A NOTE ON THE ANALYSIS OF NONLINEAR DYNAMICS OF
ELASTIC MEMBRANES BY THE FINITE ELEMENT METHOD

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ELASTIC MEMBRANES BY THE FINITE ELEMENT METHOD

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Abstract - The construction of finite-element/difference approximations
of the equations governing large-amplitude motions and waves in thin,
isotropic elastic membranes is described. Numerical solutions to
representative problems in nonlinear dynamics of hyperelastic sheets
are given.

INTRODUCTION

The analysis of large-amplitude motions, shock and acceleration
waves, and other nonlinear dynamic phenomena in highly-elastic bodies
at finite strain has stood outside the realm of available analytical
and numerical methods and computing machinery for some time. In
recent years, modern numerical techniques have managed to penetrate
some of the simpler problems of finite static deformations of elastic
bodies (e.g. [1]), but available information on nonlinear membrane
dynamics apparently is purely qualitative or is based upon experimental
data.

The present note describes recent progress in the numerical
analysis of a class of nonlinear problems in finite elastodynamics by
the finite-element method. The formulation of fully-discrete models
of the transient behavior of thin, isotropic, incompressible, sheets of hyperelastic material is described. Simplex approximations of the spatial (material) variation of the displacement field are used, and a velocity-formulated control difference scheme is incorporated to represent the temporal behavior. Numerical results are obtained for the problem of finite-amplitude oscillations and two-dimensional (flexural and extensional) waves in an isotropic beam and for the problems of wave phenomena in a thin flat membrane subjected to time-dependent loading normal to its plane.

DESCRIPTION OF THE FINITE ELEMENT MODEL

A finite-element model of a thin, homogeneous, isotropic membrane of incompressible elastic material is developed following procedures described in [1]. Minor variations in the present formulation deserve a few comments.

An arbitrary membrane of the type mentioned above is represented by a collection of thin "constant-strain" triangles, of initial thickness \(d_0\) and undeformed plane area \(A_0\). Over a typical finite element \(e\), the local displacement components \(u_i(x)\) and the velocity components \(v_i(x)\) \((i = 1, 2, x = (x_1, x_2))\) are given by the local approximations

\[
\begin{align*}
  u_1 &= \psi_N(x)u^N_1 \quad ; \quad v_1 = \varphi_N(x)v^N_1
\end{align*}
\]

where the repeated index \(N\) is summed from 1 to 3, \(u^N_1\) and \(v^N_1\) are the displacement and velocity components at node \(N\), and \(\psi_N(x)\) and \(\varphi_N(x)\) are the usual local interpolation functions. In particular, \(\psi_N(x)\) are the piecewise linear functions

\[
\psi_N(x) = a_N + b_{NQ}x^Q
\]

(2)
$a = 1, 2$, where $a_n$ and $b_{n\alpha}$ are known constants dependent only upon the local element geometry (see Eqns. (106) of [1], p. 129).

Following the usual procedure for formulating finite element approximations, we arrive at the following equations of motion for a typical finite-element e in our model (cf. [1], p. 318):

$$m_{NN'} \ddot{u}_N + 2u_0 \left( (\delta_{\alpha\beta} - \lambda^2 f_{\alpha\beta}) \frac{\partial W}{\partial l_1} + [\lambda^2 \delta_{\alpha\beta} + f_{\alpha\beta}(1 - 2\lambda^4) - 2\lambda^4 \delta_{\beta\mu} (\delta_{\mu\kappa} + \frac{1}{2} b_{n\kappa} u_{n\kappa}^N) \frac{\partial W}{\partial l_2} \right) b_{n\alpha} (\delta_{\beta1} + b_{p\beta} u_{p1}^N) = p_{n1}(t)$$

