Mixed Conjugate Finite-Element Approximations of Linear Operators

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ABSTRACT

The concept of conjugate approximation functions is generalized to mixed formulations of linear boundary value problems. Biorthogonal basis functions are generated using two distinct finite-element models of the domain of a given function. Various special cases are considered. It is shown that four distinct models can be developed corresponding to a single linear operator. A comparison of the concepts of the hypercircle and conjugate approximation is given.

I. INTRODUCTION

In earlier investigations of the theory of conjugate finite-element approximations [1-6] it has been assumed that the global approximation functions \( \varphi_n(X) \) and their corresponding conjugate approximation functions \( \varphi^*(X) \) belong to the same space. This has led to the concept of "consistent stress approximations" for finite-element models of displacement fields in linear elasticity [3-6].
For example, suppose that the approximation \( \tilde{u} \) of a function \( u \) is a linear combination of a set of \( G \) linearly independent functions \( \varphi_s(X) \):

\[
\begin{align*}
\tilde{u} & = \sum a_s \varphi_s(X) \\
& = a^x \varphi_x(X) + a^y \varphi_y(X)
\end{align*}
\]

(there \( \alpha \) is summed from 1 to \( G \)). Then, if \( \langle \varphi_s, \varphi_t \rangle = \delta_{st} \) is an inner product on \( \mathcal{H}(\tilde{u} \in \mathcal{H}) \), a system of biorthogonal basis functions \( \varphi^s(X) \), having the property

\[
\langle \varphi_s, \varphi^t \rangle = \delta_{st}
\]

can be constructed using the relations

\[
\varphi^s(X) = C^{s*} \varphi_s(X)
\]

wherein \( C^{s*} \) is the inverse of the \( n \times n \) fundamental matrix

\[
C_{st} = \langle \varphi_s, \varphi_t \rangle
\]

Then, for a given \( u \), the proper coefficients \( a^x \) in (1) are immediately given by

\[
a^x = \langle u, \varphi^x \rangle
\]

Alternately,

\[
\tilde{u} = a^x \varphi_x = a_s \varphi^s
\]

wherein \( a_s = \langle \varphi_s, u \rangle \). We can then proceed to use (2)–(6) to derive approximations of linear operators, solutions of various boundary-value problems, derivatives, integrals, stresses, etc. (e.g., [3]).

By introducing (3), however, we have limited the choice of conjugate approximation functions to the \( G \)-dimensional subspace spanned by \( \varphi_s(X) \). In the present paper, we investigate the consequences of removing this restriction. In particular, we develop herein a generalization of the theory of conjugate approximations in which (2) holds, but \( \varphi^s \) are not determined by a given set of functions \( \varphi_s \). We show that this approach leads to “mixed” finite-element approximations and that, depending upon the properties of the functions \( \varphi_s \) and \( \varphi^s \), at least four different finite-element models of a given linear operator can be derived. The latter observation leads to the question of stability of the finite-element operators; however, we shall postpone consideration of this question until a later paper. Finally, we discuss the relationship between the concept of conjugate approximations and the method of the hypercircle [7, 8].
II. DUAL AND CONJUGATE SPACES

To fix ideas, we shall review briefly certain definitions and concepts related to dual and conjugate spaces.

Consider two linear vector spaces, $\mathcal{U}$ and $\mathcal{V}$, and suppose that there is defined on $\mathcal{U} \otimes \mathcal{V}$ a linear mapping that associates with ordered pairs of elements $u, v (u \in \mathcal{U}, v \in \mathcal{V})$ real numbers. If $\langle u, v \rangle$ denotes the real number so defined, corresponding to the pair $u$ and $v$, then $\mathcal{V}$ is the dual space of $\mathcal{U}$ (denoted $\mathcal{V}^* = \mathcal{U}'$) if the following hold:

\begin{align*}
\langle u, \alpha v_1 + \beta v_2 \rangle &= \alpha \langle u, v_1 \rangle + \beta \langle u, v_2 \rangle \\
\langle \alpha u_1 + \beta u_2, v \rangle &= \alpha \langle u_1, v \rangle + \beta \langle u_2, v \rangle \\
\langle u, v \rangle &= 0 \text{ for fixed } u \text{ and all } v \Rightarrow u = 0 \\
\langle u, v \rangle &= 0 \text{ for fixed } v \text{ and all } u \Rightarrow v = 0
\end{align*}

Clearly $\langle u, v \rangle$ is a linear functional on $\mathcal{U}$. It follows that $\mathcal{V}$ (or $\mathcal{U}'$) is equivalent to (isomorphic to) the space $\mathcal{U}^*$ of all linear functionals on $\mathcal{U}$. The space $\mathcal{U}^*$ is called the conjugate space of $\mathcal{U}$ and it suffices our purposes here to treat $\mathcal{U}'$ as synonymous with $\mathcal{U}^*$.

