]  \hspace{1cm} (7.4)

where \( F: \mathbb{R}^+ \to \mathbb{R}^+ \) is a continuous real-valued function with the property that

\[ F(x) \geq 0 \ , \ F(0) = 0 \Rightarrow x = 0 \ , \ x \in \mathbb{R}^+ \]  \hspace{1cm} (7.5)

\( H(\cdot, \cdot) \) is a continuous, non-negative real-valued function satisfying (3.6), and \( B_\mu(0) \) is an open ball of radius \( \mu \) in \( U \) as in (3.4). Then
(i) There exists at least one solution \( u \in \mathcal{U} \) to problem (7.1) for the operator \( A \).

Moreover, let (7.3) define a Galerkin approximation to (7.1) on a subspace \( U_h \) belonging to a family satisfying (7.2). Then,

(ii) There exists at least one solution \( u_h \in U_h \) to (7.3) \( \forall h \), \( 0 < h < 1 \).

(iii) If \( \{ u_h \} \) is a sequence of solutions to the approximate problem (7.3) obtained as \( h \to 0 \), there exists a subsequence \( \{ u_{h'} \} \) such that

\[
  u_{h'}, \rightharpoonup u \quad \text{weakly in } \mathcal{U}
\]
as \( h' \to 0 \), where \( u \) is a solution of (7.1).

(iv) If \( F \neq 0 \) in (7.4), and \( \{ u_h \} \) is a sequence of solutions of the approximate problems (7.3) obtained as \( h \to 0 \), then there is a subsequence \( \{ u_{h''} \} \) such that

\[
  u_{h''}, \to u \quad \text{strongly in } \mathcal{U}
\]
as \( h \to 0 \), where \( u \) is a solution of (7.1). \( \Box \)

We remark that (i) and (ii) follow immediately from Theorem 3.1.

If \( F = 0 \) in (7.4), the Galerkin approximation (7.3) is still solvable, but only weak convergence of the approximate solutions can be guaranteed.

Let us now turn to the specific problem (2.6). We wish to study Galerkin approximations to this problem which are generated using finite element methods. Let us suppose, for simplicity, that \( \Omega \) is a bounded
convex polygon in \( \mathbb{R}^n \). We introduce a (finite) partition \( \mathcal{Q}_h \) of \( \Omega \) satisfying

\[
G \subseteq \bar{\Omega} \quad \forall G \in \mathcal{Q}_h \cup \quad \bigcup_{G \in \mathcal{Q}_h} G = \bar{\Omega} ,
\]

\[
G, G' \in \mathcal{Q}_h \Rightarrow \text{int } G \cap \text{int } G' = \emptyset
\]

\[\exists G \text{ is Lipschitzian } \forall G \in \mathcal{Q}_h\]

The angles between faces of \( G \) are bounded below by \( \theta_o > 0 \)

We take

\[
h = \max_{G \in \mathcal{Q}_h} \text{dia} (G)
\]

and approximate \( W^{1,p}_0(\Omega) \) by a family of subspaces

\[
U_h = \mathcal{S}_h^{k,r}(\Omega) = \{ v_h : v_h \in C^r(\bar{\Omega}), r \geq 0, v_h|_{\partial \Omega} = 0 \}
\]

\[
\left. v_h \right|_{G} \in P_k(G) \quad \forall G \in \mathcal{Q}_h\}
\]

where \( P_k(G) \) is the space of polynomials of degree \( \leq k \) on \( G \). For second-order problems of the type under investigation, we generally take

\[
U_h = \mathcal{S}_h^{1,0}(\Omega)
\]
Under conditions (7.6) - (7.8), and regular refinements of the mesh, it is known (see, e.g., CIARLET and RAVIART [5], CIARLET [4], or ODEN and REDDY [13]) that the spaces $S_{h}^{k,r}(\Omega)$ have the following interpolation property:

If $u \in W^{s,p}(\Omega) \cap W^{1,p}_{0}(\Omega)$, there exists a $\phi_{h} \in S_{h}^{k,r}(\Omega)$ such that as $h \to 0$,

$$
\|u - \phi_{h}\|_{l,p} \leq C h^{\nu} \|u\|_{s,p}
$$

where $C$ is a positive constant independent of $u$ and $h$. \(7.10\)

In general, $\phi_{h}$ can be taken to be the projection of $u$ onto $S_{h}^{k,r}(\Omega)$. We will assume that (7.10) holds in subsequent discussions.

