ON SOME FINITE ELEMENT METHODS
FOR CERTAIN NONLINEAR SECOND
ORDER HYPERBOLIC EQUATIONS

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Abstract Two computational schemes for the numerical solution of a class of nonlinear second-order hyperbolic equations are investigated. Both involve the use of finite element concepts for approximating the spatial variation of the dependent variable, and difference schemes for the temporal behavior. One of the schemes, a shock-smearing method based on finite element concepts, appears to be effective for studying shock waves. Convergence of the schemes is studied and error estimates are derived.

1. Introduction

Convergence and accuracy of finite-element approximations of certain linear time dependent problems has been the subject of much study in recent years. Mathematical properties of finite element-Galerkin approximations of linear parabolic equations, in which various Crank-Nicolson schemes are suggested for representing temporal behavior, have been studied extensively by Douglas and Dupont (e.g. [1-4]), Wheeler [5], Thomee [6], and others (for a summary, see [7] or [8]). Certain features of finite element schemes for solving hyperbolic equations have also been recently investigated by Fujii [9], Dupont [10,11], and Oden and Post [12], but a theory of finite-element-Galerkin approximations of wave
propagation is far from complete. On the other hand, applications of the method to the solution of hyperbolic systems have been extensive (e.g. [13,14]).

In the present paper, we investigate the accuracy and convergence of certain finite-element approximations of a class of linear and nonlinear hyperbolic equations of second order. In applications of the method to time-dependent problems, it is generally accepted that finite elements are to be used only to represent the solution spatially, and that by "lumping" masses, both explicit and implicit integration schemes can be used to depict the behavior of the solution in time (there are exceptions, e.g. [14]). We show that such finite element-difference schemes can be quite effective -- even for studying shocks. Following this introduction, we lay the groundwork for a study of Galerkin approximations of hyperbolic equations by first citing some standard notation and then describing classes of linear and nonlinear variational wave problems. We next review briefly some aspects of finite element models of these problems and then provide a section on schemes for discretization in time. Two schemes are emphasized; one, a simple two-point central-difference scheme for linear hyperbolic problems or mildly nonlinear problems which do not involve shocks, and a second dissipative scheme which is based on a parabolic regularization of the problem. For a certain choice of the regularization parameter, this scheme reduces to a Lax-Wendroff/finite element which is derived from physical arguments. We study these schemes for linear as well as nonlinear problems, and in each case we develop error estimates and prove convergence.
2. Some Preliminaries

We shall record here for future reference certain notations and definitions used throughout the paper.

We denote by $\Omega$ a bounded open domain in $n$-dimensional euclidean space, $\mathbb{R}^n$ with boundary $\partial \Omega$.

We denote by $H^m(\Omega)$ the Sobolev space of order $M$. This is the space of functions whose generalized derivatives of order $\leq m$ are in $L_2(\Omega)$. The norm on $H^m(\Omega)$ is

$$||u||^2_{H^m(\Omega)} = \sum_{|\alpha| \leq m} ||D^\alpha u||^2_{L_2(\Omega)}$$

(2.1)

Here we have used the multi-index notation; $\alpha$ is an ordered $n$-tuple of non-negative integers: $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_i =$ integer $\geq 0$. Also, we employ the conventions,

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}; \ |\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$$

$$D^\alpha u = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \ldots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}$$

where $x = (x_1, x_2, \ldots, x_n) \in \Omega$. $H^m(\Omega)$ is the completion of $C^m(\Omega)$ the space of functions with continuous derivatives of order $m$ relative to the norm (2.1).

In addition, $H^m_0(\Omega)$ shall denote the completion of $C^\infty_0(\Omega)$, the space of infinitely differentiable functions with compact support in $\Omega$, with respect to the norm (2.1). For a sufficiently smooth boundary $\partial \Omega$, $H^m_0(\Omega)$ is the set of functions in $H^m(\Omega)$ which satisfy homogeneous boundary conditions of order $m-1$; i.e., $D^\alpha u(\chi) = 0$, $\chi \in \partial \Omega$, $|\alpha| \leq m-1$. Note that if $u \in H^m_0(\Omega)$, $u$ satisfies a Friedrich's inequality of the form (for some $B \geq 0$)
Let $u(X,t)$ be a function defined on $\Omega \times [0,T]$ such that $u: [0,T] \to H^m(\Omega)$. We shall say that $u \in L^2(H^m(\Omega))$ if $u \in L^2(0,T)$ in the temporal variable $t$ and $u \in H^m(\Omega)$ in the spatial variable $X$. The norm on $L^2(H^m(\Omega))$ is

$$
\|u\|_{L^2(H^m(\Omega))}^2 = \int_0^T \|u(t)\|_{H^m(\Omega)}^2 dt
$$

Normally this space is denoted $L^2(a,b;H^m(\Omega))$, where $(a,b) \in \mathbb{R}^1 = (-\infty, \infty)$, but since we shall always be dealing with the fixed time interval $(a,b) = (0,T)$, the notation $L^2(H^m(\Omega)) \equiv L^2(0,T;H^m(\Omega))$ is used. We say that $u \in L^\infty(H^m(\Omega))$ if $u \in L^\infty(0,T)$ for each $X \in \Omega$ and $u \in H^m(\Omega)$ for each $t \in (0,T)$. The norm on $L^\infty(H^m(\Omega))$ is

$$
\|u\|_{L^\infty(H^m(\Omega))} = \sup_{0 \leq t \leq T} \|u(t)\|_{H^m(\Omega)}
$$

In addition, we shall use the symbol $\langle \cdot, \cdot \rangle_{\Omega}$ to denote the $L^2$ inner product:

$$
\langle u, v \rangle_{\Omega} = \int_\Omega u v \, dx \quad u, v \in L^2(\Omega)
$$

Then a special notation for the $L^2$ norm is $\|u\|_{\Omega} = (u, u)^{1/2}_{\Omega}$.

3. The Variational Problem

First consider a classical nonlinear hyperbolic problem of the second order characterized as follows: Find a function $u(X,t), \ (X,t) \in \Omega \times [0,T]$, such that
\[ \rho \frac{\partial^2 u}{\partial t^2}(x,t) - \nabla \cdot (C^2(x,t,u) \nabla u(x,t)) = f(x,t) \text{ in } \Omega \times (0,T] \]

\[ u(x,0) = g_1(x) \text{ in } \Omega \]

\[ \frac{\partial u(x,0)}{\partial t} = g_2(x) \text{ in } \Omega \] (3.1)

\[ u(x,t) = 0 \text{ in } \partial \Omega \times (0,T] \]

Even under reasonable assumptions on the functions \( C^2(x,t,u) \), \( f(x,t) \), \( g_1(x) \), and \( g_2(x) \), we often can be assured that a solution exists for this problem only for certain choices of \( T \). For the moment, assume that such a solution exists for all \( t \in (0,T] \).

We remark that the notation \( C(x,t,u) \) is used in recognition of this function as the square of the intrinsic wave speed at which "disturbances" are propagated in problem (3.1).

It is well known that a solution to the problem (3.1) is also a solution of the following weaker problem: Find

\[ u(x,t) \in L_2(H^1_0(\Omega)), (x,t) \in \Omega \times [0,T] \text{ such that} \]

\[ (\rho \frac{\partial^2 u}{\partial t^2}, v) + a(u,u,v) = (f,v) \text{ in } \Omega \]

\[ (u(\cdot , 0), v) = (g_1,v) \text{ in } \Omega \]

\[ (\frac{\partial u}{\partial t}(\cdot , 0), v) = (g_2,v) \text{ in } \Omega \] (3.2)

where

\[ a(u,u,v) = \int_\Omega C^2(x,t,u) \nabla u \cdot \nabla v \, dx \] (3.3)
Here we complete the problem description by characterizing the real valued function $C^2(\tilde{\chi}, t, u)$: we assume that there exist positive constants $M_1$, $M_2$, and $M_3$ such that (see [15])

\begin{enumerate}
\item $C^2(\tilde{\chi}, t, u) \geq M_1 \quad \forall (\tilde{\chi}, t) \in \Omega \times [0, T]$
\item $C^2(\tilde{\chi}, t, u) \leq M_2 \quad \forall (\tilde{\chi}, t) \in \Omega \times [0, T]$
\item $|C^2(\tilde{\chi}, t, u) - C^2(\chi, t, \tilde{u})| \leq M_3 |u - \tilde{u}| \quad \forall (\chi, t) \in \Omega \times [0, T]$
\end{enumerate}

These restrictions imply positiveness, boundedness, and Lipschitz continuity respectively, of the wave speed function $C^2$.

A coercive property of $a(u, u, v)$ can now be established through the following lemma:

**Lemma 3.1:** Let $v, \omega \in H^1_0(\Omega)$, then there exists a positive constant $\mu$ such that

$$a(v, \omega, \omega) \geq \mu \|\omega\|_{H^1(\Omega)}^2$$

(3.5)

**Proof:** From (3.4), $C^2(\tilde{\chi}, t, v) \geq M_1$. Thus

$$a(v, \omega, \omega) \geq M_1 \int_{\Omega} \tilde{\nabla}v \cdot \tilde{\nabla}\omega \, dx$$

But by use of Sobolev's imbedding theorem (see, for example, [16]) and Freidrick's inequality (2.2),

$$\|\omega\|_{H^1(\Omega)}^2 = \|\omega\|_\Omega^2 + \sum_{i=1}^n \|\frac{\partial \omega}{\partial x_i}\|_\Omega^2$$

$$\leq B \sum_{i=1}^n \|\frac{\partial \omega}{\partial x_i}\|_\Omega^2 + \sum_{i=1}^n \|\frac{\partial \omega}{\partial x_i}\|_\Omega^2$$

$$= \frac{M_1}{\mu} \sum_{i=1}^n \|\frac{\partial \omega}{\partial x_i}\|_\Omega^2$$

$$\leq \frac{1}{\mu} a(v, \omega, \omega)$$
where \( \mu = M_1/(B+1) \).

Now suppose we identify a finite dimensional subspace \( M \) of \( H_0^1(\Omega) \). Then the semidiscrete Galerkin approximation \( U \) of the weak solution \( u \) of (3.2) is that \( U \in M \) such that

\[
(\rho \frac{\partial^2 U}{\partial t^2}, V)_\Omega + a(U,U,V) = (f,V)_\Omega \quad \forall V \in M
\]

\[
(U(\cdot,0), V)_\Omega = (g_1,V)_\Omega \quad \forall V \in M
\]

\[
\left( \frac{\partial U}{\partial t}, V \right)_\Omega = (g_2,V)_\Omega \quad \forall V \in M
\]

Under the stated assumptions, it can be shown that \( U \) is unique. We must now describe more precisely how the subspace \( M \) can be constructed in a systematic and computationally effective way. Toward this end, we establish some basic properties of the finite element method.

4. Finite Element Models

To develop finite-element models of our problem, the region \( \Omega \) is partitioned into a finite number \( E \) of disjoint open sets \( \Omega_e \) called finite elements:

\[
\Omega = \bigcup_{e=1}^{E} \bar{\Omega}_e \quad \Omega_e \cap \Omega_f = 0 \quad e \neq f
\]

Here \( \bar{\Omega}_e \) is the closure of \( \Omega_e \). Within each element a set of local \( \alpha(e) \) basis functions \( \psi_N^e(x) \) having the following properties is identified

\[
\beta^e \alpha(e) \nabla^e \psi_N^e(x) = \delta_{N}^{M} \delta_{f}^{e} \delta_{e}^{f}
\]

\[
\alpha(e) \psi_N^e(x) = 0 \quad x \notin \Omega_e
\]

\[
\alpha, \beta \in \mathbb{Z}_+^N ; M,N = 1, 2, \ldots, N_e ;
\]

\[
e,f = 1,2,\ldots,E ; \quad |\alpha| \leq k
\]
Here $x^M_f$ is a nodal point labeled $M$ in element $\Omega_f$, $\delta_n^e$, $\delta^M_n$, $\delta_f^e$ are kronecker deltas, $N_e$ is the number of nodes in the element $\Omega_e$. The local representation of a function in terms of the basis functions $\psi_N^e(x)$ is

$$u_e(x) = \sum_{|\alpha| \leq k} \sum_{N=1}^{N_e} u_N^e(\alpha) \psi_N^e(x) \quad (4.3)$$

$$u_N^e(\alpha) = D^e u_e(x)^N \quad (4.4)$$

and the global representation is of the form

$$U(x) = \bigcup_{e=1}^E U(x^e) = \sum_{|\alpha| \leq k} \sum_{\Lambda=1}^G U^\Lambda(\alpha) \chi_\alpha^\Lambda(x) \quad (4.5)$$

Here $\chi_\alpha^\Lambda(x)$ are global basis functions given by

$$\chi_\alpha^\Lambda(x) = \bigcup_{e=1}^E \sum_{N=1}^{N_e} \bigcup_{\Lambda=1}^G \Lambda \psi_N^e(x) \quad (4.6)$$

where $\Lambda$ defines a Boolean transformation of the disconnected system of elements into the connected model $\Omega$(i.e., $\Omega^\Lambda = 1$ if node $N$ of $\Omega_e$ coincides with node $x^\Lambda$ of $\Omega$ and $\Omega^\Lambda = 0$ if otherwise).