Now suppose that the linearly independent elements $\varphi_1, \varphi_2, \ldots, \varphi_n$ provide a basis for the space $\mathcal{U}$ and that the elements $\chi^1, \chi^2, \ldots, \chi^n$ provide a basis for $\mathcal{U}' = \mathcal{V}$. Then every element $u \in \mathcal{U}$ and $v \in \mathcal{V}$ is of the form

$$u = a^1 \varphi_1 + a^2 \varphi_2 + \cdots + a^n \varphi_n \quad v = b^1 \chi^1 + b^2 \chi^2 + \cdots + b^n \chi^n$$

To determine the coefficients $a^x, b^x \ (x = 1, 2, \ldots, n)$ for given $u$ and $v$, we assume that $\{\varphi_x\}$ and $\{\chi^x\}$ constitute dual or biorthogonal bases: i.e.,

$$\langle \varphi_x, \chi^p \rangle = \delta^p_x$$

where $\delta^p_x$ is the Kronecker delta. Then it follows from (8) that

$$a^x = \langle u, \chi^x \rangle \quad b^x = \langle \varphi_x, v \rangle$$

It is clear that if $\chi^x \equiv \varphi^x$, where $\varphi^x$ is given by (3), then (9) and (10) are satisfied. Then the theory presented in [1-5] follows. However, (10) is ob-
tained from (9) without assuming that (3) holds; that is, $\chi^a$ need not belong to $\mathcal{H}$.

III. GENERALIZED CONJUGATE FINITE-ELEMENT INTERPOLATION FUNCTIONS

We now wish to examine the application of the notion of dual and conjugate spaces discussed previously to finite-element approximations of functions and of solutions to linear boundary-value problems. The present development seeks to generalize the theory of conjugate approximations presented earlier [1–5]. Consider a region $\mathcal{R}$ of $k$-dimensional Euclidean space $\mathbb{R}^k$ which is the domain of functions $u(X)$, $X = (X_1, X_2, \ldots, X_k) \in \mathcal{R}$, of a class to be specified. In general, the values of $u(X)$ at a point $X \in \mathcal{R}$ may be scalars, vectors, or tensors of any order; but, for simplicity, it is convenient to temporarily regard $u(X)$ as scalar-valued.

To construct finite-element approximations of $u(X)$, we proceed as described in [3] and replace $\mathcal{R}$ by a finite-element model $\mathcal{F}$ which consists of a collection of $E$ subregions $\Omega$, called finite elements, connected continuously together at nodal points and along interelement boundaries. We identify $G$ nodal points $X^a$ ($a = 1, 2, \ldots, G$) in the connected model $\mathcal{F}$. Other properties then follow the plan discussed in [3].

In the present study, however, it is essential to first consider the possibility of two different finite-element models of the same region $\mathcal{R}$—one model $\mathcal{R}$ representing the domain of approximations to functions belonging to a function space $\mathcal{U}$ and a second model $\mathcal{F}_\mathcal{Y}$ representing the domain of functions belonging to the dual or conjugate space $\mathcal{Y}$ of $\mathcal{U}$.

Let $\mathcal{U}$ and $\mathcal{Y}$ denote linear vector spaces, the elements of which are functions $u(X)$, and $v(X)$, respectively. The domain of the elements $u(X) \in \mathcal{U}$ and $v(X) \in \mathcal{Y}$ is the same closed region $\mathcal{R} \subset \mathbb{R}^k$. Now let $\mathcal{U}$ and $\mathcal{Y}$ denote finite dimensional subspaces of $\mathcal{U}$ and $\mathcal{Y}$, respectively, which are spanned by the sets of $G$ linearly independent basis functions $\phi_1(X), \phi_2(X), \ldots, \phi_G(X) \in \mathcal{U}$ and $\chi^1(X), \chi^2(X), \ldots, \chi^G(X) \in \mathcal{Y}$. Then the projection $\Pi u = \tilde{u}$ of an element $u \in \mathcal{U}$ into the subspace $\mathcal{U}$ is of the form

$$\tilde{u} = a^a \phi_a(X)$$  \hspace{1cm} (11)

where $a$ is summed from 1 to $G$, whereas the projection $\Pi v = \tilde{v}$ of an element $v \in \mathcal{Y}$ into the subspace $\mathcal{Y}$ is of the form

$$\tilde{v} = b^a \chi^a(X)$$  \hspace{1cm} (12)
We assume further that there is defined on \( \mathcal{U} \otimes \mathcal{V} \) (and \( \mathcal{W} \otimes \mathcal{W} \)) a bilinear mapping \( \langle u, v \rangle \) of ordered pairs of elements into real numbers that satisfies (7). Finally, and most important, we assume that the functions \( \varphi_k(X) \) and \( \chi^l(X) \) are biorthogonal basis of \( \mathcal{W} \) and \( \mathcal{W}^* \); i.e., they satisfy (9).

If (11) and (12) are to describe finite-element approximations on \( \mathcal{A} \), the basis functions \( \varphi_k(X) \) and \( \chi^l(X) \) must be endowed with the usual finite-element properties.

To obtain \( \varphi_k(X) \):

1. \( \mathcal{A} \) is replaced by a finite-element model \( \mathcal{A} \) consisting of \( E \) connected subregions \( r_e \), called finite elements, joined continuously together at inter-element boundaries.

\[
\mathcal{A} = \bigcup_{e=1}^{E} r_e
\]  

2. A finite number \( G \) of points in \( \mathcal{A} \) are identified called nodes; these are labeled \( X^a \)

\[
a = 1, 2, \ldots, G
\]

3. A typical element \( r_e \) is isolated from the collection \( \mathcal{A} \), and local coordinate system \( x_e \) is established. A finite number \( N_e \) of local nodal points \( N \) are identified \( r_e \) and are labeled \( x_e^N \), \( N = 1, 2, \ldots, N_e \).

