II

FINITE ELEMENT APPROXIMATIONS IN NONLINEAR THERMOVISCOELASTICITY

J. T. Oden
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Summary. It is the purpose of this lecture to describe the important features of general theories of thermomechanical behavior of materials, to demonstrate the construction of general finite-element models of these theories, to investigate special forms of these models which are appropriate for the numerical analysis of nonlinear problems in thermovisco-elasticity, thermoelasticity, and heat conduction in solids, and to cite numerical results obtained by applying the finite element equations to representative nonlinear problems. The bulk of this lecture deals with so-called simple materials, particularly thermoviscoelastic materials, with consideration of thermomechanical dissipation, heat conduction, and certain wave phenomena.

1. INTRODUCTION

It is a common subject of elementary physics that various types of energy can be converted from one form to another. Every child is aware of the fact that if two objects are rubbed together fast enough and long enough, they get hotter, indicating a conversion of purely mechanical energy into thermal energy or heat. Sometimes such conversions of energy are manifested internally in materials; that is, the physical character of the material is such that a great deal of heat can be generated by mechanical working. Likewise, we may produce changes in mechanical energy (work) by heating bodies; but if the same amount of heat generated in working a body is applied to the body, it may not produce the same amount of mechanical work. Then the process is an irreversible one, and energy is dissipated in the system.

In the physical world, all of these phenomena take place over a finite period of time. The "instant" recovery of an elastic body upon unloading is a mathematical abstraction; the response of different materials to the

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same conditions generally takes place over different periods of time. The physical properties of many materials are time-dependent, as is illustrated by the continuing deformation of, say, concrete, certain plastics, metals, and biological tissues under constant load. Moreover, the rate at which such materials respond to given loads may be very sensitive to changes in temperature.

While such thermomechanical phenomena are experienced frequently in everyday life, their complete mathematical description has eluded scientists for centuries. Only in recent times have rather general continuum theories of thermoviscoelasticity emerged, and, in order to achieve sufficient generality to account for most of the phenomena mentioned above, these theories are necessarily nonlinear and highly complicated. While general theories are valuable in their own right for providing general and qualitative information on the character of certain classes of materials, they are of little use in controlling, predicting, or studying specific thermomechanical phenomena or in the design of specific machines and structures if quantitative information cannot be extracted from them. It is toward this goal, that is, the quantitative description of the nonlinear thermomechanical behavior of materials, that the powerful concept of finite elements and modern high-speed computers may find their most interesting and useful applications.

It is the purpose of this lecture to describe the important features of general theories of thermomechanical behavior of materials, to demonstrate the construction of general finite-element models of these theories, to investigate special forms of these models which are appropriate for the numerical analysis of nonlinear problems in thermoviscoelasticity, thermoelasticity, and heat conduction in solids, and to cite numerical results obtained by
applying the finite element equations to representative nonlinear problems. Portions of this lecture are based on earlier articles and papers; the discussion of simple materials and the corresponding finite element models follows closely that in Finite Elements of Nonlinear Continua [1], and some of the numerical results presented are taken from pertinent papers [2,3]. The bulk of this lecture deals with so-called simple materials, particularly thermoviscoelastic materials, with ample consideration of thermomechanical dissipation, heat conduction, and certain wave phenomena.

2. THERMOMECHANICAL PRELIMINARIES

Kinematics. Consider a body $B$, the elements of which are material particles $X$. We associate with each particle an ordered triple of real intrinsic particle labels $X^i, i = 1, 2, 3$, called intrinsic coordinates of $X$. Since this association is one-to-one and onto, we use interchangeably $X$ and $X = (X^1, X^2, X^3)$. Let $C_o$ denote a subregion of three-dimensional Euclidean space, the points $X = (x^1, x^2, x^3)$ or which are in one-to-one correspondence with the particles $X$ of $B$. We regard $C_o$ as the portion of $E^3$ occupied by $B$ at time $t = t_0$ (or $\tau = t - t_0 = 0$) and we choose a correspondence $\chi_w: B \rightarrow C_o$ such that the numbers $X^i$ coincide with the coordinates $x^i$. The motion of the body is traced relative to the fixed reference configuration $C_o$ by