Here $M, N, P, R, i, k = 1, 2, 3; \alpha, \beta, \lambda, \mu = 1, 2; \nu_0(e) = d_0 A_0$ is the undeformed volume of the element, $\lambda$ is the transverse extension ratio (i.e., $\lambda = d/d_0$), $f_{\alpha\beta}$ is a surface tensor depending on the displacement gradients $u_{i\kappa}$, $W = W(l_1, l_2)$ is the strain energy per unit undeformed volume given as a function of the first and second principal invariants $l_1$ and $l_2$ of Green's deformation tensor, and $p_{n1}$ are the components of generalized force at node $N$. For the discrete model,

$$\lambda^2 = [1 + 2b_{n\alpha} (\delta_{\alpha\beta} + b_{n\beta} u_{n\beta}^N) u_{n\alpha}^N + 2\varepsilon_{\alpha\lambda} \varepsilon_{\beta\mu} \gamma_{\alpha\lambda} \gamma_{\beta\mu}]^{-1}$$

$$f_{\alpha\beta} = \delta_{\alpha\beta} + 2\varepsilon_{\alpha\lambda} \varepsilon_{\beta\mu} \gamma_{\lambda\mu}$$

$$\gamma_{\alpha\beta} = \frac{1}{2} (b_{n\alpha} u_{n\beta}^N + b_{n\beta} u_{n\alpha}^N + b_{n\alpha} b_{n\beta} u_{n\alpha}^N u_{n\beta}^N)$$

where $\varepsilon_{\alpha\lambda}$ is the two-dimensional permutation symbol.

The array $m_{NN}$ in (3) is the mass matrix

$$m_{NN} = \int_{\nu_0(e)} \rho_0 \varphi_N \varphi_N d\nu_0$$

where $\rho_0$ is the initial mass density. If we take $\varphi_N = \psi_N$, (5) yields
the usual consistent mass matrix. If we take \( \varphi(x) = \delta(x^N - x) \), \( \delta(x) \) being the Dirac delta function, \( m_{NM} \) reduces to the lumped mass matrix, 
\[
m_{NM} = \frac{1}{3} \rho \sum_\sigma \delta_{NM}.
\]
For simplicity, and for concrete theoretical reasons to be mentioned later, we shall use the lumped mass approximation hereafter, and set \( \dot{u}_1^N = \dot{v}_1^N \).

**TEMPORAL APPROXIMATIONS**

Upon assembling elements and applying boundary conditions, (4) leads to a large system of highly nonlinear second-order ordinary differential equations in the unknown nodal displacements \( u_1^N \). Our only hope for extracting data from such systems is to attempt to solve them numerically. Toward this end, we may choose between the two classes of direct numerical time-integration methods for systems of nonlinear equations: the conditionally stable explicit schemes, and implicit schemes, which are often unconditionally stable for linearized problems. Obviously, the basis for our selection is the optimization of both economy and accuracy.

Here, the principal advantage of implicit schemes (unconditionally stable for large time steps) is overshadowed by the necessity of solving a set of simultaneous nonlinear equations at each time step. Moreover, since we are also interested in nonlinear wave phenomena, the high-frequency response is very important—especially in the region of the propagated discontinuities. Hence, even if an implicit scheme is used for this type of problem, experience with linear problems suggests that the associated time step required to retain higher frequencies of the model is usually just as small as that required for numerical stability of explicit schemes.
All time integration schemes may involve the inversion of the mass matrix. If the consistent mass matrix is used, the inverse is a full matrix which, in turn, produces infinite wave speeds in the discrete model. However, if a diagonal mass matrix is used, obtaining the inverse is trivial, and the explicit solution process is extremely fast and simple. (Implicit schemes involve inverting a weighted sum of the mass and stiffness matrices (e.g., cf. [2]) so that a diagonal mass matrix affords little advantage in the nonlinear case if implicit schemes are used). Krieg and Key [3] have shown that for a number of representative linear elastodynamics problems the diagonal mass matrix and the explicit central difference time-integration scheme provide the most practical means of computing a transient response.