4. The connectivity and decomposition, respectively, of the model \( \mathcal{A} \) is established by the mappings

\[
x_e^N = \Omega_{x_e}^N X^a
\]

\[
X^a = \Lambda_{x_e}^a x_e^N
\]

To obtain \( \chi^l(X) \):

1. \( \mathcal{A} \) is replaced by a finite-element model \( \mathcal{A} \) consisting of \( E \) connected finite elements \( \bar{r}_e \), joined continuously together at inter-element boundaries

\[
\mathcal{A} = \bigcup_{e=1}^{E} \bar{r}_e
\]  

2. A finite number \( G \) of nodal points in \( \mathcal{A} \) are identified and labeled \( \bar{X}^a \)

\[
a = 1, 2, \ldots, G
\]

3. A typical element \( \bar{r}_e \) is isolated from the collection \( \mathcal{A} \), and a local coordinate system \( x^e \) is established. A finite number \( \bar{N}_e \) of local nodal points \( \bar{N} \) are identified in \( \bar{r}_e \) and are labeled \( \bar{x}_e^N \), \( \bar{N} = 1, 2, \ldots, \bar{N}_e \).

4. The connectivity and decomposition, respectively, of the model \( \mathcal{A} \) is established by the mappings

\[
\bar{x}_e^N = \Omega_{x_e}^N \bar{X}^a
\]

\[
\bar{X}^a = \Lambda_{x_e}^a \bar{x}_e^N
\]  

### Notes

1. The symbols \( \mathcal{U}, \mathcal{V}, \mathcal{W} \) represent the spaces of functions.
2. The symbols \( \varphi, \chi \) represent the basis functions.
3. The symbols \( \Omega, \Lambda \) represent the mappings that establish the connectivity and decomposition, respectively.
4. The symbol \( g \) represents a generic function.
where

$$
\Omega^e_N = \begin{cases} 
1 & \text{if node } N \text{ of element } r_e \text{ is coincident with node } z \text{ of the connected model} \\
0 & \text{if otherwise}
\end{cases}
$$

and

$$
\Lambda^e_N = \begin{cases} 
1 & \text{if node } z \text{ of } \mathcal{R} \text{ is coincident with node } N \text{ of } r_e \text{ in the connected model} \\
0 & \text{if otherwise}
\end{cases}
$$

and $\Lambda^e_N$ is the transpose of $\Omega^e_N$.

5. If $\bar{u}(X) \in \mathcal{U}$ and $u_e(x)$ is the restriction of $\bar{u}(X)$ to $r_e$, we approximate $u_e(x)$ locally according to

$$
u_e(x) = u_e^N \psi_e^N(x)
$$

where $u_e^N$ is the value of $u_e(x)$ at node $N$ of element $r_e$, and $N$ is summed from 1 to $N_e$, and the local interpolation functions have the properties

$$
\psi_e^N(x) = 0, \ x \notin r_e \\
\psi_e^N(x^M_e) = \delta^M_N \\
\sum_{N=1}^{N_e} \psi_e^N(x) = 1
$$

6. Globally (except on a set of measure zero), we have

$$
\bar{u}(X) = u^e \varphi_e(X) = \sum_{e} u_e(x)
$$

where

$$
\Omega^f_N = \begin{cases} 
1 & \text{if node } N \text{ of element } r_f \text{ is coincident with node } \bar{z} \text{ of the connected model} \\
0 & \text{if otherwise}
\end{cases}
$$

and $\Lambda^f_N$ is the transpose of $\Omega^f_N$.

5. If $\bar{v}(X) \in \mathcal{V}$ and $v_e(x)$ is the restriction of $\bar{v}(X)$ to $\bar{r}_e$, we approximate $v_e(x)$ locally according to

$$
u_e(x) = v_e^N \psi_e^N(x)
$$

where $v_e^N$ is the value of $v_e(x)$ at node $N$ of element $\bar{r}_e$, and $N$ is summed from 1 to $N_e$, and the local interpolation functions have the properties

$$
\psi_e^N(x) = 0, \ x \notin \bar{r}_e \\
\psi_e^N(x^M_e) = \delta^M_N \\
\sum_{N=1}^{N_e} \psi_e^N(x) = 1
$$

6. Globally (except on a set of measure zero), we have

$$
\bar{v}(X) = v^e \chi_e^e(X) = \sum_{e} v_e(x)
$$
Examples of such models are shown in Fig. 1. Note that at this point we require only that the models of $u(X)$ and $v(X)$ have the same number of "degrees of freedom." For first-order representations of the type described in (18), this property can be insured by endowing $\mathcal{R}$ with the same number of nodes as $\mathcal{R}$; however, the location of these nodal points in $\mathcal{R}$ need not coincide with those of $\mathcal{R}$. We shall add further restrictions on the topology of $\mathcal{R}$ subsequently.

Fig. 1 Finite-element models of a two-dimensional region $\mathcal{R}$.

The basic requirement of the basis functions $\phi_4(X)$ and $\chi'(X)$ is that they satisfy (9). Later, we prove that such functions do indeed exist by citing specific examples.