$$x = \chi(X, t)$$

(2.1)

where $\chi$ is the mapping that carries $X$ in the reference configuration onto the place $x$ at time $t$. The deformation function $\chi$ has the property $\chi(X, t_0) = X$ (through a change of variables we may also write $x = \chi(X, \tau)$ such that $\chi(X, 0) = X$). For each $\tau, \chi(X, \tau)$ may define a different configuration of the body. The configuration $C_t$ corresponding to $\chi(X, t)$ is referred to as the current configuration. In addition to (2.1), we introduce as the deformation gradient $F$ of the motion $\chi$ the second-order tensor
\[ F = \nabla_X (X, t) = F(X, t) \]  
(2.2)

where \( \nabla \) is the material gradient (i.e., \( \nabla = G \frac{\partial}{\partial X} \)). If \( g_i \) denotes a system of basis vectors tangent to the material lines \( X^i \) in \( C_o \) and \( G \) denotes a system of natural basis vectors tangent to the material lines \( X^i \) of \( S \) while in \( C \), then it is clear that

\[ G = (\partial x^i / \partial x^j) g_j = F^j_i g_j \]  
(2.3)

where \( F^j_i \) are the components of \( F \). The tensor

\[ G = F^j_i F^i_j \]  
(2.4)

is called the Green-Saint Venant deformation tensor. Clearly, its components are

\[ G_{ij} = G^i_j \cdot G^j_i = g_{ij} F^i_k F^k_j = x_i^1 x_j^1 \]  
(2.5)

wherein commas denote differentiation with respect to the material coordinates \( (x_i = \partial x^i / \partial x^j) \). The Green-Saint Venant deformation tensor is positive definite and nonsingular; it relates the differentials \( dX^i \) to the square of a material line element \( ds^2 \) according to

\[ ds^2 = G_{ij} dX^i dX^j = dX^i G_{ij} \]  
(2.6)

The Green-Saint Venant strain tensor \( \gamma \) is defined by

\[ \gamma = \frac{1}{2} (F_{ij} F^{ij} - I) = \frac{1}{2} (G - I) \]  
(2.7)

where \( I \) is the unit tensor (i.e., \( I \) has components \( g_{ij} \cdot g_{ij} = g_{ij}, g^{ij}, g_{ij} \), \( g_{ij} \) being the material metric coefficients in \( C_o \)). Thus, if \( ds_o \) is a material line element in \( C_o \) and \( ds \) is the same line element in \( C \),
\[ ds^2 - ds_o^2 = 2\gamma_{ij} dx^i dx^j = 2dx^i \cdot \gamma dx^j \]  
\[ ds^2 - ds_o^2 = 2\gamma_{ij} dx^i dx^j = 2dx^i \cdot \gamma dx^j \]  

where the superimposed dot indicates a time-rate-of-change (i.e. \( \dot{\gamma}_{ij} = \partial \gamma_{ij}(\vec{x},t)/\partial t \)).

The vector

\[ \vec{u}(\vec{x},t) = \vec{\chi}(\vec{x},t) - \vec{x} \]  

is the displacement vector of the particle \( \vec{x} \). The deformation gradient \( \vec{F} \) is given in terms of gradients of \( \vec{u} \) by

\[ \vec{F}(\vec{x},t) = \nabla \vec{\chi}(\vec{x},t) = \vec{H}(\vec{x},t) - I = \nabla \otimes \vec{u} \]  

Consequently,

\[ G = I + \vec{H}^\top + \vec{H} + \vec{H}^\top \vec{H} \]  

and

\[ \gamma = \frac{1}{2} (\vec{H} + \vec{H}^\top + \vec{H}^\top \vec{H}) \]  
\[ \dot{\gamma} = \frac{1}{2} (\vec{H} + \dot{\vec{H}} + \vec{H}^\top \vec{H} + \dot{\vec{H}}^\top \vec{H}) \]  

or, in component form,

\[ \gamma_{ij} = \frac{1}{2} (u_{i;i} + u_{j;j} + u_{k;i} u_{k;j}) \]  
\[ \dot{\gamma}_{ij} = \frac{1}{2} (\dot{u}_{i;i} + \dot{u}_{j;j} + u_{k;i} \dot{u}_{k;j} + \dot{u}_{k;i} u_{k;j}) \]  

Here \( \vec{H} \) denotes the (material) velocity gradients and the semi-colon denotes covariant differentiation in \( \Omega \).