Returning to (2.6), suppose that $U_{h}$ is a subspace of $W^{1,p}_{0}(\Omega)$ satisfying (7.8) (and (7.10)). A finite element approximation of (2.6) consists of seeking $u_{h} \in U_{h}$ such that

$$
\int_{\Omega} \left[ |v_{u_{h}}|^{p-2} v_{u_{h}} \cdot v_{v_{h}} + \alpha(1 + |v_{u_{h}}|^{2})^{-1/2} v_{h} \right] dx = \langle f, v_{h} \rangle
$$

$\forall v_{h} \in U_{h}$ \(7.11\)

We have:

**Theorem 7.2.** Let (7.8) and (7.10) hold with $\nu > 0$, and let the conditions of Theorem 6.1 hold. Then there exists at least one solution to
(7.11) for every $h > 0$. Moreover, if $\{u_h\}$ denotes a sequence of finite element approximations to (2.6) obtained from (7.11) as $h$ tends to zero, then there exists a subsequence $\{u_{h_k}\}$ which converges strongly to a solution $u$ of (2.6).

**Proof:** This follows immediately from Theorems 6.1 and 7.1 and from the fact that the operator $A$ in (2.6) satisfies (7.4) (by virtue of (5.22) and (5.25)) with $F(x) = \left(\frac{1}{2}\right)^{p-1} - \varepsilon \right) x^p$ for $p \geq 2$ and $F(x) = [(p-1)(2\mu)^{p-2} - \varepsilon] x^2$ for $1 < p < 2$. □

We remark that solutions to (2.6) and (7.11) are not, in general, unique. Indeed, suppose $u_1 \neq u_2$ are solutions, i.e., $A(u_1) = f$ and $A(u_2) = f$ for given $f$ in $W^{-1,p'}(\Omega)$. Then, for example,

$$0 = \langle A(u_1) - A(u_2), u_1 - u_2 \rangle \geq C\|u_1 - u_2\|_1^p - \gamma(\mu)\|u_1 - u_2\|_0^{p'}$$

Then

$$\|u_1 - u_2\|_1^p \leq \left(\frac{C}{\gamma}\right)^{p-p'}$$

from which we cannot conclude that $u_1 = u_2$. When the right side of the Gårding inequality is strictly positive, of course, we then have $u_1 = u_2$. 
As a preliminary to our study of error estimates in the next section, we consider here an auxiliary linear boundary value problem that leads to some useful estimates. Most of the present analysis is based on the mean-value formula,

\[ \langle A(u) - A(v), w \rangle = \langle D(A(\theta u + (1-\theta)v) \cdot (v-u), w \rangle \]

for some \( \theta \in [0,1] \), \( v, w \in W_0^{1,p}(\Omega) \) \( (8.1) \)

Here, for any \( u \in W_0^{1,p}(\Omega) \), \( D(A(u) \cdot v, w) \) is a linear operator from \( W_0^{1,p}(\Omega) \) into \( W^{-1,p'}(\Omega) \) defined by

\[ \langle D(A(u) \cdot v, w) = \lim_{t \to 0} \frac{3}{3t} \langle A(u+tv), w \rangle \] \( (8.2) \)

In the case of the operator \( A \) of (2.5), it is easily shown that the limit in (8.2) exists and that

\[ \langle D(A(u) \cdot v, w) = \int_{\Omega} \left\{ |\nabla u|^{p-2} \nabla u \cdot \nabla w + (p-2) |\nabla u|^{p-4} (\nabla u \cdot \nabla v)(\nabla u \cdot \nabla w) \right. \]

\[ - \alpha m(u)^3 u \nabla u \cdot \nabla v w + \alpha m(u) \nabla w \right \} dx \] \( (8.3) \)

where, for simplicity in notation, we have denoted

\[ m(u) = (1 + |\nabla u|^2)^{-1/2} \]
To further simplify notation, let $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ and $\eta = (\eta_1, \eta_2, \ldots, \eta_n)$ be vectors in $\mathbb{R}^n$ and let $a_{ij}(u), b_i(u), 1 \leq i, j \leq n$, denote

$$
\begin{align*}
\sum_{i,j=1}^{n} a_{ij}(u) \xi_i \eta_j &= \sum_{i,j=1}^{n} \left[ |\nabla u|^{p-2} \xi_i \eta_i \right. \\
&\left. + (p-2) |\nabla u|^{p-4} u, , \xi_i \eta_j \right]
\end{align*}
$$