IV. SOME PROPERTIES OF GENERALIZED CONJUGATE FINITE-ELEMENT APPROXIMATIONS

A. General

Without further restrictions on the character of the models $\mathcal{R}$ and $\mathcal{R}$, we
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can only list rather general properties of the finite-element approximation \( \tilde{u} \) and \( \tilde{v} \) of (17). Consider, for example, the form of the scalar product (7) on the subspaces \( \tilde{W} \) and \( \tilde{V} \).

\[
\langle \tilde{u}, \tilde{v} \rangle = \langle u^\alpha \phi_\alpha, v^\beta \chi_\beta \rangle = u^\alpha v^\beta \left( \sum_{e=1}^E \sum_{f=1}^F \Omega_N^{(e)} \psi_N^{(e)}, \sum_{f=1}^F \sum_{f=1}^F \Omega_N^{(f)} \psi_N^{(f)} \right)
\]

\[
= u^\alpha v^\beta \sum_{e}^{E} \sum_{f}^{F} \Omega_N^{(e)} \Omega_N^{(f)} \langle \psi_N^{(e)}, \psi_N^{(f)} \rangle
\]

(19)

wherein \( \alpha, \beta = 1, 2, \ldots, G; N = 1, 2, \ldots, N_e \). Due to the localized character of the functions \( \psi_N^{(e)} \) and \( \psi_N^{(f)} \) described in (17), it is clear that

\[
\langle \psi_N^{(e)}(x), \psi_N^{(f)}(x) \rangle = 0, x \notin \bar{r}_e \cap \bar{r}_f
\]

(20)

Otherwise, the scalar product (20) defines an \( N_e \times N_f \) matrix

\[
\Delta_N = \langle \psi_N^{(e)}, \psi_N^{(f)} \rangle
\]

(21)

and (19) reduces to

\[
\langle \tilde{u}, \tilde{v} \rangle = u^\alpha v^\beta \sum_{e}^{E} \sum_{f}^{F} \Omega_N^{(e)} \Omega_N^{(f)} \Delta_N
\]

(22)

B. Restricted Models

With the discrete model of the scalar product now given by (22), we can proceed to generate a variety of conjugate finite-element approximations. However, we shall postpone a discussion of such general models to a later paper and, henceforth, we confine our attention to the important special case in which

\[
\bar{N}_e = N_e, \bar{r}_e \subseteq \bar{r}_e, \bar{r}_e \subseteq \bar{r}_e, E = E
\]

(23)

that is, essentially the same model \( \mathcal{G} \) is used to model domain of functions in \( \mathcal{V} \) as is used to model the domain \( \mathcal{G} \) of elements of \( \tilde{W} \). Then \( \Omega_N^{(e)} \) is simply the transpose of \( \Omega_N^{(e)} \) and we may write

\[
\Omega_N^{(e)} = \Omega_N^{(e)}
\]

(24)
where $\Lambda_N^e$ is the connectivity matrix described in [3]. It follows that (22) reduces to

$$
\langle \tilde{u}, \tilde{v} \rangle = u^e v_p \sum_{e}^{E} \Omega_N^e \Lambda_M^e \Delta_N^e
$$

(25)

where

$$
\Delta_N^e = \langle \psi_N^{(e)}, \psi_M^{(e)} \rangle = 0 \text{ if } x \notin r_e
$$

(26)

In view of the biorthogonality condition (9),

$$
\delta_x^e = \sum_{e=1}^{E} \Omega_N^x \Lambda_M^e \Delta_N^e
$$

(27)

so that

$$
\langle \tilde{u}, \tilde{v} \rangle = u^x v_x
$$

(28)

If we design the functions $\psi_N^{(e)}(x)$ and $\psi_M^{(e)}(x)$ so that (9) is satisfied locally, i.e.,

$$
\Delta_N^e = \delta_N^e
$$

(29)

then (27) reduces to the composition

$$
\delta_x^e = \sum_{e=1}^{E} \Omega_N^x \Lambda_N^e
$$

(30)

Moreover, if (29) holds, we have (except possibly on a set of measure zero)

$$
\langle \tilde{u}, \tilde{v} \rangle = \sum_{e=1}^{E} u_N^{(e)} v_e^{(e)} = u^x v_x
$$

(31)

where $u_N^{(e)}$ and $v_e^{(e)}$ are the values of $u_{(e)}$ and $v_{(e)}$ at node $N$ [see (17)].

**C. Other Properties**

We outline briefly a number of intrinsic properties of the generalized conjugate approximation functions.
1. Distributions. Suppose we take \( u = \delta(X - A) \), \( A \in \mathcal{R} \) where \( \delta(X - A) \) is the Dirac delta function defined by

\[
\delta(X - A) = \begin{cases} 
\infty, & X = A \\
0, & X \neq A 
\end{cases}
\]

and

\[
\int_{\mathcal{R}} f(X) \delta(X - A) \, dX = f(A)
\]  

where \( A \) is a specified point in the domain \( \mathcal{R} \). Then the projection of \( \delta(X - A) \) on the space \( \mathcal{W} \) is

\[
\pi \delta(X - A) = \Delta^\ast \varphi_\beta(X) \\
\pi^\ast \delta(X - A) = \Delta_x \chi^\ast(X)
\]  

wherein, according to (10)

\[
\Delta^\ast = \langle \delta(X - A), \chi^\ast \rangle \\
\Delta_x = \langle \varphi_\beta, \delta(X - A) \rangle
\]