Now the set of functions $\{\chi_\alpha^\Lambda(x)\}_{\Lambda=1}^G; |\alpha| \leq k$ defines a finite dimensional subspace of $H^1(\Omega)$ which we denote by $S_h^\Lambda(\Omega)$. Here $h$ is the mesh parameter of the finite element mesh. For economy in notation, we shall relabel the global basis functions $\chi_\alpha^\Lambda(x)$ as $\phi^e_N(x)$, $N = 1, 2, \ldots, N^e$. Then the global representation is of the form

$$U(x) = \sum_{N=1}^{N^e} U^N \phi_N(x) \quad (4.7)$$
The functions \( \{\phi_n\}_{N=1}^{N_0} \) form a basis for the subspace \( S_h(\Omega) \), and the subscript \( h \) is used to designate that \( S_h(\Omega) \) depends upon the conventional finite-element mesh parameter \( h \); that is, if

\[
h_e = \text{dia}(\bar{\Omega}_e)
\]

then

\[
h = \max_{1 \leq e \leq E} \{h_e\}
\]

The finite-element subspaces \( S_h(\Omega) \) will be used for the space in the Galerkin approximation (3.6), and it shall be assumed that \( S_h(\Omega) \) has the following properties (Cf. [16], [17], [18]):

(i) Let \( P_j(\Omega) \) be the space of polynomials of degree \( j \) on \( \Omega \). Then there exists an integer \( k \) such that \( p(x) \in P_j(\Omega) \) is in \( S_h(\Omega) \) as long as \( j \leq k \).

(ii) Let \( h \to 0 \) uniformly (i.e., for each refinement of the mesh let the radius \( \rho_e \) of the largest sphere that can be inscribed in \( \bar{\Omega}_e \) be proportional to \( h_e \)). Then there is a constant \( K \) independent of \( h \) such that

\[
\inf_{W \in S_h(\Omega)} \|u - W\|_{H^m(\Omega)} \leq Kh^{k+l-m} \|u\|_{H^{k+1}(\Omega)} \quad (4.8)
\]

(iii) \( S_h(\Omega) \) satisfies an inverse hypothesis [16] of the following form: there exists a constant \( C^* \) independent of \( h \) such that

\[
\|v\|_{H^j(\Omega)} \leq C^*h^{-j} \|v\|_{L^2(\Omega)} \quad \forall v \in S_h(\Omega), \ j \leq k+1
\]

\[ (4.9) \]
The finite-element Galerkin model is formulated by setting $M = S_h(\Omega)$ and letting $V = \phi_N, N = 1, \ldots, N_0$ in (3.6).

$$
\left( \frac{\partial^2 U}{\partial t^2}, \phi_N \right)_O + a(U, U, \phi_N) = (f, \phi_N)_O \quad \phi_N \in S_h(\Omega),
$$

$$
(U(\cdot,0), \phi_N)_O = (g_1, \phi_N)_O \quad N = 1, \ldots, N_0
$$

$$
\left( \frac{\partial U}{\partial t} (\cdot,0), \phi_N \right)_O = (g_2, \phi_N)_O
$$

These equations describe a system of nonlinear, second-order ordinary differential equations in the coefficients $A^N(t)$ of the Galerkin approximation,

$$U(\bar{x}, t) = \sum_{n=1}^{N_0} A^N(t) \phi_n(\bar{x})$$

In practical calculations, we must, of course, also identify temporal approximations so as to numerically integrate the equations in time.

5. The Temporal Discretization

In this section, several temporal discretization procedures for (4.10) using finite difference methods are considered. In particular, a central difference and a parabolic regularization method are introduced. We show that the parabolic regularization method is the natural generalization of a physically derived Lax-Wendroff type model. We have used this Lax-Wendroff scheme in [15].
In particular, the central-difference scheme can be used for non-linear response problems in which shock formation is precluded. The parabolic regularization scheme, on the other hand, seems effective for the calculation of certain shock- and acceleration-wave problems.

Let $P$ be a partition of the time domain $[0,T]$ of the form 

$$
\{t_0, t_1, \ldots, t_N\} \text{ where } 0 < t_0 < t_1 < \ldots < t_N = T \text{ and } t_{n+1} - t_n = \Delta t \text{ for } 0 < n < N-1. 
$$

The values of the dependent variable $U(t)$ at the points of the partition $P$ are denoted by 

$$
\{U^n\}_{n=0}^N. \text{ In order to construct a difference approximation for (4.10), we introduce the central difference operator }

\begin{equation}
\delta_t^2 U^n = \frac{U^{n+1} - 2U^n + U^{n-1}}{\Delta t^2} \tag{5.1}
\end{equation}

The nonlinear central difference approximation problem corresponding to (4.10) is to find the sequence $\{U^n\}_{n=0}^N$, where $U^n \in S_h(\Omega)$, such that

$$
(\rho \delta_t^2 U^n, V) + a(U^n, U^n, V) = (f, V) \quad \forall \; V \in S_h(\Omega) \tag{5.2}
$$

We must, of course, add appropriate approximations of initial conditions. For example,

$$
(U^0, V) = (g_1, V), \quad (\delta_t U^k, V) = (g_2, V) \quad \forall \; V \in S_h(\Omega)
$$

where $\delta_t U^k$ represents a forward difference operator $\frac{U^1 - U^0}{\Delta t}$. In previous calculations [15,19] we have used exclusively a "divided" difference scheme in which the temporal mesh for $U$ and $\partial U/\partial t$ are displaced $\Delta t/2$ relative to one another.
In order to construct a Lax-Wendroff type approximation, we initially expand \( U^n \) and \( \dot{U}^n \) in Taylor's series expansions. Setting \( Y^n = \dot{U}^n \) for clarity, we have

\[
U^{n+1} = U^n + \Delta t \, Y^n + \frac{\Delta t^2}{2} \, U^n + O(\Delta t^3)
\]  
(5.3)

\[
Y^{n+1} = Y^n + \Delta t \ddot{Y}^n + \frac{\Delta t^2}{2} \, \ddot{U}^n + O(\Delta t^3)
\]  
(5.4)

where for example \( \dot{X}^n = \frac{\partial X}{\partial t} \bigg|_{t=n\Delta t} \). From (5.4)

\[
\ddot{U}^n = \frac{Y^{n+1} - Y^n}{\Delta t} - \frac{\Delta t}{2} \ddot{Y}^n + O(\Delta t^3)
\]  
(5.5)

Differentiating (4.10) with respect to time and evaluating the resulting expression at \( t = n\Delta t \)

\[
(\rho \ddot{U}^n, V)_o + a(U^n, Y^n, V) + \left( \frac{\partial}{\partial t} \right) \Delta^2(x, t, U^n) \nabla U^n, \nabla V)_o
\]  
(5.6)

\[
= \left( \frac{\partial}{\partial t} f(x, t), V \right)_o, \quad \forall \, V \in S_h(\Omega)
\]

Introducing (5.3) into (4.10) evaluated at \( t = n\Delta t \), we get

\[
(\rho \frac{U^{n+1} - U^n}{\Delta t}, V)_o - (\rho Y^n, V)_o + \frac{\Delta t}{2} a(U^n, U^n, V)
\]  
(5.7)

\[
= \frac{\Delta t}{2} (f, V)_o, \quad \forall \, V \in S_h(\Omega)
\]

For convenience, we assume that the last two terms in (5.6) are negligible compared to the first two. Then introducing (5.5) into (4.10), using (5.6), and neglecting terms of order \( \Delta t^3 \) or higher, we get
Equations (5.7) and (5.8) define a nonlinear finite-element/Lax-Wendroff Approximation for the second order hyperbolic problem (3.1).

The natural generalization of the nonlinear finite-element/Lax-Wendroff scheme is the parabolic regularization method defined by

\[
(\rho \frac{y^{n+1} - y^n}{\Delta t}, v)_o + \frac{\Delta t}{2} a(u^n, y^n, v) + a(u^n, u^n, v) = (f, v)_o, \quad \forall \, v \in S_h(\Omega)
\]

(5.9)

and

\[
(\rho \frac{y^{n+1} - y^n}{\Delta t}, v)_o + \frac{\Delta t^2}{2} a(u^n, Y^n, v) + a(u^n, u^n, v) = (f, v)_o
\]

\[
\forall \, v \in S_h(\Omega)
\]

(5.10)

where \( \alpha > 0 \).

6. Regularity

In this section we develop regularity results for a system of equations of the form

\[
(\rho \frac{y^{n+1} - y^n}{\Delta t}, v)_o + \frac{\Delta t}{2} a(u^n, y^n, v) + a(u^n, u^n, v) = (f, v)_o
\]

\[
\forall \, v \in S_h(\Omega)
\]
\[(\rho \frac{\partial \tilde{u}}{\partial t}, v) \big|_0 - (\rho \tilde{y}, v) \big|_0 + \varepsilon a(\tilde{u}, \tilde{u}, v) = 0 \quad \forall v \in H^1(\Omega) \] (6.1)

\[(\rho \frac{\partial \tilde{y}}{\partial t}, v) \big|_0 + a(\tilde{u}, \tilde{u}, v) + \varepsilon a(\tilde{u}, \tilde{y}, v) = 0 \quad \forall v \in H^1(\Omega) \]

where \(\varepsilon\) is a real parameter \(\geq 0\). This system is useful in studying the convergence and accuracy of the parabolic regularization approximation described previously. We assume that

\[|\frac{\partial^m}{\partial t^m} c^2(x,t,u)| \text{ is bounded by positive constant } M_4 \text{ for all } m \in \mathbb{Z}_+ \text{ and that } \tilde{u} \in L_2(H^1(\Omega)) \] which clearly corresponds to the case described by (6.1) when \(\varepsilon = 0\).

Initially we examine the equation (6.1)\(2\). Let \(v = \tilde{y}\) in (6.1)\(2\). Then

\[(\rho \frac{\partial \tilde{y}}{\partial t}, \tilde{y}) \big|_0 + a(\tilde{u}, \tilde{u}, \tilde{y}) + \varepsilon a(\tilde{u}, \tilde{y}, \tilde{y}) = 0 \] (6.2)

Estimating the terms in (6.2) using the Cauchy-Schwarz inequality, Lemma 3.1, the definition of the \(L_2\) norm, and the elementary inequality \(ab \leq \frac{\alpha}{2} a^2 + \frac{1}{2\alpha} b^2\) for \(\alpha > 0\) (henceforth to be called inequality E), we see that

\[
\frac{1}{2} \frac{d}{dt} \left( ||\rho \tilde{y}||^2 \right) \big|_0 + \mu \varepsilon ||\tilde{y}||^2_{H^1(\Omega)} \leq \frac{C}{\varepsilon \eta} \left( ||\tilde{u}||^2_{H^1(\Omega)} \right)
\]

\[+ \frac{C \varepsilon \eta}{2} \left( ||\tilde{y}||^2 \right)_{H^1(\Omega)} + ||\tilde{y}||^2 \big|_0 \] (6.3)
This leads to a regularity result of the form

$$||\tilde{y}||_{L^2(H^1(\Omega))} \leq C\left(\frac{1}{\epsilon} ||\tilde{u}||_{L^2(H^1(\Omega))} + \frac{1}{\sqrt{\epsilon}} ||\tilde{y}(0)||_0\right)$$  \hspace{1cm} (6.4)$$

By combining the two equations in (6.1), we find that

$$\left(\rho \frac{\partial^2 \tilde{u}}{\partial t^2}, v\right)_o + a(\tilde{u}, \tilde{u}, v) + \epsilon a(\tilde{u}, \frac{\partial \tilde{u}}{\partial t}, v) + \epsilon a(\tilde{u}, \tilde{y}, v)$$

$$+ \epsilon \left(\frac{\partial}{\partial t} C^2(x, t, \tilde{u}) \tilde{y}, \tilde{y}, v\right)_o = 0 \quad \forall \ v \in H^1(\Omega)$$  \hspace{1cm} (6.5)$$

Now let \( v = \frac{\partial \tilde{u}}{\partial t} \) and \( \rho = \text{const.} \) in (6.5). Then

$$\frac{1}{2} \frac{d}{dt} \left(\rho ||\frac{\partial \tilde{u}}{\partial t}||^2_o\right) + a(\tilde{u}, \tilde{u}, \frac{\partial \tilde{u}}{\partial t}) + \epsilon a(\tilde{u}, \frac{\partial \tilde{u}}{\partial t}, \frac{\partial \tilde{u}}{\partial t}) + \epsilon a(\tilde{u}, \tilde{y}, \frac{\partial \tilde{u}}{\partial t})$$

$$+ \epsilon \left(\frac{\partial}{\partial t} C^2(x, t, \tilde{u}) \tilde{y}, \frac{\partial \tilde{u}}{\partial t}\right)_o = 0$$  \hspace{1cm} (6.6)$$

