ON THE CONVERGENCE AND ACCURACY
OF FINITE ELEMENT APPROXIMATIONS
IN NONLINEAR ELASTICITY

J. T. ODEN

Texas Institute for Computational Mechanics
The University of Texas at Austin
Abstract

In this paper, we examine some new results and some open questions in the theory of finite element approximations of nonlinear boundary-value problems in finite elasticity.

Introduction

A constructive existence theory for linear elliptic boundary-value problems in elasticity lies at the foundation of the mathematical theory of finite element approximations of such problems. It provides a means for studying the consistency and stability of approximate methods and, in general, it provides many of the tools necessary for studying convergence and for developing error estimates. By the same token, the absence of a constructive existence and regularity theory for nonlinear elasticity problems stands as the principal barrier in the way of developing a complete theory of finite element approximations for these problems.

However, while much remains to be learned about existence theory for general problems in finite elasticity, some progress has been made in this subject in recent months, and these new results have already shed some light on the qualitative behavior of certain classes of nonlinear elastostatics problems. The objective of this presentation is to describe some of these new results and to identify some of the major issues that remain open.

Boundary-Value Problems in Finite Elasticity

Let us begin by considering a class of boundary value problems in finite elasticity involving the deformation of a homogeneous hyperelastic body subjected to body forces $\rho_0 f$, $\rho_0$ being the mass density in a reference configuration, and surface tractions $\mathbf{s}_0$. The total potential energy evaluated at a displacement field $u$ is given by

$$\Pi(u) = \int_\Omega \left( \rho_0 \sigma(u) - \rho_0 f \cdot u \right) d\mathbf{v} - \int_{\partial\Omega} \mathbf{s}_0 \cdot u \, d\mathbf{s} \quad (1)$$
wherein

\[ \sigma(u) = \text{the strain energy function} \]
\[ \Omega = \text{an open bounded region in } \mathbb{R}^3; \Omega \text{ is homeomorphic to the interior of the set of particles comprising the body and is the domain of} \]
\[ \text{the displacement field } u = u(x), \quad x \text{ being a point (particle)} \]
\[ \text{in } \Omega \]
\[ dV, ds = \text{material elements of volume and surface area in the reference configuration} \]
\[ \partial \Omega = \text{the boundary of } \Omega; \quad \partial \Omega = \partial \Omega_1 \cup \partial \Omega_2, \quad \text{where the displacement } u = 0 \]
\[ \partial \Omega_1 \]

We will consider cases in which the displacement fields \( u \) are elements of a reflexive Banach space \( \mathcal{U} \) equipped with a norm \( ||u|| \). Frequently, (e.g. when \( \sigma \) is given as a polynomial in invariants of appropriate deformation measures) \( \mathcal{U} \) will be the Sobolev space

\[ \mathcal{W}^{1,p}(\Omega) = \{ u; \ u_i, \partial u_i/\partial x_j \in L^P(\Omega) \] \[ i, j = 1, 2, 3, \ 1 < p < \infty \} \]

equipped with the "energy" norm

\[ ||u||_{1, p} = \left\{ \int_{\Omega} [(u \cdot u)^{p} + (\nabla u \cdot \nabla u)^{p}] \, dv \right\}^{1/p} \]

where \( \nabla u : \nabla u = \text{trace } (\nabla u)^T \nabla u \) and \( \nabla \) is the material gradient (e.g. \( (\nabla u)_{ij} = \partial u_i/\partial x_j \) in Cartesian components)

In general, \( \mathcal{U} \) must have the property that

\[ (1) \quad ||\Pi(u)|| < \infty \text{ for all } u \in \mathcal{U} \]
\[ (2) \quad u \in \mathcal{U} \implies u = 0 \text{ on } \partial \Omega_1 \]

However, the functional \( \Pi \) in (1) is not defined on all of \( \mathcal{U} \); rather, \( \Pi \) is defined on a complicated subset \( K \) of \( \mathcal{U} \). For compressible materials, \( K \) describes the constraint of local invertibility:

\[ K = \{ u \in \mathcal{U} : J(u(x)) > 0 \quad \text{a.e. in } \Omega \} \]

Here \( J \) is the determinant of the deformation gradient
\[ J = \det (1 + V_u) \] (6)

1 being the unit tensor. For incompressible materials,

\[ K = \{ y \in U : J(y(x)) - 1 = 0 \text{ a.e. in } \Omega \} \] (7)