If we define the scalar product of two functions \( u \) and \( v \) by

\[
\langle u, v \rangle = \int_{\mathcal{R}} uv \, d\mathcal{R}
\]

then it follows from (34) that

\[
\Delta^\ast = \chi^\ast(A) \quad \Delta_x = \varphi_\beta(A)
\]

that is, the values of the conjugate basis functions \( \chi^\ast \) and \( \varphi_\beta \) at the nodal points \( X^\ast \) are the coefficients of the projections of the delta functions \( \delta(X - X^\ast) \) in \( \mathcal{V}^\ast \) and \( \mathcal{W} \), respectively.

2. Nodal Values. Suppose we define, as in Eq. (17),

\[
\varphi_\beta(X^\rho) = \delta_\alpha^\beta \\
\chi^\ast(X^\rho) = \alpha^\ast^\beta
\]  

(37)
Then
\[ u(X) = u^* \phi_\alpha(X) \]
\[ u(X^\beta) = u^* \phi_\alpha(X^\beta) = u^\beta \] \hspace{1cm} (38)

and
\[ v(\bar{X}) = v_x \bar{X}(\bar{X}) \]
\[ v(\bar{X}^\beta) = v_x \bar{X}(\bar{X}^\beta) = v x^{a^\beta} \]

Then
\[ v_x = v(\bar{X}^x) \] \hspace{1cm} (39)

if and only if \( a^{x^\beta} = \delta^{x^\beta} \).

3. Moments and Volumes. To demonstrate another property of the conjugate approximation functions, we introduce moments \( M_x \) and \( M^x \) defined by

\[ M_x \equiv \langle \phi_x, 1 \rangle \]
\[ = \sum_{e}^{(e)} \Omega^N_x \langle \psi^N, 1 \rangle \]
\[ = \sum_{e}^{(e)} \Omega^N_x m^N_e \] \hspace{1cm} (40)

where
\[ m^N_e = \langle \psi^N, 1 \rangle \] \hspace{1cm} (41)

Similarly, we may define
\[ M^x \equiv \langle 1, \chi^x \rangle \]
\[ = \sum_{e}^{(e)} \Lambda^N_x \langle 1, \phi^{N(x)}_e \rangle = \sum_{e}^{(e)} \Lambda^N_x m^N_{e(x)} \] \hspace{1cm} (42)

where
\[ m^N_{e(x)} = \langle 1, \psi^{N(x)}_e \rangle \] \hspace{1cm} (43)

Then, if \( \langle . . . \rangle \) is given by (35),
\[ \int_{\mathcal{R}} d\mathcal{R} = \int_{\mathcal{R}} 1 \cdot d\mathcal{R} = \sum_{x=1}^{G} \int_{\mathcal{R}} \phi_\alpha(X) \ d\mathcal{R} \]
because the functions $\phi_\alpha(X)$ satisfy the condition (17). Now observe that
\[
\int_\mathcal{R} \, d\mathcal{R} = \sum_{\alpha=1}^{G} \int_\mathcal{R} \Omega_N^{(e)} \psi_N^{(e)}(X) \, d\mathcal{R} = \sum_{\alpha=1}^{G} \Omega_N^{(e)} m_{(e)}^{(e)} = \sum_{\alpha=1}^{G} M_{\alpha} = V \tag{44}
\]
where $V$ is the volume of $\mathcal{R}$. Similarly,
\[
\int_\mathcal{R} \, d\mathcal{R} = \sum_{\alpha=1}^{G} \Lambda_N^{(e)} m_{(e)}^{(e)} = \sum_{\alpha} M_{\alpha} = V \tag{45}
\]

V. LINEAR OPERATORS

We now arrive at the important problem of conjugate finite-element approximations of linear operators. Consider the linear operator $\mathcal{L}$ encountered in boundary-value problems characterized by equations of the form
\[
\mathcal{L}(u) = f \tag{46}
\]
in which, for the moment, $u$ is to satisfy homogeneous boundary conditions.

We seek approximate solutions of (46) in the subspaces $\mathcal{W}$ and $\mathcal{F}$ of the forms
\[
\bar{u} = a^* \phi_\alpha(X) \quad \bar{u} = b^* \chi^*(X) \tag{47}
\]
where the repeated index $\alpha$ is summed from 1 to $G$.