Using a procedure similar to the one used in equation (6.4), we find that

$$\frac{1}{2} \frac{d}{dt} \left(\rho ||\frac{\partial \tilde{u}}{\partial t}||^2_o\right) + \epsilon u ||\frac{\partial \tilde{u}}{\partial t}||^2_{H^1(\Omega)} \leq \frac{C_1}{\epsilon} \left( ||\tilde{u}||^2_{H^1(\Omega)} + \epsilon^2 ||\tilde{y}||^2_{H^1(\Omega)} + \epsilon^2 ||\tilde{u}||^2_{H^1(\Omega)} \right)$$

$$+ \epsilon \left(\frac{\partial}{\partial t} \tilde{y}, \tilde{y}, v\right)_o + \epsilon ||\tilde{u}||^2_{H^1(\Omega)}$$

$$+ C_2 \epsilon ||\frac{\partial \tilde{u}}{\partial t}||^2_{H^1(\Omega)} + ||\frac{\partial \tilde{u}}{\partial t}||^2_o$$
Then integrating and applying the Gronwall inequality, we find that

\[
| \frac{\partial \tilde{u}}{\partial t} |_{L^2(H^1(\Omega))} \leq C \left( \frac{1}{\varepsilon} | \tilde{u} |_{L^2(H^1(\Omega))} + \frac{1}{\sqrt{\varepsilon}} | \tilde{y}(0) |_{L^2(\Omega)} + \frac{1}{\sqrt{\varepsilon}} | \frac{\partial \tilde{u}}{\partial t}(0) |_{L^2(\Omega)} \right)
\]

(6.7)

Now differentiating (6.1) with respect to time

\[
\left( \rho \frac{\partial^2 \tilde{y}}{\partial t^2}, v \right)_o + a(\tilde{u}, \frac{\partial \tilde{u}}{\partial t}, v) + \left( \frac{\partial}{\partial t} C^2(\chi, \tilde{u}) \frac{\partial \tilde{u}}{\partial \tilde{t}}, \tilde{y}, \tilde{v} \right)_o
\]

+ \varepsilon a(\tilde{u}, \frac{\partial \tilde{y}}{\partial t}, v) + \varepsilon \left( \frac{\partial}{\partial t} C^2(\chi, \tilde{u}), \tilde{y}, \tilde{v}, \tilde{v} \right) = 0

Now setting \( v = \frac{\partial \tilde{y}}{\partial t} \) and using the procedure which has become standard we can show, using (6.4) and (6.7), that

\[
\left| \left| \frac{\partial \tilde{y}}{\partial t} \right| \right|_{L^2(H^1(\Omega))} \leq C \left( \frac{1}{\varepsilon^2} | \tilde{u} |_{L^2(H^1(\Omega))} + \frac{1}{\varepsilon^2} | \frac{\partial \tilde{u}}{\partial t}(0) |_{L^2(\Omega)} \right)
\]

\[+ \frac{1}{\varepsilon^2} | \tilde{y}(0) |_{L^2(\Omega)} + \frac{1}{\varepsilon^2} | \frac{\partial \tilde{y}}{\partial t}(0) |_{L^2(\Omega)} \}
\]

(6.8)

Differentiating (6.5) with respect to time, we get

\[
\left( \rho \frac{\partial^3 \tilde{u}}{\partial t^3}, v \right)_o + a(\tilde{u}, \frac{\partial \tilde{u}}{\partial t}, v) + \varepsilon a(\tilde{u}, \frac{\partial^2 \tilde{u}}{\partial t^2}, v)
\]

+ \varepsilon \left( \frac{\partial}{\partial t} C^2(\chi, \tilde{u}) \frac{\partial \tilde{u}}{\partial \tilde{t}}, \tilde{y}, \tilde{v} \right)_o + \varepsilon a(\tilde{u}, \frac{\partial \tilde{y}}{\partial t}, v)
\]
Now setting $v = \frac{\partial^2 \tilde{u}}{\partial t^2}$, we get, using the standard procedure

$$
|\frac{\partial^2 \tilde{u}}{\partial t^2}|_{L_2(H^1(\Omega))} \leq C\left\{ \frac{1}{\varepsilon^2} |\tilde{u}|_{L_2(H^1)} + \frac{1}{\varepsilon} |\frac{\partial \tilde{u}}{\partial t}(0)| \right. \\
+ \frac{1}{\varepsilon^2} |\tilde{y}(0)| \left. + \frac{1}{\varepsilon^2} |\frac{\partial \tilde{y}}{\partial t}(0)| \right\} 
$$

Expression for the higher derivatives can be obtained by using similar methods. We state the general result in terms of a theorem:

**Theorem 6.1.** Let $|\frac{\partial^m}{\partial t^m} C^2(\tilde{x}, \tilde{t}, \tilde{u})| \leq M_4$ for $0 \leq m \leq i+1$. Then, if $\frac{\partial \tilde{u}}{\partial t}(0) \in L_2(\Omega)$ for $0 \leq \ell \leq i+1$ and $\frac{\partial \tilde{y}}{\partial t}(0) \in L_2(\Omega)$ for $0 \leq \ell \leq i$, the regularity of the solution $(\tilde{u}, \tilde{y})$ to (6.1) is governed by (for some positive constant $C$)
If we need estimates for these temporal derivatives in other Sobolev norms, we use a similar procedure. Initially we assume that \((\tilde{u}, \tilde{y})\), the solution to (6.1), is periodic on the boundary (this assumption is implicit in our formulation henceforth). Then it is possible to show that

\[
\| \ddfrac{\partial^i \tilde{y}}{\partial t^i} \|_{L^2(H^1(\Omega))} \leq C \left( \frac{1}{\varepsilon^{i+1}} \right) \| \tilde{u} \|_{L^2(H^1(\Omega))} + \sum_{k=0}^{i-1} \frac{1}{\varepsilon^{i-k+\varepsilon}} \left( \| \ddfrac{\partial^k \tilde{y}}{\partial t^k}(0) \|_0 + \| \ddfrac{\partial^{k+1} \tilde{u}}{\partial t^{k+1}}(0) \|_0 \right) + \frac{1}{\varepsilon^i} \left( \| \ddfrac{\partial^i \tilde{y}}{\partial t^i}(0) \|_0 \right)
\]

(6.11)

\[
\| \ddfrac{\partial^{i+1} \tilde{u}}{\partial t^{i+1}} \|_{L^2(H^1(\Omega))} \leq C \left( \frac{1}{\varepsilon^{i+1}} \right) \| \tilde{u} \|_{L^2(H^1(\Omega))} + \sum_{k=0}^{i} \frac{1}{\varepsilon^{i-k+\varepsilon}} \left( \| \ddfrac{\partial^k \tilde{y}}{\partial t^k}(0) \|_0 + \| \ddfrac{\partial^{k+1} \tilde{u}}{\partial t^{k+1}}(0) \|_0 \right)
\]

(6.12)

\[
\| \tilde{y} \|_{L^2(H^\ell(\Omega))} \leq C \left( \frac{1}{\varepsilon} \right) \| \tilde{u} \|_{L^2(H^\ell(\Omega))} + \frac{1}{\sqrt{\varepsilon}} \| \tilde{y}(0) \|_{H^{\ell-1}(\Omega)} \quad \ell > 2
\]

(6.13)
Then it can be shown that (6.13) and (6.14) lead to the more general estimate

**Theorem 6.2.** Let $|\frac{\partial^m}{\partial t^m} C^2(x,t,\bar{u})| \leq M_4$, $|\frac{\partial^j}{\partial x_k^j} C^2(x,t,\bar{u})| \leq M_5$ for $1 \leq k \leq n$; $0 \leq m \leq i+1$; and $0 \leq j \leq n-1$. Then if $\frac{\partial^l \bar{u}}{\partial t^l} \in H^{n-1}(\Omega)$ for $0 \leq l \leq i+1$ and $\frac{\partial^r \bar{y}}{\partial t^r}(0) \in H^{n-1}(\Omega)$ for $0 \leq r \leq i$, the regularity of the solution $(\bar{u}, \bar{y})$ to (6.1) is given for some positive constant $C$ by

$$
|| \frac{\partial^i \bar{y}}{\partial t^i} ||_{L^2(H^n(\Omega))} \leq C \frac{1}{\epsilon} \left( \frac{1}{2i + 3n - 1} \right) \frac{1}{2} || \bar{u} ||_{L^2(H^1(\Omega))} + \sum_{k=0}^{n-2} \frac{1}{\epsilon} \frac{1}{2i + 1 + 3k} || \bar{y}(0) ||_{H^{n-k-2}(\Omega)}
$$

(6.14)
\[ + \sum_{k=0}^{n-2} \frac{1}{2i + 3 + 3k} \epsilon |\tilde{u}(0)|_{H^{n-k-1}(\Omega)} \]

\[ + \sum_{k=0}^{i-1} \frac{1}{\epsilon^{i-k+2} \epsilon^2} \left[ |\frac{\partial^k \tilde{y}(0)}{\partial t^k}|_{H^{n-1}(\Omega)} + |\frac{\partial^{k+1} \tilde{u}(0)}{\partial t^{k+1}}|_{H^{n-1}(\Omega)} \right] \]

\[ + \frac{1}{\epsilon^2} |\frac{\partial^i \tilde{y}(0)}{\partial t^i}|_{H^{n-1}(\Omega)} \quad i \geq 0 \]

\[ n \geq 2 \quad (6.15) \]

\[ |\frac{\partial^{i+1} \tilde{u}}{\partial t^{i+1}}|_{L^2(H^n(\Omega))} \leq C \left( \frac{1}{\epsilon^{2i + 3n - 1}} \epsilon \right) \frac{1}{\epsilon^{2i + 3n - 1}} \left[ |\tilde{u}|_{L^2(H^i(\Omega))} \right] \]

\[ + \sum_{k=0}^{n-2} \frac{1}{2i + 4 + 3k} \epsilon |\tilde{y}(0)|_{H^{n-k-2}(\Omega)} \]

\[ + \sum_{k=0}^{n-2} \frac{1}{2i + 3 + 3k} \epsilon |\tilde{u}(0)|_{H^{n-k-1}(\Omega)} \]

\[ + \sum_{k=0}^{i} \frac{1}{\epsilon^{i-k+2} \epsilon^2} \left[ |\frac{\partial^k \tilde{y}(0)}{\partial t^k}|_{H^{n-1}(\Omega)} \right. \]

\[ \left. + |\frac{\partial^{k+1} \tilde{u}(0)}{\partial t^{k+1}}|_{H^{n-1}(\Omega)} \right] \quad i \geq 0 \]

\[ n \geq 2 \quad (6.16) \]
Estimates in terms of the $L_\infty$ norm can also be obtained. For instance

**Theorem 6.3.** Let $|\frac{\partial^m}{\partial t^m} C^2(x,t,\bar{u})| \leq M_4$, $|\frac{\partial^j}{\partial x_k^j} C^2(x,t,\bar{u})| \leq M_5$ for $1 \leq k \leq n$; $0 \leq m \leq i+1$; and $0 \leq j \leq n-1$. Then if $\frac{\partial \bar{u}}{\partial t}(0) \in H^{n-1}(\Omega)$ for $0 \leq \ell \leq i+1$ and $\frac{\partial \bar{y}}{\partial t}(0) \in H^{n-1}(\Omega)$ for $0 \leq r \leq i$, the regularity of the solution $(\bar{u},\bar{y})$ to (6.1) for some positive constant $C$ is given by

$$
\left|\frac{\partial^i \bar{y}}{\partial t^i}\right|_{L_\infty(H^{n-1}(\Omega))} \leq C \left\{ \frac{1}{2i+3n-2} \left| \bar{u} \right|_{L_2(H^1(\Omega))} + \sum_{k=0}^{n-2} \frac{1}{2i+3+3k} \left| \bar{y}(0) \right|_{H^{n-k-2}(\Omega)} + \sum_{k=0}^{n-2} \frac{1}{2i+2+3k} \left| \bar{u}(0) \right|_{H^{n-k-1}(\Omega)} + \sum_{k=0}^{i-1} \frac{1}{\epsilon^{i-1-k}} \left[ \left| \frac{\partial^{k+1} \bar{y}}{\partial t^{k+1}}(0) \right|_{H^{n-1}(\Omega)} + \left| \frac{\partial^{k+1} \bar{u}}{\partial t^{k+1}}(0) \right|_{H^{n-1}(\Omega)} \right] \right\}
$$
\[ + \left| \frac{\partial^{i+1} u}{\partial t^{i+1}} \right|_{L^\infty(H^{n-1}(\Omega))} \leq C \frac{1}{\varepsilon} \frac{1}{2i+3n-2} \left| \frac{\partial^{i} y}{\partial t^{i}}(0) \right|_{H^{n-k-2}(\Omega)} \]

\[ + \sum_{k=0}^{n-2} \frac{1}{2i+3+3k} \frac{1}{\varepsilon} \left| \frac{\partial^{i-k} y}{\partial t^{i-k}}(0) \right|_{H^{n-k-2}(\Omega)} \]

\[ + \sum_{k=0}^{n-2} \frac{1}{2i+2+3k} \frac{1}{\varepsilon} \left| \frac{\partial^{i-k} u}{\partial t^{i-k}}(0) \right|_{H^{n-k-1}(\Omega)} \]

\[ + \sum_{k=0}^{i} \frac{1}{\varepsilon} \left[ \left| \frac{\partial^{k} y}{\partial t^{k}}(0) \right|_{H^{n-k}(\Omega)} \right] \]

\[ + \left| \frac{\partial^{i+1} u}{\partial t^{i+1}} \right|_{H^{n-1}(\Omega)} \]

\[ + \left| \frac{\partial^{i} y}{\partial t^{i}}(0) \right|_{H^{n-1}(\Omega)} \]

\[ i \geq 0 \quad n \geq 2 \]