(8.4)

$$
\begin{align*}
\sum_{i=1}^{n} b_i(u) \xi_i &= -\sum_{i=1}^{n} \alpha m(u)^3 u, , \xi_i
\end{align*}
$$

where $u, , j = \partial u / \partial x_j$. Then

$$
\langle DA(u) \cdot v, w \rangle = \int_{\Omega} \left\{ \sum_{i,j=1}^{n} a_{ij}(u) v, , \xi_i \eta_j + \sum_{i=1}^{n} b_i(u) v, , \xi_i + \alpha m(u) v \right\} dx
$$

(8.5)

Our local analysis will be based on the following assumptions:

(i) $u^0$ is a solution of (2.5) such that $u^0 \in W^{1,\infty}(\Omega) \cap W_0^{1,p}(\Omega)$, and, in particular, $a_{ij}(u), b_i(u) \in L^\infty(\Omega), 1 \leq i, j \leq n$ for all $u$ in a neighborhood $N(u^0)$ of $u^0$.

(ii) If $p > 2$, then there exists a constant $\mu > 0$ such that

$$
\mu \leq |\nabla u|^{p-2} + \alpha m(u) - \mu^2/[4(p-2) |\nabla u|^{p-4}] < \infty
$$

a.e. in $\Omega$ for all $u \in N(u^0)$.

(8.6) (continued)
(iii) If \( p = 2 \), then there exists a constant \( \mu > 0 \) such that

\[
\mu \leq \alpha (m(u) - \frac{\alpha}{4} u^2 |v|^2) < \infty
\]

a.e. in \( \Omega \) for all \( u \in N(u^0) \).

We first describe some algebraic results for the linearized operator \( DA(u) \). Using Young's inequality,

\[
ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2 \quad \forall \varepsilon > 0
\]

we have

\[
\langle DA(u) \cdot v, v \rangle \geq \int_{\Omega} \left\{ |\nabla u|^{p-2} |\nabla v|^2 + (p-2) |\nabla u|^{p-4} |\nabla u \cdot \nabla v|^2 - \alpha \varepsilon |\nabla u \cdot \nabla v|^2 + \alpha (m(u) - \frac{u^2}{4\varepsilon}) v^2 \right\} \, dx
\]

(8.7)

By assumption (i), \( \varepsilon \) can be taken so that

\[
(p-2) |\nabla u|^{p-4} - \alpha \varepsilon = 0 \quad \text{i.e.,} \quad \varepsilon = \frac{1}{\alpha} (p-2) |\nabla u|^{p-4}
\]

for \( p > 2 \). If condition (ii) of (8.6) holds, then there exists a constant \( \rho \) such that

\[
\langle DA(u) \cdot v, v \rangle \geq \rho \int_{\Omega} |\nabla v|^2 \, dx
\]

(8.8)

For \( p = 2 \),

\[
\langle DA(u) \cdot v, v \rangle \geq \int_{\Omega} \left\{ (1 - \alpha \varepsilon |\nabla u|^2) |\nabla v|^2 + \alpha (m(u) - \frac{u^2}{4\varepsilon}) v^2 \right\} \, dx
\]

(8.9)
Thus, if condition (iii) of (8.6) holds, there exists a positive constant \( \hat{\mu} > 0 \) such that

\[
\langle DA(u) \cdot v, v \rangle \geq \hat{\mu} \int_{\Omega} |\nabla v|^2 \, dx
\]  

(8.10)

Moreover, if condition (i) of (8.6) holds,

\[
\langle DA(u) \cdot v, w \rangle \leq C(|v|_{0,\infty}, |u|_{0,\infty}, p, \alpha, n) \|v\|_{1,2} \|w\|_{1,2}
\]  

(8.11)

where \( C \) is a bounded continuous function of its arguments. Then the bilinear form

\[
B(v, w) = \langle DA(u) \cdot v, w \rangle
\]  

(8.12)

is continuous on the Hilbert space \( W^{1,2}_0(\Omega) \times W^{1,2}_0(\Omega) \). By (8.8) and (8.10), the bilinear form \( B \) is coercive on the Hilbert space \( W^{1,2}_0(\Omega) \times W^{1,2}_0(\Omega) \) under the assumptions (8.6). Thus, by the Lax-Milgram theorem, there exists a unique solution \( \eta \in W^{1,2}_0(\Omega) \) to the problem

\[
B(v, \eta) = f(v) \quad \forall v \in W^{1,2}_0(\Omega)
\]  