It is natural to examine projections of $\mathcal{L}(u)$ into $\mathcal{W}$ and $\mathcal{F}$ in which $\bar{u}$ appears as the argument of $\mathcal{L}(u)$ rather than $u$; that is,
\[
\pi[\mathcal{L}(u) - f] = \pi[\mathcal{L}(\bar{u})] - f^* \phi_\alpha(X) \approx 0 \tag{48}
\]
or
\[
\pi^*[\mathcal{L}(u) - f] = \pi^*[\mathcal{L}(\bar{u})] - f^* \chi^*(X) \approx 0 \tag{49}
\]
wherein
\[
f^* = \langle f, \chi^* \rangle \quad \text{and} \quad f_\alpha = \langle \phi_\alpha, f \rangle \tag{50}
\]

Following [2], we assume (at least approximately) the commutativity of the
operators $\pi$ and $L$ and $\pi^*$ and $L^*$:

$$L\pi \approx \pi L \quad L\pi^* \approx \pi^* L$$  \hspace{1cm} (51)

Then

$$\pi[L(\tilde{u})] = a^\beta \pi L(\varphi_a) = a^\beta L_z^\beta \varphi_b(X) = b_\alpha \pi L(\chi^\alpha) = b_\alpha M_{\beta \gamma} \varphi_b(X)$$  \hspace{1cm} (52)

and

$$\pi^*[L(\tilde{u})] = a^\beta \pi^* L(\varphi_a) = a^\beta N_{\beta \gamma} \chi^\gamma(X) = b_\alpha \pi^* L(\chi^\alpha) = b_\alpha P_{\beta \gamma} \chi^\gamma(X)$$  \hspace{1cm} (53)

where

$$L_z^\beta = \left\langle L \varphi_a, \chi^\beta \right\rangle \quad M_{\beta \gamma} = \left\langle L \chi^\beta, \chi^\gamma \right\rangle$$  \hspace{1cm} (54)

$$N_{\beta \gamma} = \left\langle \varphi_a, L \varphi_b \right\rangle \quad P_{\beta \gamma} = \left\langle \varphi_b, L \chi^\gamma \right\rangle$$

Therefore, if $\tilde{u}$ and $\tilde{v}$ are distinct, there are four distinct approximations of the operator $L$ consistent with (47) and (51).

Introducing (52) and (53) into (48) and (49), and equating like coefficients of the basis functions $\varphi_a(X)$ and, separately, $\chi^\alpha(X)$, we obtain the following discrete analogues of (46):

$$L_z^\beta a^\alpha - f^\beta = 0$$  \hspace{1cm} (55)

$$M_{\beta \gamma} b_\gamma - f^\beta = 0$$  \hspace{1cm} (56)

$$N_{\beta \gamma} a^\alpha - f_\gamma = 0$$  \hspace{1cm} (57)

$$P_{\beta \gamma} b_\gamma - f_\gamma = 0$$  \hspace{1cm} (58)

It is clear from (55)–(58) that

$$L_z^\beta a^\alpha = M_{\beta \gamma} b_\gamma$$  \hspace{1cm} (59)

$$N_{\beta \gamma} a^\alpha = P_{\beta \gamma} b_\gamma$$  \hspace{1cm} (60)

Define

$$[L_z^\beta]^{-1} \equiv L_z^{-1 \cdot \beta}$$  \hspace{1cm} (61)

$$[M_{\beta \gamma}]^{-1} \equiv M_{\beta \gamma}$$  \hspace{1cm} (62)

$$[N_{\beta \gamma}]^{-1} \equiv N_{\beta \gamma}$$  \hspace{1cm} (63)

$$[P_{\beta \gamma}]^{-1} \equiv P_{\beta \gamma}^{-1 \cdot \gamma}$$  \hspace{1cm} (64)
Hence, from Eqs. (59) and (60) it follows that

$$L_\beta^{-1} \mu M^{\mu \beta} = N^{\mu \beta} P_\beta^\gamma$$

(65)

where $N^{\mu \beta}$ is given by Eq. (63).

A. Local Approximations

We assume that the element $r_e$ coincides with $\bar{r}_e$. Then, locally, we can define

$$m^{MN} = \langle \mathcal{L} \psi^M_N, \psi^{(e)}_N \rangle$$

$$P_\gamma^M = \langle \psi^{(e)}_N, \mathcal{L} \psi^{(e)}_N \rangle$$

Then, from (54), it follows that

$$L_2^\beta = \langle \mathcal{L} \varphi, \chi^\beta \rangle = \langle \mathcal{L} \sum_{e=1}^E \Omega^N_e \psi^M_{N_e}, \sum_{f=1}^E \Omega^M_F \psi^{(e)}_{F(f)} \rangle$$

$$= \sum_{e=1}^E \sum_{f=1}^E \Omega^N_e \Omega^M_F \langle \mathcal{L} \psi^{(e)}_N, \psi^{(e)}_{F(f)} \rangle$$

(67)

and

$$M^{\mu \beta} = \langle \mathcal{L} \chi, \chi^\beta \rangle = \sum_{e=1}^E \sum_{f=1}^E \Omega^N_e \Omega^M_F \langle \mathcal{L} \chi^\beta, \psi^{(e)}_{F(f)} \rangle$$

$$N_{\delta \beta} = \langle \varphi, \mathcal{L} \varphi \beta \rangle = \sum_{e=1}^E \sum_{e=1}^E \Omega^N_e \Omega^M_F \langle \psi^{(e)}_N, \mathcal{L} \psi^{(e)}_M \rangle$$

$$P_{\gamma}^\alpha = \langle \varphi, \mathcal{L} \chi^\beta \rangle = \sum_{e=1}^E \sum_{f=1}^E \Omega^N_e \Omega^M_F \langle \psi^{(e)}_N, \mathcal{L} \chi^\beta \rangle$$

(68)

(69)

(70)

In the special case in which the conditions (23) and (24) hold, we have the important results

$$L_2^\beta = \sum_{e=1}^E \Omega^N_e \delta^{M}_{M} I^{\mu}_{N_e}$$

$$M^{\mu \beta} = \sum_{e=1}^E \delta^{(e)}_{M} M^{(e)}_{MN}$$

(71)

(72)
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\[ N_{\beta} = \sum_e^{(e)} \Omega_{\beta}^{(e)} n_{MN} \]  
(73)

\[ P_{\beta} = \sum_c^{(e)} \Omega_{\beta}^{(e)} n_{MN} \]  
(74)

where, for example, \( n_{MN}^{(e)} \) represents the local component of \( L_{z\beta} \) relative to the element \((e)\).