Therefore, due to its computational speed and more accurate representation of a finite wave speed, a central difference time-integration scheme (sometimes referred to as the "velocity" formulated central difference scheme) is used together with a lumped-mass representation for the numerical examples discussed in the next section. For this method, the general second-order nodal equations of motion, of the form \( \ddot{u}_i^N(t) = F_i^N(u_i^N(t)) \) are first separated into two first-order equations by introducing again the nodal velocities \( v_i^N = du_i^N/dt \). By partitioning the time interval into equally spaced time steps \( \Delta t \) and denoting \( u_i^N(n) = u_i^N(n\Delta t) \), etc., our resulting difference equations are of the form

\[
\begin{align*}
    v_i^N(n + \frac{1}{2}) &= v_i^N(n - \frac{1}{2}) + \Delta t F_i^N(u_i^N(n)) \\
    u_i^N(n + 1) &= u_i^N(n) + \Delta t v_i^N(n + \frac{1}{2})
\end{align*}
\]  

(6)
Direct substitution shows that (6) is equivalent to using the usual second central difference approximation for $\ddot{u}_t(t)$. However, (6) generally admits less round-off error into the solution process (cf. [2]).

**STABILITY AND CONVERGENCE**

Although at the present time we do not have stability and convergence criteria for the two-dimensional nonlinear wave equation, we conjecture that our recent work [4] on one-dimensional nonlinear problems furnishes some insight into at least the order of magnitude expected of the two-dimensional estimates. The plausibility of this conjecture can be argued by noting that for the linear case, the time step associated with stability estimates of Richtmyer and Morton [5] (pp. 304, 361) decreases by only about 30% in going from one to two dimensions. Also, the two-dimensional stability estimates of Fujii [6] seem to require a time-step twice as small as that needed for one-dimensional stability. The principal results of the nonlinear one-dimensional investigation [4] are summarized below:

The stability of the central difference scheme in energy is assured if

$$\frac{h}{\Delta t} > \frac{\sqrt{2} C_{x_{\text{max}}}}{\sqrt{2}},$$

where $h$ is the minimum mesh length for the finite-element model, $\nu_\alpha (\alpha = 1,2)$ are constants, $\nu_1 = 2/3$ corresponding to a consistent mass formulation and $\nu_2 = 2$ to a lumped mass formulation, and $C_{x_{\text{max}}}$ is the maximum speed of propagation of acceleration waves relative to the material at the $i$th time increment. Obviously, this remarkably simple result reduces to similar criteria obtained for linear difference approximations when $C_{\text{max}} = \text{constant}$. 
While the stability criteria give only sufficient conditions for stability, they suggest that for a fixed $h$ it is sufficient to use a smaller time step for the consistent mass formulation than for the lumped mass formulation since $\nu_1 > \nu_2$. (Moreover, use of lumped masses avoids ringing in front of a wave front and maintains the finite character of wave speeds in the model.)

Under the stated assumptions, the square of the $L_2$-norm, $\| \nabla e^{(t)} \|^2$, of the gradient of the error at each time step $i$ is $O(h^2 + (\Delta t)^2)$. (Similar accuracies are obtained after $R$ time steps in the linear case.) Uniform convergence of the error $e$ is also obtained.

The same rates-of-convergence for the consistent mass formulation are obtained for the lumped mass formulation.

**NUMERICAL RESULTS**

In this section, we present numerical results obtained by applying the procedures described earlier to the problem of a beam with a central load and a square sheet with a transverse load applied suddenly at the center and then removed.