7. Approximation Theory Results and The Gronwall Lemma

Certain approximation theory results are reviewed in this section to provide a complete theory of convergence. These results will be presented as a series of known lemmas, the proofs of which can be found in the literature cited.
Suppose we define an element $w$ of the subspace $S_h(\Omega)$ through the weighted $H^1(\Omega)$ projection introduced by Wheeler [5]. Then $W$ satisfies

$$a(u, u - W, V) = 0 \quad \forall \ V \in S_h(\Omega) \quad (7.1)$$

Let $E$ denote the spatial projection error,

$$E = u - W \quad (7.2)$$

Then the behavior of $E$ and its time derivatives in various norms is given in the following lemma:

**Lemma 7.1.** Let $u, \frac{\partial u}{\partial t} \in L_\infty(H^{k+1}(\Omega))$ and $\frac{\partial^2 u}{\partial t^2} \in L_2(H^{k+1}(\Omega))$. Then there exists a constant $C$, independent of the discretization parameters, such that

$$\|E\|_{L_\infty(L_2(\Omega))} + \|\frac{\partial E}{\partial t}\|_{L_\infty(L_2(\Omega))} + \|\frac{\partial^2 E}{\partial t^2}\|_{L_2(L_2(\Omega))} \leq C(h^{k+1}\|u\|_{L_\infty(H^{k+1}(\Omega))} + h^{k+1}\|\frac{\partial u}{\partial t}\|_{L_\infty(H^{k+1}(\Omega))}$$

$$+ \ h^{k+1}\|u\|_{L_2(H^{k+1}(\Omega))} \} \quad (7.3)$$

This Lemma has been established, for example, by Wheeler [5]. The second Lemma is the discrete version of the classical Gronwall inequality (Cf. Lees [21]).
Lemma 7.2. (The Discrete Gronwall Inequality) If $\phi(t)$ and $\psi(t)$ are nonnegative functions with $\psi(t)$ nondecreasing, and

$$\phi(N\Delta t) \leq \psi(N\Delta t) + C\Delta t \sum_{n=0}^{N-1} \phi(n\Delta t)$$

Then

$$\phi(N\Delta t) \leq \psi(N\Delta t)e^{CN\Delta t}$$

We can now pass on to the investigation of the first scheme proposed in Section 5.

8. The Linear Central Difference Approximation

In this section we briefly outline the method of establishing a priori bounds for the approximation error involved in modeling the linearized version of (3.2), with the linearized scheme (5.2). The methods used in this section are of the $L_2$-type, and represent an extension of the results of Dupont [10] to the explicit case. Our results are in some ways similar to those obtained by Fujii [9]. In the section following this one, we expand our results to the nonlinear problem.

Initially we evaluate (3.2) at $t = n\Delta t$ and set $v = v_\Omega S_h(\Omega)$. Then adding $(\rho_\delta^2 u_n, V)_\Omega$ to each side of (3.2), we have

$$(\rho_\delta^2 u_n, V)_\Omega + a(u_n, V) = (f, V)_\Omega + (\epsilon_n, V)_\Omega \quad \forall \ V \in S_h(\Omega)$$

(8.1)
where \( u_n \) is the exact solution evaluated at time point \( t = n\Delta t \),

\[
a(u_n, v) = \int_{\Omega} \nabla^2(x, t) \nabla u_n \cdot \nabla v \, dx,
\]

and

\[
\epsilon_n = \frac{\partial^2 u_n}{\partial t^2} \bigg|_{t = n\Delta t}
\]

(8.2)

We assume in this development that the regularity property,

\[
\frac{\partial^4 u}{\partial t^4} \in L_2(L_2(\Omega)) \text{ holds. This assumption precludes the existence}
\]

of certain physical phenomena such as shock and acceleration waves

in the solution. For \( \frac{\partial^4 u}{\partial t^4} \in L_2(L_2(\Omega)) \), Dupont [10] has shown that

an estimate for \( \epsilon_n \) is

\[
||\epsilon_n||_{L_2(\Omega)}^2 \leq C\Delta t^3 \int_{t_{n-1}}^{t_{n+1}} ||\frac{\partial^4 u}{\partial t^4}(\tau)||_{L_2(\Omega)}^2 \, d\tau
\]

(8.3)

Setting \( e_n = u_n - U^n \) and subtracting the linearized version

of (5.2) from (6.1), we get

\[
(\rho \delta_t^2 e_n, v) + a(e_n, v) = (\rho e_n, v) \quad \forall \, v \in S_h(\Omega)
\]

(8.4)

We again identify an element \( W^n \in S_h(\Omega) \) through the weighted

\( H^1(\Omega) \) projection introduced by Wheeler [5]

\[
a(u_n - W^n, v) = 0, \quad \forall \, v \in S_h(\Omega)
\]

(8.5)

We decompose \( e_n \) by letting \( e_n = E_n + E^n \) where \( E_n = u_n - W^n \)

and \( E^n = W^n - U^n \). In addition we define certain auxiliary variables by
\[
\begin{align*}
\delta_t u_{n+\frac{1}{2}} &= \frac{1}{2} (u_{n+1} + u_n) \\
\delta_t u_{n+\frac{1}{2}} &= \frac{u_{n+1} - u_n}{\Delta t} \\
\delta_t u_{n+\frac{1}{2}}(x) &= \frac{X|_{t = (n+1)\Delta t} - X|_{t = n\Delta t}}{\Delta t}
\end{align*}
\] (8.6)

The behavior of the error component \(E_n\) is given by the following Theorem:

**Theorem 8.1:** If \(\frac{3}{4} u \in L_2(L_2(\Omega))\) and \(\frac{\Delta t^2}{h^2} < -\frac{2\rho}{c'C*2}\), then there exist positive constants \(C_1, C_2\), not depending on the discretization parameters, such that

\[
\|\delta_t E\|_{L_\infty(L_2(\Omega))} + C_1 \|E\|_{L_\infty(H^1(\Omega))} \\
\leq C_2 \{ \|E_0\|_{H^1(\Omega)} + \|E_1\|_{H^1(\Omega)} + \|\delta_t E_1\|_{L_\infty(\Omega)} \\
+ \|\frac{3}{4} E \|_{L_2(L_2(\Omega))} + \Delta t^2 \|\frac{3}{4} u \|_{L_2(L_2(\Omega))} \} (8.7)
\]

where

\[
\|\delta_t E\|_{L_\infty(L_2(\Omega))} = \sup_{0 \leq n < N} \|\delta_t E_{n+\frac{1}{2}}\|_0 (8.8)
\]

**Proof:** It follows from the decomposition of \(e_n\) and (8.4) that

\[
(\rho \delta_t E_n, V) + a(E_n, V) = - (\rho \delta_t E_n, V) \\
- a(E_n, V) + (\rho e_n, V) \in S_h(\Omega) (8.9)
\]

Now let \(V = \delta_t E_{n+\frac{1}{2}} + \delta_t E_{n-\frac{1}{2}}\); then
(\rho \delta_t^2 E_n', \delta_t E_n' + \delta_t E_n' \cdot o + a(E_n', \delta_t E_n' + \delta_t E_n')

\begin{align*}
&= - (\rho \delta_t^2 E_n', \delta_t E_n' + \delta_t E_n' \cdot o \\
&- a(E_n', \delta_t E_n' + \delta_t E_n' + \delta_t E_n' \cdot o)
\end{align*}

(8.10)

But

\begin{align*}
a(E_n', \delta_t E_n' + \delta_t E_n') &= -\frac{\Delta t}{2} a(\delta_t E_n' + \delta_t E_n', \delta_t E_n') \\
&+ \frac{1}{2} \delta_t E_n' a(E, E)
\end{align*}

(8.11)

Similarly

\begin{align*}
a(E_n', \delta_t E_n' - \delta_t E_n') &= \frac{\Delta t}{2} a(\delta_t E_n' + \delta_t E_n', \delta_t E_n') \\
&+ \frac{1}{2} \delta_t E_n' a(E, E)
\end{align*}

(8.12)

and

\begin{align*}
(\rho \delta_t^2 E_n', \delta_t E_n' + \delta_t E_n' \cdot o) &= \frac{1}{\Delta t} \left[ |\delta_t E_n' + \delta_t E_n' |^2 - |\delta_t E_n' - \delta_t E_n' |^2 \right]
\end{align*}

(8.13)

Introducing these results into (8.11), we get

\begin{align*}
&\frac{1}{\Delta t} \left[ |\delta_t E_n' + \delta_t E_n' |^2 - |\delta_t E_n' - \delta_t E_n' |^2 \right] - \frac{\Delta t}{2} a(\delta_t E_n' + \delta_t E_n', \delta_t E_n') \\
&+ \frac{\Delta t}{2} a(\delta_t E_n' + \delta_t E_n', \delta_t E_n') + \frac{1}{2} \delta_t E_n' a(E, E) + \frac{1}{2} \delta_t E_n' a(E, E)
\end{align*}

(8.14)
Eliminating the second term on the right hand side using (8.5), estimating the remaining terms on the right hand side using the Cauchy-Schwarz inequality and the elementary inequality E, multiplying by $\Delta t$, and summing from $1$ to $N - 1$,

$$
\frac{1}{|\rho^2 \delta_{E, N-\frac{1}{2}}|^2} - \frac{1}{|\rho^2 \delta_{E, 1}|^2} - \frac{\Delta t^2}{2} a(\delta_{E, N-\frac{1}{2}}, \delta_{E, N-\frac{1}{2}})
+ \frac{\Delta t^2}{2} a(\delta_{E, 1}, \delta_{E, 1}) + \frac{1}{2} a(E_n, E_n) + \frac{1}{2} a(E_{n-1}, E_{n-1})
- \frac{1}{2} a(E_1, E_1) - \frac{1}{2} a(E_0, E_0) \leq \Delta t \sum_{i=1}^{N-1}\left\{ \kappa \left| \rho^2 \delta_{E, n}^2 \right|_0^2 \right\}
+ \xi \left| \rho^2 \delta_{E, n} \right|_0^2
+ \Delta t \sum_{i=1}^{N-1}\left\{ \alpha \left| \rho^2 \delta_{E, n+\frac{1}{2}} \right|_0^2 \right\}
+ \nu \left| \rho^2 \delta_{E, n-\frac{1}{2}} \right|_0^2
(8.15)
$$

where $\kappa$, $\xi$, $\alpha$, and $\nu$ are positive constants. Then using the Cauchy-Schwarz inequality, we conclude that there exists a positive constant $C'$ such that

$$
a(\delta_{E, N-\frac{1}{2}}, \delta_{E, N-\frac{1}{2}}) \leq C' \left| \delta_{E, N-\frac{1}{2}} \right|_{H^1(\Omega)}^2
$$

and using the inverse hypothesis on the subspace $S_n(\Omega)$ (4.9), we have

$$
a(\delta_{E, n-\frac{1}{2}}, \delta_{E, n-\frac{1}{2}}) \leq \frac{C' c^2}{h^2} \left| \delta_{E, n-\frac{1}{2}} \right|_0^2
(8.16)
$$

Introducing (8.14) into (8.13), using (3.4), and applying the Cauchy-Schwarz inequality.
\[(1 - \frac{C'C^*}{2\rho} \frac{\Delta t^2}{h^2}) ||\frac{1}{\rho} \delta_t E_{n-1/2}||^2_o + \frac{\mu}{2} ||E_N||^2_{H^1(\Omega)} \]

\[+ \frac{\mu}{2} ||E_{N-1}||^2_{H^1(\Omega)} \leq \frac{C'}{2} ||E_N||^2_{H^1(\Omega)} + \frac{C'}{2} ||E_1||^2_{H^1(\Omega)} \]

\[+ \frac{1}{2} ||\rho^2 \delta_t E_1||^2_o \]

\[+ \Delta t \sum_{n=1}^{N-1} \{\kappa ||\frac{1}{\rho^2} \delta_t^2 E_n||^2_o + \xi ||\rho^2 E_n||^2_o \} \]

\[+ \Delta t \sum_{n=1}^{N-1} \{\alpha ||\frac{1}{\rho^2} \delta_t E_{n+1/2}||^2_o + \nu ||\rho^2 \delta_t E_{n-1/2}||^2_o \} \]

(8.17)

As a condition of stability we require that

\[(1 - \frac{C'C^*}{2\rho} \frac{\Delta t^2}{h^2}) = C'' > 0 \] (8.18)

which, as expected, places a constraint on the permissible values of the discretization parameters.

Finally, applying the discrete Gronwall inequality (Lemma 7.2), and the inequality

\[\Delta t \sum_{n=1}^{N-1} ||\delta_t^2 v_n|| \leq ||\frac{\delta^2 v}{\delta t^2}||_{L^2(L^2(\Omega))} \]

(8.19)

we obtain (8.8).