(8.13)

for every functional \( f \) defined on \( W^{1,2}_0(\Omega) \). Moreover, by standard regularity results (e.g., NEČAS [8]) for coercive bilinear forms, the solution \( \eta \) belongs to \( W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega) \) whenever

\[
f(v) = \int_{\Omega} \psi \, w \, dx, \quad \psi = L^2(\Omega)
\]  

(8.14)

That is,

\[
\|\eta\|_{2,2} \leq C(\Omega, \hat{\mu}) \|\psi\|_{0,2}
\]  

(8.15)
where the constant $C$ depends upon only the domain $\Omega$ and the coercivity constants $\rho$ or $\hat{\mu}$ in (8.8) or (8.10). Here the boundary of the domain $\Omega$ is assumed to be sufficiently smooth; for example, the boundary of class $C^2$.

**Theorem 8.1.** Let conditions (8.6) be satisfied and suppose that the boundary of the domain $\Omega$ is $C^2$. Then the problem (8.13) has a unique solution $\eta \in W^{1,2}_o(\Omega) \cap W^{2,2}(\Omega)$, which satisfies the estimate (8.15), for every $\psi \in L^2(\Omega)$.

Let us now consider the local behavior of the error in the finite element approximation under the assumptions of Theorem 8.1. Suppose that $u^0$ is a solution to (2.6) and that $u^0_h$ is a solution of the finite element approximation (7.11). Then, from the mean value theorem (8.1),

\[
\langle A(u^0) - A(u^0_h), w \rangle = \langle DA(z_{\theta h}) \cdot e_h, w \rangle
\]

\[
z_{\theta h} = \theta u^0 + (1 - \theta) u^0_h, \quad \theta \in [0,1]
\]

\[
e_h = u^0 - u^0_h
\]

for any $w \in W^{1,p}_o(\Omega)$. Then, for $\psi \in L^{q'}(\Omega)$, $q' \geq 2$, we know that there exists a unique $\eta \in W^{1,2}_o(\Omega) \cap W^{2,2}(\Omega)$ such that

\[
\langle DA(z_{\theta h}) \cdot w, \eta \rangle = \int_{\Omega} \psi w \, dx \quad \forall \psi \in L^{1,2}_o(\Omega)
\]

(8.17)

if conditions (i), (ii), and (iii) in (8.6) are satisfied for $z_{\theta h} = u^0 + \theta(u^0_h - u)$, where $\theta$ is an arbitrary number in $[0,1]$. If
u^0 \in \mathring{W}_0^{1,\infty} \cap \mathring{W}_0^{1,p}(\Omega), z_{\theta h} \in \mathring{W}_0^{1,\infty} \cap \mathring{W}_0^{1,p}(\Omega) \text{ for any } \theta \in [0,1]. \text{ If } u^0 \text{ satisfies condition (ii) and (iii) in (8.6), for } u_h^0 \text{ sufficiently close to } u^0, z_{\theta h} \text{ also satisfies condition (ii) and (iii) for any } \theta \in [0,1].

Thus, we may proceed as follows:

\begin{enumerate}
\item For a solution \( u^0 \) of (2.5) satisfying (i) of (8.6), let \( \varepsilon > 0 \) exist such that every \( w \in B_\varepsilon(u^0) = \{ v \in \mathring{W}_0^{1,p}(\Omega) : ||v - u^0||_{1,p} < \varepsilon \} \) satisfies (ii) of (8.6) if \( p > 2 \) or (iii) of (8.6) if \( p = 2; a_{ij}(w), b_i(w) \in L^\infty(\Omega) \).

\item \( \{ u_h^0 \} \) is a sequence of finite element approximations converging strongly to \( u^0 \) in \( \mathring{W}_0^{1,p}(\Omega) \) as \( h \to 0 \), (the existence of such a sequence being guaranteed by Theorem under the conditions of Theorem 6.1).
\end{enumerate}

Then there exists an \( h_{\varepsilon} > 0 \) such that for all \( h < h_{\varepsilon} \), \( z_{\theta h} \in B_\varepsilon(u^0) \) for any \( \theta \in [0,1] \). Hence, whenever (8.18) holds, we can take \( w = e_h \in \mathring{W}_0^{1,p}(\Omega) \subset \mathring{W}_0^{1,2}(\Omega) \), \( p > 2 \), and set
\[
\langle DA(z_{\theta h}) \cdot e_h, \eta \rangle = \int_{\Omega} \psi e_h \, dx
\]

If \( q' > p \), from (8.16),
\[
\int_{\Omega} \psi e_h \, dx \leq \langle A(u^0) - A(u_h^0), \eta \rangle
\]

Using the orthogonality condition,
\[
\langle A(u^0) - A(u_h^0), \eta_h \rangle = 0, \quad \forall \eta_h \in U_h
\]
we get
\[ \int_{\Omega} \psi \, e_h \, dx \leq \langle A(u^0) - A(u_h^0), \eta - \eta_h \rangle \]
for every \( \eta_h \notin U_h \).