To further complicate matters, \( \sigma \) should possess the singular behavior

\[ \sigma \to +\infty \text{ as } J \to 0 \] (8)

The classical minimization problem in nonlinear elasticity is to find the minima of \( \Pi \) in \( K \); i.e. find \( \psi \in K \) such that, for given \( \rho_0 \), \( f \) and \( S_0 \),

\[ \Pi(\psi) = \inf_{\psi \in K} \Pi(\psi) \] (9)

If \( \Pi \) is differentiable on \( K \) in the sense of Fréchet (see, e.g., Oden and Reddy (1976a)), and \( K \) is given by (5), then (9) leads to the abstract boundary value problem:

\[ \langle A(u), \psi \rangle = \langle F, \psi \rangle \text{ for every } \psi \in K \] (10)

where \( \langle \cdot, \cdot \rangle \) denotes duality pairing on \( U' \times U \) and

\[ \langle A(u), \psi \rangle = \int_{\Omega} \frac{\partial \sigma}{\partial V_u} : V \psi \, dv \] (11)

\[ \langle F, \psi \rangle = \int_{\Omega} \rho_0 \psi \cdot \psi \, dv + \int_{\partial \Omega} S_0 \psi \cdot V \, ds \]

Of course, \( \partial \sigma / \partial V_u \) corresponds to the first Piola-Kirchhoff stress tensor. If \( K \) is given by (6), several alternatives are possible. If we introduce a Lagrange multiplier \( p \), then (9) is converted to the problem of finding a pair \( (\psi, p) \in U \times M \) such that

\[ \inf_{u \in U} \sup_{q \in M} L(u, q) = L(v, p) \] (12)

\[ L(u, q) = \Pi(u) + (J(u) - 1, p) \]

where \( M \) is a subset of a reflexive Banach space \( V \) and \( (\cdot, \cdot) \) denotes duality pairing on \( V' \times V \). Alternatively, we could consider the problem of finding \( \psi \) such that
\[ \Pi(u) = \min_{\mathcal{U}} \Pi_e(u); \quad \Pi_e(u) = \Pi(u) + \frac{1}{\varepsilon} P(u) \]

\[u \in \mathcal{U} \]

where \( \varepsilon > 0 \) and \( P \) is a penalty functional corresponding to (6). Then \( \psi_e \to \psi \), as \( \varepsilon \to 0 \). Formulation (12) leads to the system

\[ \mathinner{\left< A(u), v \right> + \left< B(u), p \right>} = \mathinner{\left< F, v \right>} \quad \text{for all } v \in \mathcal{U} \]

\[ (J(u) - 1, q) = 0 \quad \text{for all } q \in \mathcal{M} \]

where

\[ \left< B(u,p), v \right> = \int_{\Omega} p \text{adj} \left( 1 + \nabla u \right)^T : v \, dv \]

In (15), \( \text{adj} \) denotes the transpose of the cofactors of a tensor \( A \).

Some Difficulties and Open Questions

Different methods must be used to determine the existence of solutions of each of the different formulations (9), (10), (12), (13), and (14). All present formidable mathematical problems which have not been completely resolved.

1. Problem (9): The generalized Weierstrass minimization theorem (e.g. Vainberg (1973)) asserts that \( \Pi \) achieves its minimum on a weakly sequentially closed (unbounded) set \( K \in \mathcal{U} \) if \( \Pi \) has a growth property with respect to a point in \( K \) and if \( \Pi \) is weakly lower semicontinuous. The set \( K \) in (5) is not weakly sequentially closed, and the establishment of the weak lower semicontinuity of \( \Pi \) is far from trivial. The problem concerning \( K \) is simplified if, for some \( \varepsilon > 0 \), \( K \) can be chosen as the set \( \{ u \in \mathcal{U} : J(u(x)) \geq \varepsilon \text{ a.e. in } \Omega \} \), but, apparently, this can be justified only when \( u \) is sufficiently regular, and no regularity theory exists for nonlinear problems of this type.