Then using Theorem 8.1, Lemma 7.1, and the triangle inequality, we obtain the final error estimate:

Theorem 8.2: If \(u, u_t \in L_\infty(H^{k+1}(\Omega))\), \(\frac{\delta^2 u}{\delta t^2} \in L_2(H^{k+1}(\Omega))\),

\(\frac{\delta^4 u}{\delta t^4} \in L_2(L^2(\Omega))\), and \(\frac{\Delta t^2}{h^2} \leq \frac{2\rho}{c'C^*}\), there exist positive constants \(C_3\) and \(C_4\) such that
9. The Nonlinear Central Difference Approximation

In this section, a priori bounds for the error involved in approximating (3.2) with (5.2) are established. Initially we 
evaluate (3.2) at \( t = n\Delta t \) and set \( v = V \). The effect of the tem-
poral approximation is determined by adding \( (\rho \delta^2_t u_n, V) \) to each 
side of the equation.

\[
\begin{align*}
||\delta_t e||_{L^\infty(L^2(\Omega))} + C_3 ||e||_{L^\infty(L^2(\Omega))} \\
\leq C_4 \left( ||e_0||_{H^1(\Omega)} + ||e_1||_{H^1(\Omega)} + ||\delta_t e_2||_{L^2(\Omega)} \\
+ h^{k+1} ||u||_{L^\infty(H^{k+1}(\Omega))} + h^{k+1} ||\frac{\partial u}{\partial t}||_{L^\infty(H^{k+1}(\Omega))} \\
+ h^{k+1} ||u||_{L^2(H^{k+1}(\Omega))} + \Delta t^2 ||\frac{\partial^4 u}{\partial t^4}||_{L^2(\Omega)} \right)
\end{align*}
\]
We use a decomposition of the error of the approximation described by (8.6). Let $w^n \in S^h_\Omega$ be defined by the nonlinear energy projection introduced by Wheeler [5]:

$$a(u_n, u_n - w_n, v) = 0 \quad \forall \ v \in S^h_\Omega \quad (9.2)$$

Then subtracting (5.2) from (9.1), we have

$$a(u_n, u_n - w_n, v) + a(u_n, u_n, v) - a(U_n, U_n, v) = (\rho \varepsilon_n, v)_o$$

$$\forall \ v \in S^h_\Omega \quad (9.3)$$

Now we introduce certain conditions which for the nonlinear central difference scheme turn out to be sufficient for convergence.

I. The Stability Condition

$$\frac{\Delta t^2}{h^2} \leq \frac{2\rho}{M_2C^*_2} \quad (9.4)$$
II. The Response Condition

Let

\[ L_n = M_3 \left| \frac{u^{n+1} - u^{n-1}}{2\Delta t} \right|_{L_{\infty}(\Omega)} \] (9.5)

\[ Q_n = \frac{nM_3^2 \xi^2 C^2}{2} \sup_{1 \leq i \leq n} \left| \frac{\partial W^n}{\partial x_i} \right|_{L_{\infty}(\Omega)}^2 \] (9.6)

where \( n, \xi \) are positive constants. Then we require

\[ \frac{h}{2} (\| E_{N-1} \|_{H^1(\Omega)}^2 + \| E_N \|_{H^1(\Omega)}^2) - \sum_{n=1}^{N-2} \Delta t L_n \| E_n \|_{H^1(\Omega)}^2 \]

\[ - \sum_{n=1}^{N-1} \frac{\Delta t}{h^2} Q_n \| E_n \|_{H^1(\Omega)}^2 \geq \phi \| E_N \|_{H^1(\Omega)}^2 \] (9.7)

where \( \phi > 0 \).

The behavior of the error component \( E_n \) is given in the following theorem:

**Theorem 9.1.** If \( \frac{\partial^4 u}{\partial t^4} \notin L_2(L_2(\Omega)) \) and the stability condition I and the response condition II are satisfied, then there exist positive constants \( C_1 \) and \( C_2 \), not depending on the discretization parameter such that
\[
\|\delta_t E\|_{L_\infty(L_2(\Omega))} + C_1 \|E\|_{L_\infty(H^1(\Omega))} \leq C_1 \|E_o\|_{H^1(\Omega)} + \|E_1\|_{H^1(\Omega)} + \|\delta_t E\|_{L_\infty} + \frac{1}{n} \|E\|_{L_\infty(L_2(\Omega))} + \|\frac{\partial^2 E}{\partial t^2}\|_{L_2(L_2(\Omega))} + \Delta t^2 \|\frac{\partial^4 u}{\partial t^4}\|_{L_2(\Omega)} (9.8)
\]

Proof: Decomposing \(e_n\) in (9.3)

\[
(\rho \delta_t^2 E_n, V) + a(u_n, u_n, V) - a(U_n, U_n, V)
\]

\[
\leq - (\rho \delta_t^2 E_n, V) + (\rho e_n, V) V V \leq S_h(\Omega) (9.9)
\]

But

\[
a(u_n, u_n, V) - a(U_n, U_n, V)
\]

\[
= a(u_n, E_n, V) + ((c^2(x, t, u_n) - c^2(x, t, w^n)) \|w^n, V\|_V)
\]

\[
+ ((c^2(x, t, w^n) - c^2(x, t, u^n)) \|w^n, V\|_V) + a(U_n, E_n, V)
\]

(9.10)
Introducing (9.10) into (9.9) and setting $V = \delta t^n E_{n+\frac{1}{2}} + \delta t^n E_{n-\frac{1}{2}}$.

\[
(r^\delta_2 E_n, \delta t^n E_{n+\frac{1}{2}} + \delta t^n E_{n-\frac{1}{2}}) + a(U^n, E_n, \delta t^n E_{n+\frac{1}{2}} + \delta t^n E_{n-\frac{1}{2}})
\]
\[
+ (\tilde{C}^2(x, t, w^n) - \tilde{C}^2(x, t, u^n)) \tilde{\nabla} w^n, \tilde{\nabla} (\delta t^n E_{n+\frac{1}{2}} + \delta t^n E_{n-\frac{1}{2}})
\]
\[
= - (r^\delta_2 E_n, \delta t^n E_{n+\frac{1}{2}} + \delta t^n E_{n-\frac{1}{2}}) - a(u_n, E_n, \delta t^n E_{n+\frac{1}{2}} + \delta t^n E_{n-\frac{1}{2}})
\]
\[
- (\tilde{C}^2(x, t, u^n) - \tilde{C}^2(x, t, w^n)) \tilde{\nabla} w^n, \tilde{\nabla} (\delta t^n E_{n+\frac{1}{2}} + \delta t^n E_{n-\frac{1}{2}})
\]
\[
+ (r^\delta n, \delta t^n E_{n+\frac{1}{2}} + \delta t^n E_{n-\frac{1}{2}})
\]  

(9.11)

We next repeat steps used to obtain (8.13) to simplify the first term on the left and use (9.2) to eliminate the second term on the right.

\[
\frac{1}{\Delta t} \left[ \left| \delta t^n E_{n+\frac{1}{2}} \right| - \left| \delta t^n E_{n-\frac{1}{2}} \right| \right] + a(U^n, E_n, \delta t^n E_{n+\frac{1}{2}} + \delta t^n E_{n-\frac{1}{2}})
\]
\[
= - (r^\delta_2 E_n, \delta t^n E_{n+\frac{1}{2}} + \delta t^n E_{n-\frac{1}{2}})
\]
\[
- (\tilde{C}^2(x, t, w^n) - \tilde{C}^2(x, t, u^n)) \tilde{\nabla} w^n, \tilde{\nabla} (\delta t^n E_{n+\frac{1}{2}} + \delta t^n E_{n-\frac{1}{2}})
\]
\[
- (\tilde{C}^2(x, t, u^n) - \tilde{C}^2(x, t, w^n)) \tilde{\nabla} w^n, \tilde{\nabla} (\delta t^n E_{n+\frac{1}{2}} + \delta t^n E_{n-\frac{1}{2}})
\]
\[
+ (r^\delta n, \delta t^n E_{n+\frac{1}{2}} + \delta t^n E_{n-\frac{1}{2}})
\]  

(9.12)
In order to estimate the terms in (9.12) we introduce certain useful relationships

\[
a(U^n, E_n, \delta t E_{n+\frac{1}{2}}) = -\frac{\Delta t}{2} a(U^n, \delta_t E_{n+\frac{1}{2}}, \delta_t E_{n+\frac{1}{2}}) + \frac{1}{2} \delta t_{n+\frac{1}{2}} a(U^n, E, E)
\]

and

\[
a(U^n, E_n, \delta_t E_{n-\frac{1}{2}}) = \frac{\Delta t}{2} a(U^n, \delta_t E_{n-\frac{1}{2}}, \delta_t E_{n-\frac{1}{2}}) + \frac{1}{2} \delta t_{n-\frac{1}{2}} a(U^n, E, E)
\]

Then

\[\Delta t \sum_{i=1}^{N-1} [a(U^n, E_n, \delta_t E_{n+\frac{1}{2}} + \delta_t E_{n-\frac{1}{2}})]\]

\[= -\frac{\Delta t^2}{2} a(U^{n-1}, \delta_t E_{n-\frac{1}{2}}, \delta_t E_{n-\frac{1}{2}}) + \frac{\Delta t^2}{2} a(U^0, \delta_t E_{\frac{1}{2}}, \delta_t E_{\frac{1}{2}})
\]

\[+ \frac{\Delta t^2}{2} \sum_{i=1}^{N-1} [a(U^n, \delta_t E_{n-\frac{1}{2}}, \delta_t E_{n-\frac{1}{2}}) - a(U^{n-1}, \delta_t E_{n-\frac{1}{2}}, \delta_t E_{n-\frac{1}{2}})]
\]

\[+ \frac{1}{2} a(U^{N-1}, E_N, E_N) + \frac{1}{2} a(U^{N-2}, E_{N-1}, E_{N-1}) - \frac{1}{2} a(U^2, E_1, E_1)
\]

\[- \frac{1}{2} a(U^1, E_0, E_0) + \frac{1}{2} \sum_{n=1}^{N-2} [a(U^{n-1}, E_n, E_n) - a(U^{n+1}, E_n, E_n)]
\]

(9.13)
Using (3.3)_3 and the Holder inequality, we obtain

\[ \frac{\Delta t^2}{2} \sum_{i=1}^{N-1} \left[ a(U^n, \delta_t E_{n-\frac{\Delta t}{2}}, \delta_t E_{n-\frac{\Delta t}{2}}) - a(U^{n-1}, \delta_t E_{n-\frac{\Delta t}{2}}, \delta_t E_{n-\frac{\Delta t}{2}}) \right] \]

\[ \leq \sum_{i=1}^{N-1} \frac{\Delta t^3}{h^2} K_n \| \delta_t E_{n-\frac{\Delta t}{2}} \|^2_0 \]

where

\[ K_n = M_3 C^* \frac{\| U^n - U^{n-1} \|_{L_\infty(\Omega)}}{\Delta t} \]

and

\[ \frac{1}{2} \sum_{n=1}^{N-2} \left[ a(U^{n-1}, E_{n+1}, E_n) - a(U^{n+1}, E_{n+1}, E_n) \right] \]

\[ \leq \sum_{n=1}^{N-2} \Delta t L_n \| E_n \|_{H^1(\Omega)}^2 \]

In addition, the inverse assumption (4.9) leads to

\[ \frac{\Delta t^2}{2} a(U^{N-1}, \delta_t E_{N-\frac{\Delta t}{2}}, \delta_t E_{N-\frac{\Delta t}{2}}) \leq \frac{M_2 C^*}{2} \frac{\Delta t^2}{h^2} \| \rho^{\frac{1}{2}} \delta_t E_{N-\frac{\Delta t}{2}} \|^2_0 \]

Thus from (9.13)
Similarly using the Holder inequality, the inverse assumption (4.9), and the imbedding result \( ||E_n||_{L^2(\Omega)} \leq \xi ||E_n||_{H^1(\Omega)} \) \[^{16}\], we conclude that there exists a positive constant \( \eta \) such that

\[
\Delta t \sum_{i=1}^{N-1} \left( a(U^n, E_n, \delta_t E_{n+\frac{1}{2}} + \delta_t E_{n-\frac{1}{2}}) \right) \geq \frac{M_2 c^*}{2\rho} \frac{(\Delta t)^2}{h^2} ||\rho \delta_t E_{N-\frac{1}{2}}||^2_0 + \frac{\Delta t^2}{2} a(U^0, \delta_t E_{\frac{1}{2}}, \delta_t E_{\frac{1}{2}}) - \frac{1}{2} a(U^{N-1}, E_N, E_N) \\
\sum_{n=1}^{N-1} \Delta t L_n ||E_n||^2_{H^1(\Omega)}
\]

(9.14)
where $Q_n$ is defined by (9.6). In addition

$$
\Delta t \sum_{i=1}^{N-1} \left( (c^2(x,t,u_n) - c^2(x,t,w^n)) \nabla w^n, \nabla (\delta_t E_{n+\frac{1}{2}} + \delta_t E_{n-\frac{1}{2}}) \right)_0
$$

$$
\leq \sum_{i=1}^{N-1} \frac{\Delta t}{h^2} \delta_n \| E_n \|_0^2 + \frac{\Delta t}{2n} \sum_{i=1}^{N-1} \left[ \| \delta_t E_{n+\frac{1}{2}} \|_0^2 + \| \delta_t E_{n-\frac{1}{2}} \|_0^2 \right]
$$

(9.16)

where

$$
\delta_n = \frac{\eta M_2^2 C^2}{2} \sup_{1 \leq i \leq n} \left\| \frac{\partial}{\partial x_i} w^n \right\|_{L^\infty(\Omega)}
$$

(9.17)

Multiplying (9.12) by $\Delta t$, summing from 1 to $N-1$, and using (9.14-9.18)

$$
(1 - \frac{M_2 C^2}{2\rho} \frac{\Delta t^2}{h^2}) \| \rho^{\frac{1}{2}} \delta_t E_{N-\frac{1}{2}} \|_0^2
$$

$$
+ \frac{1}{2} a(U^{N-1}, E_n, E_n) + \frac{1}{2} a(U^{N-2}, E_{N-1}, E_{N-1}) - \sum_{n=1}^{N-2} \Delta t L_n \| E_n \|_{H^1(\Omega)}^2
$$

$$
- \sum_{n=1}^{N-1} \frac{\Delta t}{h^2} Q_n \| E_n \|_{H^1(\Omega)}^2 \leq \| \rho^{\frac{1}{2}} \delta_t E_{\frac{1}{2}} \|_0^2 + \frac{1}{2} a(U, E, E)
$$
Now using the stability condition (I) and the response condition (II), estimating the terms on the right-hand side using the Cauchy-Schwarz inequality and inequality E, applying the Gronwall inequality (Lemma 7.2) and using (8.17), we obtain the result (9.8).