By the continuity of \( A \) (recall (5.3))
\[ \int_{\Omega} \psi \, e_h \, dx \leq c_1(u^0, u_h^0) \| e_h \|_{1,p} \| \eta - \eta_h \|_{1,p} \]
(8.19)

Here we have used the fact that \( W_0^{1,p}(\Omega) \subset W^{2,2}(\Omega) \cap W_0^{1,p}(\Omega) \). Taking \( \eta_h \) as the interpolant \( \Pi_h \eta \) of \( \eta \), yields
\[ \int_{\Omega} \psi \, e_h \, dx \leq c_1(u^0, u_h^0) \| e_h \|_{1,p} \| \eta \|_{2,2} \]

Then, using (8.15) and (8.19),
\[ \| e_h \|_{0,q} = \sup_{\psi \in L^{q'}(\Omega)} \frac{\langle \psi, e_h \rangle}{\| \psi \|_{0,q'}} \left( \frac{1}{q} + \frac{1}{q'} = 1 \right) \]
\[ \leq \frac{c_1(u^0, u_h^0)}{\| e_h \|_{1,p}} \frac{\| \eta \|_{2,2}}{\| \psi \|_{0,2}} \]
\[ \leq C c_1(u^0, u_h^0) \| e_h \|_{1,p} h \]
(8.20)

We note that if \( u_h^0 \) converges strongly to \( u^0 \) in \( W_0^{1,p}(\Omega) \), then \( \| u_h^0 \|_{1,p} \) is bounded and there exists a positive number \( g(u^0) \) such that
Theorem 8.2. Let the conditions (8.18) hold, and let $q' > p > 2$, $q = q'/(q' - 1)$. Then,

$$
\left\| e_h \right\|_{0,q} \leq C g(u^0) h \left\| e_h \right\|_{1,p}
$$

(8.22)

where $C$ is a constant independent of $u^0$, $u^0_h$, and $h$, $g(u^0)$ satisfies (8.21), and $e_h = u^0 - u^0_h$ is the error in the finite element approximation of the solution $u^0$ of (2.6).

9. Error Estimates

In this section we make use of Theorems 5.1, 5.4 and 8.2 in establishing the error estimates for the finite element approximations. We make the following assumptions.

(i) Let $u$ be the solution of (2.6), then

$$
u \in W^{1,p}_0(\Omega) \cap W^{s,p}(\Omega), \quad s \geq 2.
$$

(ii) Let $u_h$ be the solution of the approximate problem (7.11), then $u_h \in U_h \cap W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$ and there exists a bound $\tilde{\mu}$ such that for every $h$,

$$
\left\| u_h \right\|_{L^\infty(\Omega)} \leq \tilde{\mu}
$$

(9.1)
Theorem 9.1. Let \( A : W^{1,p}_0(\Omega) \rightarrow W^{-1,p'}(\Omega) \) be the operator defined in (2.5). Let the assumptions (9.1) and the conditions of Theorem 8.2 hold. Then for arbitrary \( \varepsilon, \delta > 0, \varepsilon + \delta < (1/2)^{p-1} \) there exist constants \( C_1(p, \delta, \varepsilon, u), C_2(p, \delta, \varepsilon, u) \) and \( C_3(p, \delta, \varepsilon, u, \tilde{\mu}) \) such that as \( h \rightarrow 0 \),

\[
\| e_h \|_{1,p} \leq C_1 \| u \|_{s,p}^{1/(p-1)} h^{\nu/(p-1)} + C_2 h^{2/(p-2)} + C_3 h^{1/(p-2)}
\]

\[
= o\left( h^{\nu/(p-1)} + h^{1/(p-2)} \right) \quad (9.2)
\]

where \( \nu \) is defined in (7.10) and

\[
C_1 = \left[ (C g(u) \| u \|_{s,p})^{p'/p} (\delta p')^{p'/p} \left( \frac{1}{2} \right)^{p-1} - \varepsilon - \delta \right]^{1/p} \quad (9.3)
\]