B. Specific Linear Operators

Here we give some specific examples of the linear operators. One of the most familiar linear operators is the partial differential operator.

\[ \mathcal{L} = \frac{\partial}{\partial X_i} \]  
(75)

We assume that the partial derivatives of the basis functions \( \varphi_\beta(X) \) and \( \chi^\beta(X) \) exist and belong to the spaces \( \mathcal{H} \) and \( \mathcal{P} \), respectively. (This requirement usually holds only approximately.) Then, according to Eq. (54),

\[ L_{z\beta} = \mathcal{L} \varphi_\beta \chi^\beta = \langle \frac{\partial \varphi_\beta}{\partial X_i}, \chi^\beta \rangle \equiv L_{z\beta} \]

\[ M_{z\beta} = \mathcal{L} \chi^\beta = \langle \frac{\partial \chi^\beta}{\partial X_i}, \chi^\beta \rangle \equiv M_{z\beta} \]

\[ N_{z\beta} = \langle \varphi_\beta, \mathcal{L} \varphi_\beta \rangle = \langle \varphi_\beta, \frac{\partial \varphi_\beta}{\partial X_i} \rangle \equiv N_{z\beta} \]

\[ P_{z\beta} = \langle \varphi_\beta, \mathcal{L} \chi^\beta \rangle = \langle \varphi_\beta, \frac{\partial \chi^\beta}{\partial X_i} \rangle \equiv P_{z\beta} \]  
(76)

Also, if

\[ \mathcal{L} = \frac{\partial}{\partial X_i} \frac{\partial}{\partial X_j} \]  
(77)

we have

\[ \pi \left( \frac{\partial u}{\partial X_i} \right) = D_{z\beta} u^\gamma \varphi_\beta(X) = \Delta_{z\beta} u^\gamma \varphi_\beta(X) \]

\[ \pi^* \left( \frac{\partial u}{\partial X_i} \right) = \nabla_{z\beta} u^\gamma \chi^\beta(X) = \delta_{z\beta} u^\gamma \chi^\beta(X) \]
and
\[
\frac{\partial^2 u}{\partial X_i \partial X_j} \approx D_{a\beta} \nabla_{i\beta} u^x \chi^y(X) = \Delta_{i\gamma} \nabla_{i\gamma} u^x \chi^y(X)
\]
\[
\approx \nabla_{i\gamma} \Delta_{j\gamma} u^x \varphi_\lambda(X) = \delta_{\mu\lambda} \Delta_{i\lambda} u^x \varphi_\lambda(X) \quad (78)
\]
Likewise, if
\[
\mathcal{L}u = \int_a^b K(s, X) u(s) \, ds \quad (79)
\]
then
\[
L_{\mu}^x = \langle \mathcal{L} \varphi_\mu, \chi^y \rangle = \left\langle \int_a^b K(s, X) \varphi_x(s) \, ds, \chi^y \right\rangle
\]
\[
M_{\mu}^x = \langle \mathcal{L} \chi^y, \chi^y \rangle = \left\langle \int_a^b K(s, X) \chi^y(s) \, ds, \chi^y \right\rangle
\]
\[
N_{\mu} = \langle \varphi_x, \mathcal{L} \varphi_x \rangle = \left\langle \varphi_x, \int_a^b K(s, X) \varphi_x(s) \, ds \right\rangle
\]
\[
P_{\mu}^x = \langle \varphi_x, \mathcal{L} \chi^y \rangle = \left\langle \varphi_x, \int_a^b K(s, X) \chi^y(s) \, ds \right\rangle \quad (80)
\]

Obviously, many other examples could be cited. Since the procedure for constructing such approximations is amply demonstrated by the above examples, we shall not elaborate further here.

**VI. RELATIONSHIP OF CONJUGATE APPROXIMATIONS TO THE HYPERCIRCLE CONCEPT**

We shall conclude this investigation with a discussion of the relationship between the concept of generalized conjugate approximations and the concept of the hypercircle of Prager and Synge [7] and Synge [8]. Consider the boundary-value problem
\[
\mathcal{L}(g(X) \mathcal{L}w(X)) = f(X) \text{ in } \mathcal{R}
\]
\[
\mathcal{B}(w) = h(X) \text{ on } \partial \mathcal{R}_1 \quad \mathcal{C}(w) = j(X) \text{ on } \partial \mathcal{R}_2 \quad (81)
\]
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where \( \mathcal{L}(g \mathcal{L}) \) is a linear elliptic partial differential operator of order \( 2p \), \( \mathcal{A}(w) \) is a linear differential operator of order \( p - 1 \) describing the principal boundary conditions, and \( \mathcal{C}(w) \) is a linear differential operator of order \( 2p - 1 \) describing the natural boundary conditions. Here \( \partial \mathcal{R} = \partial \mathcal{R}_1 + \partial \mathcal{R}_2 \).