Now using Theorem 9.1, Lemma 7.1, and the triangle inequality, we obtain the final error estimate.

**Theorem 9.2.** If \( u, \frac{\partial u}{\partial t} \in L_\infty(H^{k+1}(\Omega)), \frac{\partial^2 u}{\partial t^2} \in L_2(H^{k+1}(\Omega)), \frac{\partial^4 u}{\partial t^4} \in L_2(L_2(\Omega)) \), and the stability condition I and the response condition II are satisfied, then there exist positive constants \( C_3 \) and \( C_4 \) such that

\[
\| \delta_t e \|_{L_\infty(L_2(\Omega))} + C_3 \| e \|_{L_\infty(L_2(\Omega))}
\]
\[
\begin{align*}
40 \\
\leq C_4 \left( \| e_0 \|_{H^1(\Omega)} + \| e_1 \|_{H^1(\Omega)} ight) \\
+ \| \delta \varepsilon_k^2 \|_0 + h^k \| u \|_{L_\infty(H^{k+1}(\Omega))} + h^{k+1} \| \partial u \|_{L_\infty(H^{k+1}(\Omega))} \\
+ h^{k+1} \| \partial^2 u \|_{L_2(H^{k+1}(\Omega))} + \Delta t^2 \| \partial^4 u \|_{L_2(L_2(\Omega))}
\end{align*}
\] (9.19)

10. The Linear Parabolic Regularization Approximation

In this section of the paper we consider the approximation of the linearized version of (3.2) by the corresponding linearized versions of (5.9) and (5.10). To simplify the calculations, it is assumed that \( f = 0 \). However, the method presented here is in no way restricted to this case.

It is possible to split up the second order equation (3.2) into two coupled first order equations. This is carried out by defining the new variable \( y = \frac{\partial u}{\partial t} \). Then (3.2) is fully equivalent to the system

\[
\begin{align*}
(p \frac{\partial y}{\partial t}, v)_0 + a(u, v) &= 0 \quad \forall \ v \in H^1(\Omega) \\
(p \frac{\partial u}{\partial t}, v)_0 - (p y, v)_0 &= 0 \quad \forall \ v \in H^1(\Omega)
\end{align*}
\] (10.1)

Thus, when we discuss the convergence of the parabolic regularization scheme, we mean convergence to the solution of a problem (10.1) which is equivalent to (3.2).
Now we pose an auxiliary problem. Let \((\tilde{u}, \tilde{y})\) be the solution to the system

\[
\begin{align*}
(\rho \frac{\partial \tilde{y}}{\partial t}, v) + a(\tilde{u}, v) + \frac{\Delta t^*}{2} a(\tilde{y}, v) &= 0 \quad \forall \, v \in H^1(\Omega) \\
(\rho \frac{\partial \tilde{u}}{\partial t}, v) - (\rho \tilde{y}, v) + \frac{\Delta t^*}{2} a(\tilde{u}, v) &= 0 \quad \forall \, v \in H^1(\Omega)
\end{align*}
\]

(10.2)

We obtain an approximate auxiliary problem by introducing the forward difference operator in (10.2). Then the solution to the approximate auxiliary problem \((\tilde{U}^n, \tilde{Y}^n)\) satisfies

\[
\begin{align*}
\frac{\rho \tilde{Y}^{n+1} - \tilde{Y}^n}{\Delta t} + a(\tilde{U}^n, v) + \frac{\Delta t^*}{2} a(\tilde{Y}^n, v) &= 0 \quad \forall \, v \in S_h(\Omega) \\
\frac{\rho \tilde{U}^{n+1} - \tilde{U}^n}{\Delta t} - (\rho \tilde{Y}^n, v) + \frac{\Delta t^*}{2} a(\tilde{U}^n, v) &= 0 \quad \forall \, v \in S_h(\Omega)
\end{align*}
\]

(10.3)

To demonstrate the convergence and determine the rate of convergence, we will show that, for stable schemes,

\[
\begin{align*}
U^n &\rightarrow \tilde{U}^n \rightarrow \tilde{u} \rightarrow u \\
\Delta t + \Delta t^* &\rightarrow \Delta t + 0 \rightarrow \Delta t^* + 0 \\
Y^n &\rightarrow \tilde{Y}^n \rightarrow \tilde{y} \rightarrow y \\
\Delta t + \Delta t^* &\rightarrow \Delta t + 0 \rightarrow \Delta t^* + 0 \\
h &\rightarrow 0
\end{align*}
\]
and this implies that

\[ u^n \rightarrow u \]
\[ \frac{\Delta t}{h} \rightarrow 0 \]

\[ u^n \rightarrow y \]
\[ \frac{\Delta t}{h} \rightarrow 0 \]

Initially, we will investigate the convergence of (10.3) to (10.2). Then using the regularity results of section 6, the convergence of (10.2) to (10.1) will be determined. As a final step \( \Delta t^* \) in (10.3) will be made to approach \( \Delta t \), and the convergence of (5.9) and (5.10) to (10.1) will be the result.

If (10.2)\(_2\) is evaluated at time \( t = n\Delta t \) and \( v \) is equated to \( v \), then it can be seen that

\[ (\rho \frac{\partial \tilde{u}_n}{\partial t}, \mathbf{V})_0 - (\rho \tilde{y}_n, \mathbf{V})_0 + \frac{\Delta t^* \alpha}{2} a(\tilde{u}_n, \mathbf{V}) = 0, \quad \mathbf{V} \in S_h(\Omega) \quad (10.3) \]

Now adding \( \frac{\tilde{u}_{n+1} - \tilde{u}_n}{\Delta t} \) to each side of (10.3) gives

\[ (\rho \frac{\partial \tilde{u}_{n+1}}{\partial t}, \mathbf{V})_0 - (\rho \tilde{y}_n, \mathbf{V})_0 + \frac{\Delta t^* \alpha}{2} a(\tilde{u}_n, \mathbf{V}) = (\rho \psi_N, \mathbf{V})_0 \quad \mathbf{V} \in S_h(\Omega) \quad (10.4) \]

where

\[ \psi_n = \frac{\tilde{u}_{n+1} - \tilde{u}_n}{\Delta t} - \frac{\partial \tilde{u}}{\partial t} \bigg|_{t=n\Delta t} \]
It can be shown that

$$\psi_n = \int_{t_n}^{t_{n+1}} (t_{n+1} - t) \frac{\partial^3 \bar{u}}{\partial t^3} \, dt \quad (10.5)$$

and an index of the accumulated temporal approximation error is

$$\psi = \sum_{n=0}^{N-1} \| \psi_n \|_2^2 = \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} (t_{n+1} - t)^2 \| \frac{\partial^3 \bar{u}(t)}{\partial t^3} \|_2^2 \, dt \quad (10.6)$$

Using the Cauchy inequality, we get

$$\psi \leq \sum_{n=0}^{N-1} (\int_{t_n}^{t_{n+1}} (t_{n+1} - t)^2 \, dt) \int_{t_n}^{t_{n+1}} \| \frac{\partial^3 \bar{u}(t)}{\partial t^3} \|_2^2 \, dt$$

$$= \frac{\Delta t^3}{3} \| \frac{\partial^3 \bar{u}}{\partial t^3} \|_2^2 \quad (L_2(L_2(\Omega)))$$

Thus

$$\Delta t \psi = \Delta t \sum_{n=0}^{N-1} \| \psi_n \|_2^2 \leq \frac{\Delta t^4}{3} \| \frac{\partial^3 \bar{u}}{\partial t^3} \|_2^2 \quad (10.7)$$

Now set \( e_n = \bar{u} - \bar{u}^n \) and \( f_n = \bar{y} - \bar{y}^n \). Then subtracting (10.3)2 from (10.4) gives

$$\left( \frac{e_{n+1} - e_n}{\Delta t}, V \right)_o - (\rho f_n, V) + \frac{\Delta t \alpha}{2} (e_n, V) = (\rho \psi_n, V) \quad V \in S_h(\Omega) \quad (10.8)$$
We identify elements $W^n, P^n \in S_h(\Omega)$ through the weighted $H^1(\Omega)$ projection, as was done previously,

$$a(\tilde{u}_n - W^n, v) = 0, \quad \forall v \in S_h(\Omega)$$  \hspace{1cm} (10.9)

$$a(\tilde{y}_n - P^n, v) = 0, \quad \forall v \in S_h(\Omega)$$

Thus, we perform the normal decomposition of the approximation error $e_n$ and $f_n$. Let $e_n = E_n + E$ where $E_n = \tilde{u}_n - W^n$ and $E = u - \tilde{u}$, and $f_n = F_n + f$ where $F_n = \tilde{y}_n - P^n$ and $f = y - \tilde{y}$.

The following theorem describes the behavior of $E_n$:

**Theorem 10.1.** Let \( \frac{3^3\tilde{u}}{\partial t^3} \in L_2(L_2(\Omega)) \), and

$$\Delta t \frac{\alpha}{h^2} < \frac{8\beta}{C'C^*^2}$$

where $\beta$ is a positive constant. Then there exists a constant $C_1$ such that

$$||E||_{L_\infty(L_2(\Omega))} \leq C_1 \left( ||E_0||_0 + ||F||_{L_\infty(L_2(\Omega))} ||\frac{\alpha}{\partial t^3}||_{L_2(L_2(\Omega))} \\ + ||F||_{L_\infty(L_2(\Omega))} + \Delta t^2 ||\frac{\alpha}{\partial s^3}||_{L_2(L_2(\Omega))} \right)$$  \hspace{1cm} (10.10)

where $||E||_{L_\infty(L_2(\Omega))} = \sup_{0 \leq i \leq N} ||E_i||_0$.

**Proof:** Decomposing the error in (10.8) and using (8.7)$_3$, leads to
\[
\begin{align*}
\langle \rho \delta_t E_{n+\frac{1}{2}}, V \rangle_0 + \frac{\Delta t^* \alpha}{2} a(E_n, V) &= -\langle \rho \delta_t E_{n+\frac{1}{2}}, V \rangle_0 + \langle \rho F_n, V \rangle_0 \\
+ \langle \rho F_n, V \rangle_0 - \frac{\Delta t^* \alpha}{2} a(E_n, V) \\
+ \langle \rho \psi_n, V \rangle_0 & \quad \forall \, V \in S_h(\Omega)
\end{align*}
\]

We can now set \( V = E_{n+\frac{1}{2}} \) since \( E_{n+\frac{1}{2}} \in S_h(\Omega) \). Then, using (10.9), the Cauchy-Schwarz inequality, inequality \( E \), the inverse property (4.6), and the coercive property (3.4), we get for positive constants \( \psi \) and \( \beta \)

\[
\frac{1}{\Delta t} \left[ (1 - \frac{\Delta t^* \alpha C^* C^*}{8 \beta h^2}) \left| \rho \frac{E_{n+1}}{h} \right|_0^2 - \left| \rho \frac{E_n}{h} \right|_0^2 \right] + \Delta t^* \alpha \psi \left| E_n \right|_{H^1(\Omega)}^2
\leq -\langle \rho \delta_t E_{n+\frac{1}{2}}, E_{n+\frac{1}{2}} \rangle_0 + \langle \rho F_n, E_{n+\frac{1}{2}} \rangle_0 + \langle \rho F_n, E_{n+\frac{1}{2}} \rangle_0 + \langle \rho \psi_n, E_{n+\frac{1}{2}} \rangle_0
\]

As a condition of stability we require that

\[
\frac{\Delta t^* \alpha}{h^2} < \frac{8 \beta}{C^* C^*}
\]

Now estimating the terms on the right hand side of (10.12) using the Cauchy-Schwarz inequality and inequality \( E \), multiplying by \( \Delta t \), and summing from 1 to \( N-1 \), we get

\[
\left| \rho \frac{E_{N}}{h} \right|_0 - \left| \rho \frac{E_0}{h} \right|_0^2 + \Delta t \Delta t^* \alpha \psi \sum_{n=0}^{N-1} \left| E_n \right|_{H^1(\Omega)}^2
\leq \Delta t \sum_{n=0}^{N-1} \left\{ \frac{\gamma}{2} \left| \rho \frac{E_{n+\frac{1}{2}}}{h} \right|_0^2 + \frac{\eta}{2} \left| \rho \frac{F_n}{h} \right|_0^2 + \frac{\xi}{2} \left| \rho \frac{F_{n+\frac{1}{2}}}{h} \right|_0^2 + \frac{\omega}{2} \left| \rho \frac{\psi_n}{h} \right|_0^2 \right\}
\]

\[
+ \Delta t \sum_{n=0}^{N-1} \left\{ \left| \rho \frac{E_{n+1}}{h} \right|_0^2 + \left| \rho \frac{E_n}{h} \right|_0^2 \right\}
\]

(10.13)
where \( y, n, \xi, \omega, \) and \( v \) are positive constants and \( v = \frac{1}{4\gamma} + \frac{1}{4n} + \frac{1}{4\xi} + \frac{1}{4\omega} \).