2. Problem (10): There is still the problem of showing that \( \Pi \) is differentiable on \( K \) and that (8) is equivalent to (10). This is open (see Ball (1977)). The existence of solutions to (10) has not been established. However, some sufficient conditions for similar problems were proved by Oden (1978), but a proof that these conditions hold for specific elasticity problems does not exist.
3. Problems (12) and (13): Minimax methods for saddle point problems on convex sets are known (Ekeland and Temam (1976)), but some generalizations of existing theory are needed to handle (12). If (12) \(\Rightarrow\) (14), regularization methods can also be considered. No results are yet available. The principal difficulties are in proving the boundedness of the Lagrange multipliers (hydrostatic pressures) \(p\). A sufficient condition is that there exist a constant \(C > 0\) such that

\[
\sup_{v \in U} \left| \int_{\Omega} p \text{adj}(1 + \nabla v)^T : \nabla v \, dv \right| \geq C \|p\|_{\Omega}^\alpha
\] (16)

\(\alpha > 1\), but a proof that this condition holds has never been found for general elastostatics problems.

4. Problem (13): We have shown (unpublished) that solutions to (13) exist and converge weakly to solutions of (9) whenever \(\epsilon \to 0\). The difficulties arise, when \(K\) is given by (6), in determining \(p\), the hydrostatic pressure. Let \((\nu_\epsilon, p_\epsilon)\) be a solution of the penalized problem. Then the existence of a solution \((\nu, p)\) hinges on the convergence of \(\int_{\Omega} p_\epsilon \text{adj}(1 + \nabla \nu_\epsilon)^T : \nabla \nu_\epsilon \, dv\) to \(\int_{\Omega} p \text{adj}(1 + \nabla \nu)^T : \nabla \nu \, dv\) for all \(\nu \in U\). The convergence of this integral has never been proved (so far as we know). Similar difficulties are encountered in regularization methods.

5. A local analysis of, say, Problem (10) is somewhat easier but less general. Nevertheless, local existence theorems devoid of gross assumptions on the local smoothness of solutions are not available. Recently Oden and Reddy (1978) have constructed local uniqueness theorems for a class of nonlinear elasticity problems under some assumptions on the global regularity of solutions. In local analysis, some of the more important problems that await answers concern the multiplicity of solutions for fixed data (bifurcation phenomena) and the stability of solutions on post-critical equilibrium paths.

Approximations

Constructive existence theory, such as that embodied in the compactness methods of Brezis (1968) and Lions (1969), provide the most natural guides for constructing an approximation theory for problems of the type in (10). If \(U\) is the Sobolev space \(W^{1,p}(\Omega)\) and if \(u\) is an element of \(W^{1,p}(\Omega)\), it is known (see Oden and Reddy (1976b)) that finite element interpolants \(\tilde{u}_h\) of \(u\) can be constructed satisfying the asymptotic estimate
\[
\left\| u - u_h \right\|_{1,p} \leq C h^{\mu} \| u \|_{\ell,p}
\]
where \( C \) is a positive constant, \( h \) is the mesh parameter, and \( k \) is the degree of the polynomials used in constructing the interpolant \( u_h \).

We will now outline the remarkable parallel between the study of approximations of (10) and the study of existence of solutions to (10).

1. Suppose \( U \) is a separable reflexive Banach space and \([\phi_1, \phi_2, \ldots] \) is a countable everywhere dense set (a basis) of \( U \). The finite set \([\phi_1, \ldots, \phi_m] \) is a basis for a finite-dimensional subspace \( U_m \) of \( U \). We consider the finite-dimensional approximation of (10):
\[
\langle A(u_h), \phi_j \rangle = \langle F, \phi_j \rangle \quad j = 1, 2, \ldots, m
\]
In finite element theory, we can choose a family of subspaces \( \{U_h\}_{0 < h < 1} \) by (17) such that \( U_h \) is everywhere dense in \( U \). Then setting \( U_h = U_m \) gives a system such as (18).

2. Problem (18) has a solution if
   
   (i) \( A \) is coercive; i.e.
   \[
   \lim_{\| u \|_{U} \to \infty} \frac{\langle A(u), u \rangle}{\| u \|_{U}} = +\infty,
   \]
   and

   (ii) \( A : U_m \to U_m' \) is continuous

   This follows from the classical Brouwer fixed point theorem.

3. Condition (19) implies that the sequence \( \{u_h\} \) of solutions \( u_h = u_m \) is bounded and, therefore, there exists a subsequence \( \{u_{m}^{h}\} \) which converges weakly to an element \( u \in U \).