\[
C_2 = \left[ p \left( |\alpha| C g(u)^2 \right)^{p/(p-2)} (\delta p')^{2/(p-2)} \left( \frac{1}{2} \right)^{p-1} - \varepsilon - \delta \right] \quad (9.4)
\]

\[
C_3 = \left[ (p-2) \left( \tilde{\gamma}_1(\varepsilon, \tilde{\mu}) (C' g(u))^{p'} \right)^{(p-1)/(p-2)} / (p-1) (\delta (p-1))^{1/(p-2)} \left( \frac{1}{2} \right)^{p-1} - \varepsilon - \delta \right]^{1/p} \quad (9.5)
\]

\[
\tilde{\gamma}_1(\varepsilon, \tilde{\mu}) = \frac{1}{p' (\varepsilon p')} \left( |\alpha| \tilde{\mu} \right)^{p'} \quad (9.6)
\]

\( C, \tilde{C} \) and \( C' \) are generic constants appearing in (7.10), (8.22). \( g(u) \) is a positive number defined in (8.21).
Proof: From the step (5.30), for \( v = u_h \), under assumption (9.1), we get

\[
\langle A_0(u) - A_0(u_h), u - u_h \rangle \geq -|\alpha| C_\alpha \|u - u_h\|_{0,2}^2 - |\alpha| \tilde{\mu} \|u - u_h\|_{1,p} \|u - u_h\|_{0,p}.
\]

(9.7)

By the application of Young's inequality (5.34), we get

\[
\langle A(u) - A(u_h), u - u_h \rangle \geq \left[ \frac{1}{2} 2^{p-1} - \varepsilon \right] \|u - u_h\|_{1,p}^p - |\alpha| C_\alpha \|u - u_h\|_{0,2}^2
\]

\[
- \tilde{\gamma}_1(\epsilon, \tilde{\mu}) \|u - u_h\|_{0,p}^p.
\]

(9.8)

where \( \tilde{\gamma}_1(\epsilon, \tilde{\mu}) \) is defined by (9.6).

By the orthogonality property,

\[
\langle A(u) - A(u_h), u \rangle = \langle A(u) - A(u_h), u - v_h \rangle \text{ } \forall \text{ } v_h \in U_h \tag{9.9}
\]

Thus choosing \( v_h = \tilde{u}_h \), the interpolant of \( u \), and using (5.2), we get

\[
g_1(u, u_h) \|e_h\|_{1,p} \|u - \tilde{u}_h\|_{1,p} \geq \left[ \frac{1}{2} 2^{p-1} - \varepsilon \right] \|e_h\|_{1,p}^p
\]

\[- |\alpha| C_\alpha \|e_h\|_{0,2}^2 - \tilde{\gamma}_1(\epsilon, \tilde{\mu}) \|e_h\|_{0,p}^p.
\]

(9.10)

Rearranging the terms and introducing (7.10) and (8.22),
\[
\left(\frac{1}{2}\right)^{p-1} - \varepsilon \leq \frac{1}{p/(p-2) + (p-2)/(p-1)} \left[ a \, c_\alpha (c \, g(u))^2 \right]^{p/(p-2) + 2p/(p-2)} h^{1/(p-2)} \\
+ \frac{p}{(p-2)(\delta p/2) + (p-2)/(p-1)} \left[ \gamma_1 (\varepsilon, \bar{u}) (c \, g(u))^p \right]^{(p-1)/(p-2)} h^{p/(p-2)}
\]

where \( p' = p/(p-1) \).

Then

\[
\|e_h\|_{1,p} \leq C_1 \|u\|_{s,p}^{1/(p-1)} h^{\nu/(p-1)} + C_2 h^{2/(p-2)} + C_3 h^{1/(p-2)}
\]

where \( C_1(p, \delta, \varepsilon, u) \), \( C_2(p, \delta, \varepsilon, u) \) and \( C_3(p, \delta, \varepsilon, \bar{u}, u) \) are constants defined by (9.3), (9.4) and (9.5), respectively. \( \Box \)
Remark 9.1: In case of piecewise linear finite element approximations for $u$ bounded in $W^{2,p}(\Omega)$, we have

$$\|e_h\|_{1,p} = 0 \cdot h^{1/(p-1)} \quad (9.13)$$

Remark 9.2: $L^p$-estimates can be obtained by introducing the $W^{1,p}$-estimate of (9.2) into (8.22).

Acknowledgement: The support of this work by U.S. Air Force Office of Scientific Research under Contract F-49620-78-C-0083 is gratefully acknowledged.

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