A useful integral representation for terms of the form \( \delta_t E_{n+1} \) was obtained by Dupont [3]. It can be shown there that since

\[
\delta_t E_{n+1} = \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \frac{\partial E}{\partial t} \, dt
\]

then

\[
\Delta t \sum_{n=0}^{N-1} \left| \frac{\rho}{\xi} \delta_t E_{n+1} \right|_2^2 \leq \left| \frac{\rho}{\xi} \frac{\partial E}{\partial t} \right|_{L^2(L_2(\Omega))}^2
\] (10.14)

In addition

\[
\sup_{1 \leq i \leq N} \Delta t \sum_{n=0}^{i-1} K \left| E_n \right|_2^2 \leq K_N \Delta t \sup_{0 \leq n \leq N} \left| E_n \right|_2^2
\]

\[
= K_2 \left| E_n \right|_{L^2(L_2(\Omega))}^2
\] (10.15)

Using the discrete Gronwall inequality (Lemma 7.2) in (10.13), taking the supremum over all \( \nu \) in the resulting expression, using (10.14) and (10.15), and introducing the temporal error term (10.7), we obtain the result (10.10).

Now we examine the approximation error induced in the approximation of \( (10.2)_1 \) by \( (10.3)_1 \). If \( (10.2)_1 \) is evaluated at \( t = n\Delta t \) and \( \nu \) is set equal to \( \nu \in S_h(\Omega) \), then
Now adding \((\frac{\bar{y}_{n+1} - \bar{y}_n}{\Delta t}, V)\) to each side of (10.17), we get

\[
(\frac{\bar{y}_{n+1} - \bar{y}_n}{\Delta t}, V) + a(u_n, V) + \frac{\Delta t^4}{2} a(\bar{y}_n, V) = (\rho \beta_n, V) \quad \forall \, V \in S_h(\Omega)
\]

where

\[
\beta_n = \frac{\bar{y}_{n+1} - \bar{y}_n}{\Delta t} - \frac{\partial \bar{y}_n}{\partial t}
\]

An estimate for the temporal error component (see the derivative of (10.7)) is

\[
\Delta t \sum_{n=0}^{N-1} \| \beta_n \|^2_{L_2(\Omega)} \leq \frac{\Delta t^4}{3} \| \frac{\partial^3 \bar{y}}{\partial t^3} \|^2_{L_2(L_2(\Omega))}
\]

Now subtracting (10.3) from (10.18), we find that for \(\forall \, V\) in \(S_h(\Omega)\),

\[
(\frac{f_{n+1} - f_n}{\Delta t}, V) + a(e_n, V) + \frac{\Delta t^4}{2} a(f_n, V) = (\rho \beta_n, V)
\]

The approximation error \(f_n\) and \(e_n\) are then decomposed in the normal manner. The behavior of \(F_n\) is given in the following theorem:

**Theorem 10.2.** Suppose \(\frac{\partial^3 \bar{y}}{\partial t^3} \in L_2(L_2(\Omega))\) and suppose that we choose \(\Delta t\) and \(h\) so that
\[
\frac{h^2}{\Delta t} < \frac{C' C^*}{\psi}
\]
\[
\frac{\Delta t^*}{\Delta t} < \frac{\psi^2}{2 \lambda C^* \zeta C'}
\]
\[
\frac{\Delta t^*}{\Delta t} < \frac{\psi^2 \zeta}{2 \lambda C^* \zeta C'}
\]

where \( \psi, \zeta, C', C^* \) are positive constants and \( \psi, \zeta \) are arbitrarily chosen. Then there exists a constant \( C_2 \) such that

\[
\| F \|_{L^\infty(L_2(\Omega))} \leq C_2 \left\{ \| F_0 \| \| \phi \| + \| \frac{\partial F}{\partial t} \|_{L^2(L_2(\Omega))} + \Delta t^2 \left\| \frac{\partial^3 \psi}{\partial t^3} \right\|_{L^2(L_2(\Omega))} \right\}
\]

(10.21)

**Proof:** Decompose the error in (10.20) according to

\[
\left( \frac{\rho_{n+1} - \rho_n}{\Delta t} \right) \phi + a(E_n, \psi) + \frac{\Delta t^*}{2} a(F_n, \psi)
\]

\[
= - \left( \frac{F_{n+1} - F_n}{\Delta t} \right) \phi - a(E_n, \psi) - \frac{\Delta t^*}{2} a(F_n, \psi)
\]

\[
+ (\rho \beta_{n'}, \psi)
\]

(10.22)

and take note of the following identity:

\[
\delta^*_{n^*} (F, \phi) = 2 (F_n, \delta^*_{n^*} \phi) + \Delta t (\delta_{n^*} F_{n^*}, \delta_{n^*} \phi)
\]

(10.23)
Setting \( V = F_n \) in (10.22) and using (10.23), we get

\[
\delta \left[ n, \frac{1}{2} \right] (\rho F, F) - \Delta t (\rho \delta \left[ n, \frac{1}{2} \right], \delta \left[ n, \frac{1}{2} \right]) + 2a(F_n, F_n) + \Delta t^\alpha a(F_n, F_n) - 2a(F_n, F_n) - \Delta t^\alpha a(F_n, F_n) + 2(\rho \beta_n, F_n) \quad (10.24)
\]

Using (10.9), Lemma 3.1, and inequality E (for some positive constant \( \zeta \)), we get

\[
\delta \left[ n, \frac{1}{2} \right] (\rho F, F) - \Delta t (\rho \delta \left[ n, \frac{1}{2} \right], \delta \left[ n, \frac{1}{2} \right]) - \Delta t^\alpha \| F_n \|_{H^1(\Omega)}^2 + \| F_n \|_{H^1(\Omega)}^2 + 2(\rho \beta_n, F_n) \quad (10.25)
\]

Now we simplify this expression by defining an auxiliary relationship. Letting \( V = \delta \left[ n, \frac{1}{2} \right] \) in (10.22) and using (10.2), we have

\[
\| \rho \delta \left[ n, \frac{1}{2} \right] F_n \|_{H^1(\Omega)}^2 = -a(E_n, \delta \left[ n, \frac{1}{2} \right] F_n) - \frac{\Delta t^\alpha}{2} a(F_n, \delta \left[ n, \frac{1}{2} \right] F_n) + (\rho \delta \left[ n, \frac{1}{2} \right] F_n, \delta \left[ n, \frac{1}{2} \right] F_n) + (\rho \beta_n, \delta \left[ n, \frac{1}{2} \right] F_n) \quad (10.26)
\]
Using the Cauchy-Schwarz inequality, the inequality E, and the inverse assumption (4.6), we find that for \( \psi \) a positive constant

\[
(1 - \frac{C'C^2\Delta t^*}{\psi h^2}) \left| \left\| \rho^{\frac{1}{2}} \delta_t F_{n+\frac{1}{2}} \right\|_0^2 \right.
\]

\[
\leq \frac{C'\psi}{2\Delta t^*} \left\| E_n \right\|_{H^1(\Omega)}^2 + \frac{C'\Delta t^{2\alpha-1}}{2} \left\| F_n \right\|_{H^1(\Omega)}^2
\]

\[
- (\rho \delta_t F_{n+\frac{1}{2}}, \delta_t F_{n+\frac{1}{2}})_0 - (\rho \delta_t F_{n+\frac{1}{2}}, \delta_t F_{n+\frac{1}{2}})_0
\]

This implies that

\[
\left\| \rho^{\frac{1}{2}} \delta_t F_{n+\frac{1}{2}} \right\|_0^2 \leq \frac{C'\psi^2 h^2}{2\Delta t^* (\psi h^2 - C'C^2\Delta t^*)} \left\| E_n \right\|_{H^1(\Omega)}^2
\]

\[
+ \frac{C'\psi^2 h^2 \Delta t^{2\alpha-1}}{2\psi h^2 - 2C'C^2\Delta t^*} \left\| F_n \right\|_{H^1(\Omega)}^2 + BZ
\]

where

\[
Z = (\rho \delta_t F_{n+\frac{1}{2}}, \delta_t F_{n+\frac{1}{2}})_0 - (\rho \delta_t F_{n+\frac{1}{2}}, \delta_t F_{n+\frac{1}{2}})_0
\]

and

\[
B = \frac{\psi h^2}{C'C^2\Delta t^* - \psi h^2}
\]

Using (10.28) to simplify (10.25)
\[ \delta_{t_n+\xi} \left| \rho^* F \right|_o^2 + \left[ \frac{\psi^2 h^2 \Delta t}{2(1 - \frac{\psi}{C'C*^2} \frac{h^2}{\Delta t}) C*^2 \Delta t*^2} - \zeta C' \right] \left| E_n \right|_H^2 (\Omega) \]

\[ + \left[ \frac{\psi^2 h^2 \Delta t}{(1 - \frac{\psi}{C'C*^2} \frac{h^2}{\Delta t*}) 2C*^2 \Delta t*^2 - 2\alpha} \right] \left[ \Delta t^* \mu - \frac{C'}{\zeta} \right] \left| F_n \right|_H^2 (\Omega) \]

\[ \leq -2(\rho \frac{F_{n+1} - F_n}{\Delta t}, F_n)_o + 2(\rho \beta_n, F_n)_o + \Delta tBZ \quad (10.29) \]

As conditions of stability we require that

\[ \left[ \frac{\psi^2 h^2 \Delta t}{2(1 - \frac{\psi}{C'C*^2} \frac{h^2}{\Delta t}) C*^2 \Delta t*^2} - \zeta C' \right] = \lambda \geq 0 \quad (10.30) \]

and

\[ \left[ \frac{\psi^2 h^2 \Delta t}{2(1 - \frac{\psi}{C'C*^2} \frac{h^2}{\Delta t*}) C*^2 \Delta t*^2 - 2\alpha} - \frac{C'}{\zeta} \right] = \gamma \geq 0 \quad (10.31) \]

where, of course, \( \psi \) and \( \zeta \) are the arbitrary positive constants introduced previously. Clearly (10.30) and (10.31) are satisfied if the conditions (10.20) are satisfied. Then estimating the terms on the right hand side of (10.30) using the Cauchy-Schwarz inequality and inequality \( E \)
where $\gamma$ and $\phi$ are positive constants and $\xi = \frac{1}{2\gamma} + \frac{1}{2\phi}$.

Multiplying (10.33) by $\Delta t$, summing from 1 to $N-1$, applying the discrete Gronwall inequality given in Lemma 7.2, taking the supremum over $n$ in the resulting expression, and using the temporal error term (10.19), we obtain the result (10.21).

We remark that the stability restrictions (10.20) are very severe. For a given value of the spatial discretization parameter $h$, $(10.20)_2$ and $(10.20)_3$ require a temporal discretization parameter $\Delta t$ which is less than a certain positive constant. However, the condition $(10.20)_1$ specifies that $\Delta t$ cannot be too small. That is, instability can result from either the choice of too small or too large a discretization parameter.

We obtain the final error estimate for $||E||_{L_\infty(L_2(\Omega))}$ by combining Theorem 10.1 and 10.2.

**Theorem 10.3.** Suppose $\frac{3}{\Delta t^3} \frac{\partial^3 u}{\partial t^3}, \frac{3}{\Delta t^3} \frac{\partial^3 \psi}{\partial t^3} \in L_2(L_2(\Omega))$ and suppose that the conditions of Theorem 10.2 are satisfied. Then there exists
a positive constant $C_3$ such that

$$
\left\| E \right\|_{L^\infty(L^2(\Omega))} \leq C_3 \left\{ \left\| E_0 \right\|_0 + \left\| F_0 \right\|_0 + \left\| \frac{\partial F}{\partial t} \right\|_{L^2(L^2(\Omega))}
\right. \\
+ \left. \left\| F \right\|_{L^\infty(L^2(\Omega))} + \left\| \frac{\partial F}{\partial t} \right\|_{L^2(L^2(\Omega))}
\right. \\
+ \Delta t^2 \left\| \frac{\partial^3 \tilde{u}}{\partial t^3} \right\|_{L^2(L^2(\Omega))} + \Delta t^2 \left\| \frac{\partial^3 \tilde{v}}{\partial t^3} \right\|_{L^2(L^2(\Omega))}\}

(10.33)

Using the Theorem 10.3, Lemma 7.1, and the triangle inequality, we obtain the error estimate for $e$ and $f$.