4. If \( A \) is bounded \( (A : U \to U') \), then \( \{A(u_{m})\} \) converges weakly to an element \( \chi \in U' \). Moreover,
\[
\langle \chi, v \rangle = \langle F, v \rangle \quad \text{for all } v \in U
\]
5. Normally $A$ is hemicontinuous; i.e. the function $\phi(t) = \langle A(u + tv), w \rangle$ is continuous in $t \in \mathbb{R}$.

Up to this point, most of the assumptions made concerning $A$ and $U$ hold for certain classes of nonlinear elasticity problems. The remaining (and most difficult) step is to show that $\chi = A(u)$. Then (10) has a solution and this finite element approximation (18) of (10) converges weakly to this solution as $h \to 0$.

In monotone operator theory, the monotonicity of $A$; i.e. the assumption that $A$ satisfies

$$\langle A(u) - A(v), u - v \rangle \geq 0 \text{ for all } u, v \in U$$

is sufficient to guarantee that $A(u_h) = A(u)$. Unfortunately, the operators $A$ of finite elasticity are not monotone. A generalization of (21) that will work is

$$\langle A(u) - A(v), u - v \rangle \geq C||u - v||^p_U - \phi(u) + \phi(v)$$

where $C \geq 0$, $p > 1$, and $\phi$ is a compact operator from $U$ into itself.

Oden (1978) has shown that if

$$\phi(u) - \phi(v) \leq \gamma||u - v||^\alpha_U$$

for $u$ and $v$ bounded in $U$, $\gamma > 0$, $V$ is a space in which $U$ is compactly embedded, $\alpha > 1$, and $\gamma$ may depend on the bound on $u$ and $v$, then $A$ is pseudo-monotone in the sense of Brezis (1968).

Inequality (22) then becomes

$$\langle A(u) - A(v), u - v \rangle \geq C||u - v||^p_U - \gamma||u - v||^\alpha_U$$

It has been shown by Oden and Reddy (1978), subject to conditions listed in steps 1-5 and (24), that (10) has a solution $u$ and that the finite-element/Galerkin approximations $u_h$ converge weakly to $u$ if $C=0$ and strongly to $u$ if $C > 0$.

The general approximation theorem covering these cases is stated as follows.
A General Existence and Approximation Theorem [Oden and Reddy (1978)]

I. Weakly Convergent Approximations

Let $U$ and $V$ be separable reflexive Banach spaces equipped with norms $\| \cdot \|_U$ and $\| \cdot \|_V$ respectively, and let $U$ be compactly embedded in $V$. Let $A$ be an operator mapping $U$ into its dual $U'$ and let $A$ be bounded, hemicontinuous and coercive. Moreover, let $A$ be such that for every $u$ and $v$ in the ball $B_\mu(0)$ of radius $\mu$ in $U$,

$$\langle A(u) - A(v), u - v \rangle \geq -\theta(\mu, \|u - v\|_V)$$

(25)

where $\theta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is continuous in each argument and

$$\lim_{\theta \to 0} \frac{1}{\theta} \theta(x, \theta y) = 0 \quad x, y \in \mathbb{R}^+$$

(26)

Then, for every $f \in U'$, there is at least one $u \in U$ satisfying (11).

Moreover, let $\{U_h\}_{0 < h < 1}$ be a family of subspaces such that $\bigcup U_h$ is everywhere dense in $U$. Then, for an $U_h \subset U$, there is an $u_h \in U_h$ satisfying (18). Let $\{u_{h_k}\}$ be a sequence of approximate solutions obtained as $h \to 0$. Then there exists a subsequence of approximate solutions $\{u_{h_{k}}\}$ such that as $h_{k} \to 0$

$$u_{h_{k}} \rightharpoonup u \text{ weakly in } U$$

(27)

II. Strongly Convergent Approximations

In addition, if the Gårding inequality (24) holds, then there exists a subsequence of $\{u_{h_{k}}\}$ of approximate solutions such that

$$u_{h_{k}} \to u \text{ strongly in } U$$

(28)

A Priori Error Estimates

Inequality (24) is referred to as a generalized Gårding inequality (see Oden (1978)). To date, it has been possible to show that such an inequality holds only under certain assumptions on the regularity of the functions $u$ and $v$ appearing in it. However, when these regularity assumptions are valid, (24) leads to a global a priori estimate of the error in finite element approximations of (11).
We will record here an estimate of this type obtained by Oden and Reddy (1978) for a boundary value problem in elastostatics involving fixed boundaries (∂Ω = ∅) and an isotropic compressible material characterized by a strain energy function σ of the form

\[ σ = -E_0 \ln J + E_1 (I-3) + E_2 (I-3)^2 + E_3 (II-3) + E_4 (J-1) \]

where \( E_i \), \( 0 \leq i \leq 4 \), are constants, \( J \) is given by (6), and \( I \) and \( II \) are the first two principal invariants of the Green deformation tensor.