Let \( f(\vec{x},t) \) denote a continuous function of \( \vec{x} \) and a piecewise continuous function of \( t \). The function
is called the total history of \( f \) at time \( t \). It is often convenient to represent the total history by the pair \((\bar{f}^t, \bar{f}^t(0))\), where \( \bar{f}^t \), called the past history, is the restriction of \( f(X,t-s) \) to the open interval \( \mathcal{C}(0,\infty) \) and \( \bar{f}^t(0) = \bar{f}(X,t) \) is the current value of \( f \) at time \( t \). The families of tensors and vectors

\[
\begin{align*}
\gamma^t(s) &= \gamma(X,t-s) \\
u^t(s) &= u(X,t-s)
\end{align*}
\]

are called the total strain histories and displacement histories at \( X \).

Similarly, \( \gamma^t(s), u^t(s) \) are the past histories of \( \gamma(X,t) \) and \( u(X,t) \).

**Physical Laws.** In addition to the kinematical relations just described, we may postulate certain physical laws which govern the behavior of all continuous media. Let \( b(X,t), S(X,t) \) denote the body force density (i.e., the body force per unit mass) and \( S(X,t) \) the contact force per unit area of a material surface \( \partial \mathcal{E} \) in \( \mathcal{C} \). Let \( \sigma = \sigma_{ij} \mathcal{E}_{i j} \) denote the Piola-Kirchhoff stress tensor and \( \rho(X,t) \) denote the mass density of \( \mathcal{E} \) in \( \mathcal{C} \).

Then, under sufficient smoothness conditions, the principles of conservation of mass, balance of linear momentum, and angular momentum can be shown to hold at a particle \( X \) if and only if

\[
\begin{align*}
\rho &= \left| \det \mathcal{F}(X,t) \right| \rho = G \sqrt{G} \\
\text{Div} (\sigma) + \rho b &= \rho a \\
\sigma^t &= \sigma
\end{align*}
\]

wherein \( G = \det G \) and \( a \) is the acceleration field. In component form, (2.16) appears as

\( (2.16)_a \) appears as
\[ \sigma^{ij} + \rho b^j = \rho a^j \]  

(2.17a)

in which \( \sigma \) denotes covariant differentiation in \( C \). However, if we introduce the stress tensor \( \tau^{ij} = \sigma^{ij} \sqrt{g} \), and interpret \( \hat{b}(X, t) \) as the body force per unit mass in \( C_o \), then (2.17a) becomes

\[ (\tau^{ij} \hat{x}_{\tau}^i);_t + \rho_o \hat{b}^r = \rho_o \hat{u}^r \]  

(2.17b)

which is often more convenient to use in applications. If \( \hat{n} \) is a unit normal to an element of area on the material surface of \( \partial \) while in \( C_o \), and \( \hat{n} \) is a unit normal to the same area while in \( C \), we also have

\[ \tau^{ij} \hat{n}_i = S^j_0 \]  

\[ \sigma^{ij} \hat{n}_i = S^j \]  

(2.18)

where \( S^j_0 \) and \( S^j \) are contravariant components of the surface traction per unit initial and current surface area, respectively.