**Highly Elastic (Rubber-like) Beam with Central Load.** As the first example, a beam with fixed ends is subjected to a central impulse load and allowed to deform as a function of time. The undeformed beam is 18 in. long, 0.4 in. deep and 0.05 in. thick, and is constructed of an isotropic, incompressible material of the Mooney type (i.e., $W = C_1(I_1 - 3) + C_2(I_2 - 3)$) with material constants of $C_1 = 24$ psi, $C_2 = 1.5$ psi. The mass density of the material is 0.0001 lb.sec²/in⁴.
Since the loading is symmetric, a finite element model was generated for only one-half of the beam. The model consisted of 20 bays along the length and 4 bays through the depth of the beam, as shown in Fig. 1a. This resulted in a model with 105 nodes and 160 triangular elements. Upon applying boundary conditions and restricting 15 degrees of freedom, the number of unknown displacements reduced to 195. Impulse loads of 1.0, 0.9 and 0.25 lbs. were applied for 0.001 sec. to the node representing the center, the node 0.25 in. from the center and 0.6 in. from the center, respectively, to avoid stress singularities at the center.

Figure 2 shows the deformed shape of the beam computed at various times. Here the complete beam is shown for clarity.

It is emphasized that the deformation is plotted to scale. Hence, very large-amplitude motions are clearly developed. In Fig. 2a the undeformed beam is shown; Fig. 2b shows the beam just prior to termination of the impulse load, and Figs. 2c and d show the propagation of the wave toward the fixed edge. At this point in time there are two significant motions, one is a general vertical (flexural) motion of the whole beam and the other is a longitudinal wave traveling along the deformed axis of the beam. In Fig. 2e, the wave traveling along the beam is reflected off the boundary and starts moving toward the center. In Fig. 2f, the vertical motion is resisted by bending at the fixed edge and the wave traveling along the beam is reflected from the center. The remaining frames of the figure show the interaction of the slow bending wave and the faster moving wave along the beam at various times. The 12 frames presented here were selected from a 16 mm movie film containing approximately 1200 frames. It was evident from the movie that longitudinal waves (i.e., waves along the deformed x₁-axis) resulted
in very fast oscillations ahead of the much slower flexural type motions.

Approximately 12,000 integration steps were required to solve this problem. This was achieved on the UNIVAC 1108 in 35 minutes. It is interesting to note than an additional 40 minutes computer time was required to generate the 16 mm film containing 1200 frames which can be shown in 40 seconds.

Square Sheet with Normal Load. As a second example, the square sheet with clamped edges shown in Fig. 1b is subjected to a central impulse load, normal to the plane of sheet, and allowed to deform as a function of time. The membrane is a 20 in. square with a thickness of 0.05 in., and the material properties are the same as the first example. Here it was necessary to model only one-eighth of the membrane since symmetry was assumed. The finite element model of the membrane contains 400 elements and 231 nodes. The supported model resulted in 590 unknown displacements. A uniformly distributed impulse load of 3 lbs./element was applied to the 4 elements closest to the node representing the center of the sheet. The duration of the impulse load was .001 sec.

Figure 3a shows the initial finite element model of the sheet; Fig. 3b shows the sheet just after the removal of the load. The energy imparted to the membrane continues the vertical movement of the center in Fig. 3c. In Fig. 3d the internal forces overcome the inertia effects and the center of the sheet collapses and forms a standing wave which propagates toward the fixed edge in Figs. 3e through 3i. The wave which travels with a circular front strikes the nearest edge and starts to rebound, while the wave is still traveling toward the extreme
edge in Fig. 3i. The rebound wave continues to propagate in Figs. 3j through 3l. Again, a closer examination of the response indicated that "membrane" waves propagating along the deformed sheet are developed which traveled much faster than the "ripple" obvious in the figures.

Thirty minutes on the UNIVAC 1108 were required to complete 4200 integration steps. A 16 mm film containing 1300 frames was made and the 12 frames presented here were selected from the film.

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REFERENCES


Fig. 1. Finite-element models of a highly-elastic beam (a), and a square membrane (b), (not to scale).
Fig. 2. Deformed shapes of centrally loaded beam.
Fig. 3. Deformed shapes of centrally loaded sheet.