In this case, \( \bar{u} = W_0^1, p(\Omega) \), \( \bar{v} = \bar{V}_p(\Omega) \) \((p=4)\), and finite element approximations are sought in the finite dimensional subspaces

\[ S_h^k(\Omega) = \{ v_h e W_0^1, p(\Omega) : v_h e P_k(\Omega), 1 \leq e \leq E \}, \]

Here \( \{ \Omega_e \}_{e=1}^E \) denotes the partition of \( \Omega \) into \( E \) elements and \( P_k(\Omega) \) is the space of polynomials of degree \( k \) on \( \Omega_e \). The interpolation estimate (17) is assumed to hold.

The major result now available is summarized as follows:

**Error Estimation** (Oden and Reddy (1978)) Let

\[ A : W_0^1, p(\Omega) \rightarrow (W_0^1, p(\Omega))^* = W^{-1}, p(\Omega), p' = p/(p-1) \],

\[ \langle A(u) - A(v), u-v \rangle \geq C_0 \| u-v \|_{1,p}^{p'} - \gamma(\mu) \| u-v \|_{0,p}^{p'} \]

and the inequality

\[ \| A(u) - A(v), z \| \leq C_1(\mu) \| u-v \|_{1,p} \| z \|_{1,p} \]

for every \( u, v, z \) in the ball \( B_\mu(0) \) of radius \( \mu \) in \( W_0^1, p(\Omega) \), where \( C \) is a positive constant, and \( \gamma(\mu) \) and \( C_1(\mu) \) are positive constants dependent on \( \mu \).

Also, for given \( \rho \in C W^{-1}, p'(\Omega) \), let \( w \in W_0^1, p(\Omega) \) and \( v_h \in S_h^k(\Omega) \subset W_0^1, p(\Omega) \) exist.
such that
\[
\begin{align*}
\langle A(\tilde{w}), \tilde{v} \rangle &= \langle f, \tilde{v} \rangle, \\
\langle A(\tilde{w}_h), \tilde{v}_h \rangle &= \langle f, \tilde{v}_h \rangle,
\end{align*}
\]
where \( \tilde{w}_h \) is a member of a family of subspaces of \( W^{1,p}(\Omega) \) satisfying (17).

Then there is a subsequence of \( \{\tilde{w}_h\} \) for \( \tilde{w}_h \), also denoted \( \{\tilde{w}_h\} \), such that
\[
\|\tilde{w}_h\|_{1,p} \to 0 \text{ as } h \to 0
\]
Moreover, let there exist a positive function \( r(h) \), dependent on \( h \), with the property \( r(h) \to 0 \) as \( h \to 0 \), such that
\[
\|\tilde{w}_h\|_{1,p} \leq C_r |\tilde{w}_h|_{1,p} r(h)
\]
for the strongly converging sequence \( \{\tilde{w}_h\} \), and let \( w \) be bounded in \( W^{\ell,p}(\Omega) \), \( \ell \geq 1 \). Then there exist positive constants \( C_3 \) and \( C_4 \) such that
\[
\|\tilde{w}_h\|_{1,p} \leq C_3 \|w\|_{1,p}^{1/p(p-1)} h^{\frac{s}{p-1}} + C_4(r(h))^{1/(p-2)}
\]
where
\[
s = \min_0 (k, \ell-1)
\]
It is shown in Oden and Reddy (1978) that hypothesis (34) does, in fact hold for \( r(h) = h \) if \( w \) is sufficiently smooth. The verification of (34) requires an elaborate local analysis of the behavior of small solutions in a neighborhood of solutions to (10). Also note that the rate of convergence is generally controlled by the last term in (35), independent of the order of the polynomial \( k \).

Acknowledgement

This work was supported by the National Science Foundation through Grant ENG-75-0748.
References