In addition to (2.16), we introduce the principle of conservation of energy and the Clausius-Duhem inequality. Globally, these take the forms

\[ \dot{\kappa} + \dot{U} = \Omega + Q \]  

(2.19)

\[ \Gamma > 0 \]

where \( \kappa, U, \Omega, \) and \( Q \) are the kinetic energy, the internal energy, the mechanical power, and the heat energy respectively, and \( \Gamma \) is the total rate of entropy production:

\[ \kappa = \frac{1}{2} \int_U \rho \dot{v} \cdot \dot{v} \]  

\[ U = \int_U e \]  

\[ \Omega = \int_U \rho \hat{b} \cdot \dot{v} + \int_A \hat{S} \cdot \dot{v} \]  

\[ A \]
\[ Q = \int_{U} \rho h \mathbf{v} + \mathbf{q} \cdot \mathbf{\eta} \, dA \]  
\[ \mathbf{\Gamma} = \int_{U} \rho \mathbf{\eta} \mathbf{v} \, dA - \int_{U} \rho \frac{h}{\theta} \, d\mathbf{v} - \int_{A} \frac{q}{\theta} \cdot \mathbf{n} \, dA \]

Here \( \mathbf{v} \) is the velocity field, \( h \) is the heat supply density from internal sources, \( \mathbf{q} \) is the heat flux vector, \( \mathbf{\eta} \) is the internal entropy density and \( \theta \) is the absolute temperature. If (2.16) holds, we may obtain, under sufficient smoothness conditions, the following local forms of (2.19):

\[ \rho \dot{\mathbf{\varepsilon}} = \mathbf{w} + \mathbf{v} \cdot \mathbf{q} + \rho \mathbf{h} \]  
\[ \rho \dot{\mathbf{\eta}} = \mathbf{w} - \mathbf{v} \cdot \mathbf{q} - \rho \mathbf{h} + \frac{1}{\theta} \mathbf{v} \theta \cdot \mathbf{q} \geq 0 \]  

Here \( \mathbf{w} \) is the stress power,

\[ \mathbf{w} = \text{tr} (\sigma^I J^I) = \sigma^{ij} \dot{\gamma}_{ij} \]  

(2.22)

Alternately, we may introduce the free energy density

\[ \psi = \varepsilon - \mathbf{\eta} \theta \]  

(2.23)

and the internal dissipation,

\[ \sigma = \mathbf{w} - \rho (\dot{\psi} - \dot{\mathbf{\eta}}) \]  

(2.24)

and obtain the alternative equations

\[ \rho \dot{\psi} = \mathbf{w} - \mathbf{\eta} \dot{\theta} - \sigma \]  
\[ \rho \dot{\mathbf{\eta}} = \mathbf{v} \cdot \mathbf{q} + \rho \mathbf{h} + \sigma \]  

(2.25)

In subsequent calculations, two alternate global forms are convenient to use. Observing that \( Q = \int_{U} (\mathbf{v} \cdot \mathbf{q} + \rho \mathbf{h}) \, d\mathbf{v} \), we introduce (2.21), into
(2.19), and obtain

$$\dot{\mathbf{x}} + \int_{\mathcal{U}} \mathbf{w} \, d\mathbf{u} = \Omega$$

(2.26)

Likewise, let $T_o$ be a uniform reference temperature of $\mathcal{B}$ while in its reference configuration $C_o$ and denote $T(X,t) = \Theta(X,t) - T_o$. Multiplying (2.25) by $T$ and using some algebraic manipulations, we obtain

$$\int_{\mathcal{U}} (\rho \dot{T} \mathbf{n} + \nabla T \cdot \mathbf{n}) \, d\mathbf{u} = \int_{\mathcal{U}} (\rho \dot{\Theta} + \sigma) \, d\mathbf{u} + \int_{\mathcal{A}} T \mathbf{q} \cdot \mathbf{n} \, dA$$

(2.27)

All of the relations established thusfar are insufficient in number to solve specific problems in the thermomechanics of continua. We must supplement these equations with a collection of constitutive equations that characterize the material under consideration. We shall examine such constitutive laws appropriate for simple materials in Section 4.

3. THERMOMECHANICS OF A FINITE ELEMENT

We now direct our attention to the problem of constructing general finite-element models of continuous media. We follow here the general procedures outlined in previous work [1,4,5,6].