The problem (81) can be "split" by defining two linear manifolds \( \mathcal{U} \) and \( \mathcal{V} \), the elements in \( \mathcal{U} \) being those functions \( \mathcal{L}u(X) \) where the \( u(X) \) are such that

\[
\mathcal{A}(u) = h \text{ on } \partial \mathcal{R}_1
\]

(82)

Likewise, if \( v \in \mathcal{V} \), then

\[
g \mathcal{L}(v) = f \text{ in } \mathcal{R} \text{ and } \mathcal{C}(v) = j \text{ on } \partial \mathcal{R}_2
\]

(83)

It is assumed that there exists a unique solution \( w^* \) of (81) which lies at the intersection of \( \mathcal{U} \) and \( \mathcal{V} \).

The scalar product defined on \( \mathcal{U} \otimes \mathcal{V} \) is defined by

\[
[u, v] = \langle u, \mathcal{L}(g \mathcal{L}v) \rangle = \langle \mathcal{L}(u), g \mathcal{L}(v) \rangle + k(u, v)
\]

(84)

where \( k(u, v) \) is a bilinear functional representing certain boundary values of \( u \) and \( v \) and their derivatives. Then the "energy" norm is \( \|u\| = [u, u]^{1/2} \) and the squared distances from arbitrary elements \( u \in \mathcal{U} \) and \( v \in \mathcal{V} \) from the exact solution \( w^* \) are

\[
d^2_u(u, w^*) = \|u - w^*\| \quad d^2_v(v, w^*) = \|v - w^*\|
\]

(85)

To obtain approximate solutions of (81), we consider two finite-dimensional subspaces \( \mathcal{V} \) and \( \mathcal{V}' \) of \( \mathcal{U} \) and \( \mathcal{V} \) (actually, in finite-element approximations the spaces \( \mathcal{V} \) and \( \mathcal{V}' \) are selected so that (82) and (83) are only approximately satisfied; hence, \( \mathcal{V} \) and \( \mathcal{V}' \) need not actually be subsets of \( \mathcal{U} \) and \( \mathcal{V} \)). In the hypercircle method, elements in \( \mathcal{V} \) are assumed to be of the form \( \bar{u} = \bar{u}_0 + \bar{u} \), where \( \bar{u}_0 \) satisfies (at least approximately) (82) and \( \bar{u} \) satisfies homogeneous principal boundary conditions on \( \partial \mathcal{R}_1 \); i.e., \( \bar{u} \) is an element of a linear space \( \mathcal{V} \) and \( \bar{u}_0 \) describes a translation of \( \mathcal{V} \). Likewise, \( \bar{v} = \bar{v}_0 + \bar{v} \), where \( \bar{v}_0 \) satisfies (83) and \( \bar{v} \) satisfies homogeneous conditions of the form (83). Then \( \bar{v} \) belongs to a linear vector space \( \mathcal{V}' \) and \( \bar{v}_0 \) describes a translation. It can then be shown that \( \mathcal{V} \) is orthogonal to \( \mathcal{V}' \).

In contrast, the method of conjugate projections takes \( \bar{u} \) and \( \bar{v} \) to be of the form (11) and (12) and the coefficients \( a^*, b^* \), and the basis functions \( \varphi_a(X) \). \( \chi'(X) \) can be adjusted so that (82) and (83) are satisfied.

Turning now to (48), the approximate solutions \( \bar{u}, \bar{v} \) can be determined by choosing the coefficients \( a^*, b^* \) such that the following distances are mini-
It is easily shown that minimization of each member of (86) leads directly to the approximations (55)–(58). For example, introducing (11) into the first equation of (86), we have

\[
d_i^2(\bar{u}, w^*) = \langle a^2 \phi_a - w^*, a^b L(g L \phi_b) - L(g L w^*) \rangle
\]

\[
= a^2 a^b \langle \phi_a, L(g L \phi_b) \rangle - 2a^2 \langle \phi_a, f \rangle + \|w^*\|^2
\]

(87)

Or, using the notation of (50) and (54),

\[
d_i^2(\bar{u}, w^*) = a^2 a^b N_{a,b} - 2a^2 f_a + \|w^*\|^2
\]

(88)

Hence, \(d_i^2(\bar{u}, w^*)\) is a minimum if \(\partial^2 d_i/d a^b = 0\): or

\[
N_{a,b} a^b = f_b
\]

(89)

which is identical to (57). Similarly, minimization of \(d_i^2(\bar{u}, w^*)\) of (86) with respect to \(b_a\) gives (56). Equations (55) and (58) can be obtained by seeking the minimum of the second equation of (86), first holding \(a^a\) fixed and varying \(b_a\) and then holding \(b_a\) fixed and varying \(a^a\).

We remark that it is also possible to establish bounds on \(\|w^*\|\) in \(\bar{u}\) and \(\bar{v}\) as described in [8], and that a generally better approximation to \(w^*\) can often be obtained by averaging the approximations \(\bar{u}\) and \(\bar{v}\) which minimize the first and second equations of (86).

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REFERENCES


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