**Theorem 10.4.** Suppose that $\tilde{u}, \tilde{v}, \frac{\partial \tilde{u}}{\partial t}, \frac{\partial \tilde{v}}{\partial t} \in L^\infty(H^{k+1}(\Omega))$, 

$$
\frac{\partial \tilde{v}}{\partial t} \in L^2(H^{k+1}(\Omega)), \quad \frac{\partial^3 \tilde{u}}{\partial t^3} \in L^2(L^2(\Omega)) \quad \text{and} \quad \frac{\partial^3 \tilde{v}}{\partial t^3} \in L^2(L^2(\Omega))
$$

and suppose that the conditions of Theorem 10.2 are satisfied. Then there exist positive constants $C_4$ and $C_5$ such that
\[ ||e||_{L_\infty(L_2(\Omega))} \leq C_4 \{ ||e_0||_\infty + ||f_0||_\infty + h^{k+1} ||\bar{u}||_{L_\infty(H^{k+1}(\Omega))} \]

\[ + h^{k+1} ||\frac{\partial \bar{u}}{\partial t}||_{L_\infty(H^{k+1}(\Omega))} + h^{k+1} ||\bar{v}||_{L_\infty(H^{k+1}(\Omega))} \]

\[ + \Delta t^2 \left| \left| \frac{\partial^3 \bar{u}}{\partial t^3} \right| \right|_{L_2(L_2(\Omega))} + \Delta t^2 \left| \left| \frac{\partial^3 \bar{v}}{\partial t^3} \right| \right|_{L_2(L_2(\Omega))} \} \]

and

\[ ||f||_{L_\infty(L_2(\Omega))} \leq C_5 \{ ||f_0||_\infty + h^{k+1} ||\frac{\partial \bar{v}}{\partial t}||_{L_2(H^{k+1}(\Omega))} \]

\[ + \Delta t^2 \left| \left| \frac{\partial^3 \bar{v}}{\partial t^3} \right| \right|_{L_2(L_2(\Omega))} \} \]

We can refine the estimate by introducing the regularity results of Theorems 6.1, 6.2, and 6.3 with \( \varepsilon \) replaced by \( (\Delta t^*)^\alpha \).

**Theorem 10.5.** If the hypotheses of Theorem 10.4 are satisfied, then

\[ ||e||_{L_\infty(L_2(\Omega))} \leq 0 \left[ \frac{h^{k+1}}{(\Delta t^*)^\alpha} \right] \]

\[ + 0 \left[ \frac{\Delta t^2}{(\Delta t^*)^2} \right] \]
and

$$||f||_{L^\infty(L^2(\Omega))} \leq O\left[\frac{h^{k+1}}{(\Delta t^*)^2} a\left(\frac{3k+4}{2}\right)\right]$$

$$+ O\left[\frac{\Delta t^2}{2} \right]$$

If we set $\Delta t^* = \Delta t$ and vary the discretization parameters so that $
\frac{\Delta t}{h^q} = C$ where $C$ is a positive constant, we obtain the final estimate of the rate of convergence:

**Theorem 10.6.** If the hypotheses of Theorem 10.4 are satisfied and $\frac{\Delta t}{h^q} = C$, then

$$||u - U||_{L^\infty(L^2)} \leq O\left[h^{2k - 3aqk - 6aq + 2}\right]$$

$$+ O\left[\Delta t^2\right]$$

$$||y - y||_{L^\infty(L^2(\Omega))} \leq O\left[h^{2k - 3aqk - 4aq + 2}\right] + O\left[\Delta t^2\right]$$

Theorem 10.6 leads to a corollary which gives a sufficient condition for the convergence of the parabolic regularization method to shock wave solutions:
Corollary 10.6. If the hypotheses of Theorem 10.4 are satisfied and \( \frac{\Delta t}{h^q} = C \), the convergence of the parabolic regularization method (5.9-5.10) to (10.1) occurs if

\[
k + 1 > aq \left( \frac{3k+6}{2} \right)
\]

and

\[
a < \frac{4}{7}
\]

We observe that the constraint on \( a \) given in Corollary 10.6 implies that the Lax-Wendroff type scheme (5.7-5.8) (for which \( a = 1 \)) will not necessarily converge to shock wave solutions.

11. The Nonlinear Parabolic Regularization

In this section of the paper, we consider the approximation of (3.2) by (5.9-5.10). However, we can easily split up the second order equation (3.2) into two coupled first order equations by defining \( y = \frac{\partial u}{\partial t} \)

\[
\rho \frac{3\mathbf{v}}{\partial t} \cdot \mathbf{v} + a(u,u,v) = 0 \quad \forall \mathbf{v} \in H^1(\Omega)
\]

\[
(\rho \frac{3u}{\partial t},\mathbf{v})_o - (y,\mathbf{v})_o = 0 \quad \forall \mathbf{v} \in H^1(\Omega)
\]

(11.1)
Thus, an equivalent problem and the problem to be undertaken here is to show the convergence of the parabolic regularization method (5.1-5.10) to (11.1).

Now we pose an auxiliary problem. Let \((\tilde{u}, \tilde{y})\) be the solution to the system

\[
\left(\rho \frac{\partial \tilde{y}}{\partial t}, v\right)_0 + a(\tilde{u}, \tilde{u}, v) + \frac{\Delta t^{*} \alpha}{2} a(\tilde{u}, \tilde{y}, v) = 0 \quad \forall \, v \in H^1(\Omega)
\]

\[
\left(\rho \frac{\partial \tilde{u}}{\partial t}, v\right)_0 - \left(\rho \tilde{y}, v\right)_0 + \frac{\Delta t^{*} \alpha}{2} a(\tilde{u}, \tilde{u}, v) = 0 \quad \forall \, v \in H^1(\Omega)
\]

(11.2)

We obtain an approximate auxiliary problem by introducing the forward difference operator in (11.2). Then the solution to the approximate auxiliary problem \((\tilde{u}^n, \tilde{y}^n)\) satisfies

\[
\left(\rho \frac{\tilde{y}^{n+1} - \tilde{y}^n}{\Delta t}, v\right) + a(\tilde{u}^n, \tilde{u}^n, v) + \frac{\Delta t^{*} \alpha}{2} a(\tilde{u}^n, \tilde{y}^n, v) = 0 \quad \forall \, v \in S_h(\Omega)
\]

\[
\left(\rho \frac{\tilde{u}^{n+1} - \tilde{u}^n}{\Delta t}, v\right) - \left(\rho \tilde{y}^n, v\right) + \frac{\Delta t^{*} \alpha}{2} a(\tilde{u}^n, \tilde{u}^n, v) = 0 \quad \forall \, v \in S_h(\Omega)
\]

(11.3)

We follow here a procedure identical to that of Section 10. That is, we use the convergence of the auxiliary problem as an intermediate step in the proof of convergence of (5.9-5.10) to (11.1).

If (11.2) \(_2\) is evaluated at time point \(t = n\Delta t\) and \(v\) is equated to \(V\), then it can be seen that
\[
\frac{\partial \tilde{u}_n}{\partial t},(\rho \tilde{u}_n)^2 + \frac{\Delta t \psi^2}{2} a(\tilde{u}_n, \tilde{u}_n, \nu) = 0 \quad \forall \nu \in S_h(\Omega)
\]

(11.4)

Now adding \((\frac{\tilde{u}_{n+1} - \tilde{u}_n}{\Delta t}),(\rho \tilde{y}_n, \nu)\) to each side of (10.4) gives

\[
\frac{\tilde{u}_{n+1} - \tilde{u}_n}{\Delta t},(\rho \tilde{y}_n, \nu) + \frac{\Delta t \psi^2}{2} a(\tilde{u}_n, \tilde{u}_n, \nu) = (\rho \psi_n, \nu) \quad \forall \nu \in S_h(\Omega)
\]

(11.5)

where \(\psi_n\) is defined in (10.4).

Now we set \(e_n = \tilde{u} - \tilde{U}^n\) and \(f_n = \tilde{y} - \tilde{Y}^n\). Then subtracting (11.3) from (11.5)

\[
(\rho \frac{e_{n+1} - e_n}{\Delta t},(\rho \tilde{y}_n, \nu) + \frac{\Delta t \psi^2}{2} [a(\tilde{u}_n, \tilde{y}_n, \nu) - a(\tilde{u}_n, \tilde{y}_n, \nu)] = (\rho \psi_n, \nu) \quad \forall \nu \in S_h(\Omega)
\]

(11.6)

Now we identify \(\tilde{W}_n, \tilde{P}_n \in S_h(\Omega)\) through the weighted \(H^1(\Omega)\) projections

\[
a(\tilde{u}_n, \tilde{y}_n - \tilde{W}_n, \nu) = 0 \quad \forall \nu \in S_h(\Omega)
\]

\[
a(\tilde{u}_n, \tilde{y}_n - \tilde{P}_n, \nu) = 0 \quad \forall \nu \in S_h(\Omega)
\]

Then we perform the normal decomposition of the error \(e_n\) and \(f_n\).

\[
e_n = E_n + F_n \quad \text{where} \quad E_n = \tilde{u}_n - \tilde{W}_n \quad \text{and} \quad F_n = \tilde{W}_n - \tilde{U}^n
\]

and

\[
f_n = P_n + F_n \quad \text{where} \quad F_n = \tilde{F}_n - \tilde{Y}_n
\]

Then using a method of proof quite similar to the one presented in Theorem 10.1, we have
Theorem 11.1. Let \( \frac{\partial^3 u}{\partial t^3} \in L_2(L_2(\Omega)) \), then there exists a positive constant \( C_1 \) such that

\[
|E|_{L_\infty(L_2(\Omega))} \leq C_1 \{ |E_o|_{L_\infty(L_2(\Omega))} + |F|_{L_\infty(L_2(\Omega))} + \frac{\partial^3 u}{\partial t^3} \}_{L_2(L_2(\Omega))} \\
+ |F|_{L_\infty(L_2(\Omega))} + \Delta t^2 \left\| \frac{\partial^3 u}{\partial t^3} \right\|_{L_2(L_2(\Omega))} \}
\]  
(11.7)

Now we estimate the error induced in the approximation of (11.1) by (11.2). If (11.1) is evaluated at \( t = n\Delta t \) and \( v \) is set equal to \( v \in S_h(\Omega) \), then

\[
\rho \frac{\partial \bar{y}_n}{\partial t} + a(\bar{u}_n,\bar{u}_n,v) + \frac{\Delta t^2}{2} a(\bar{u}_n,\bar{y}_n,v) = 0 \quad \forall \ v \in S_h(\Omega)
\]
(11.8)

Now adding \( \frac{\bar{y}_{n+1} - \bar{y}_n}{\Delta t} \) to each side of (11.8), we get

\[
\rho \frac{\partial \bar{y}_{n+1}}{\partial t} + \frac{\bar{y}_{n+1} - \bar{y}_n}{\Delta t} + a(\bar{u}_n,\bar{u}_n,v) + \frac{\Delta t^2}{2} a(\bar{u}_n,\bar{y}_n,v) = (\rho \beta_n,v) \quad \forall \ v \in S_h(\Omega)
\]
(11.9)

where \( \beta_n \) is defined in (10.17).

Now subtracting (11.3) from (11.9)
\[
\frac{f_{n+1} - f_n}{\Delta t} + a(\tilde{u}_n, \tilde{u}_n, \nu) - a(\bar{u}_n, \bar{u}_n, \nu)
\]

\[
+ \frac{\Delta t}{2} [a(\bar{u}_n, \bar{\nu}_n, \nu) - a(\tilde{u}_n, \tilde{\nu}_n, \nu)] = (\rho \delta_n, \nu)_0 \quad (11.10)
\]

The behavior of the approximation error \( F_n \) is established using a technique similar to the one used to prove Theorem 10.2. We state the result here.

**Theorem 11.2.** Suppose \( \frac{\partial^3 \psi}{\partial t^3} \in L_2(L_2(\Omega)) \) and suppose that the stability condition \((10.20)\) is satisfied. Then there exists a positive constant \( E \) such that

\[
||F||_{L_\infty(L_2(\Omega))} \leq C_2 \left( ||F_0||_{L_2(\Omega)} + ||E||_{L_2(L_2(\Omega))} + ||\frac{\partial^3 F}{\partial t^3}||_{L_2(L_2(\Omega))} + \Delta t^2 ||\frac{\partial^3 \psi}{\partial t^3}||_{L_2(L_2(\Omega))} \right) \quad (11.11)
\]

We obtain a final estimate for the rate of convergence and a sufficient condition for convergence which are the same as Theorem 10.6 and Corollary 10.6, respectively. They will not be repeated here.

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