Following the usual finite-element philosophy, we represent $\mathcal{B}$ by a discrete model $\mathcal{B}^*$ consisting of a finite number $E$ of pieces $\mathcal{B}_e$, called finite elements, which are connected continuous together at $G$ preselected material points called nodes. Then $\mathcal{B}^* = \bigcup_{e=1}^{E} \mathcal{B}_e$. Each element may be considered to be disjoint for the purpose or describing its local behavior. Locally, we identify $N_e$ particles in element $\mathcal{B}_e$ and label them consecutively $1,2,\ldots,N_e$; the particle at node $N$ is then $X^N_e$, and if $X^A$ is the
same particle in the connected model $\bar{\mathcal{B}}$, then the decomposition of the model is accomplished by the simple Boolean mapping

$$X^N = \omega^{(e)} \Delta$$

(3.1)

where $\omega^{(e)}_\Delta = 1$ if node $N$ of $\mathcal{B}_e$ is incident on node $\Delta$ of $\bar{\mathcal{B}}$ and $\omega^{(e)}_\Delta = 0$ otherwise. Likewise, for fixed $e$,

$$X^\Delta = \omega^{(e)} N^N$$

(3.2)

$\omega^N_N$ being the transpose of $\omega^{(e)}_\Delta$, and we say that (3.2) establishes the connectivity of the model $\bar{\mathcal{B}}$. Further properties of these mappings can be found elsewhere [1,7]. What is important in regard to our present study is that individual elements can be isolated from $\bar{\mathcal{B}}$ and their behavior can be described independent of their ultimate location or mode of connection in $\bar{\mathcal{B}}$; indeed, once the behavior of a typical element is described, the discrete model of the behavior of any global collection of such elements is obtained by using the transformations (3.2) and (3.2).

With this fundamental property in mind, it suffices to confine our attention to the local behavior of a typical element, and to take for granted that equations governing the global behavior of the collection of elements can be generated by applying the transformations (3.1) and (3.2). Consider, then, a typical finite element $\mathcal{B}_e$, and let $u^{(e)}$ and $T^{(e)}$ denote the restrictions of $u(X,t)$ and $T(X,t)$ to $\mathcal{B}_e$. Let

$$u^N(t) = u^{(e)}(X^N, t)$$

$$T^N(t) = T^{(e)}(X^N, t)$$

(3.3)

denote the values of $u^{(e)}$ and $T^{(e)}$ at node $X^N$ of $\mathcal{B}_e$. Then we approximate $u^{(e)}$ and $T^{(e)}$ locally over $\mathcal{B}_e$ by functions of the form
where \( \psi^N_n(x) \) and \( \phi^N_n(x) \) are local interpolation functions which have the properties

\[
\psi^N_n(x^e) = \delta^N_n, \quad \phi^N_n(x) = 0, \quad x \notin \Omega,
\]

\[
\sum_{N=1}^{N_c} \psi^N_n(x) = 1
\]

with similar properties holding for \( \phi^N_n(x) \). Note that the repeated index \( N \) in (3.4) is summed from 1 to \( N_c \). Instead of (3.4), we could also represent \( u^e \) and \( T^e \) as linear combinations of the values of various derivatives at the nodes; this is discussed elsewhere [1,7], and since the procedure is straightforward, we limit the present discussion to first-order representations.

With the local displacement approximation and the local change in temperature given by (3.4), we can now proceed to compute all of the other kinematic variables and the absolute temperature:

\[
2 \gamma^e_{1j} = u^N_n (f^e_{N_{J1}} + f^e_{N_{1j}} + f^e_{N_{rj}} f^*_{N1r} u^N_r)
\]

\[
f^e_{N_{1j}} = \frac{\partial \psi^N}{\partial x_j} \delta^e_{1j} - \psi^N_{N_{1j}}
\]

\[
2 \gamma^e_{1j} = (f^e_{N_{1j}} + f^e_{N_{1j}}) u^N_n + f^e_{N_{1j}} f^*_{N1r} (u^N_n u^N_r)
\]

\[
\zeta^e_{1j} = 2 \gamma^e_{1j} + \eta^e_{1j}
\]

\[
\zeta^e = T^e + \phi^N_n T^N
\]
etc., where the element identification label (e) is omitted in some places for simplicity and the dependence of \( \psi_{N,1} \), \( \varphi_N \) on \( x \) and \( u^N \), \( T^N \) on \( t \) is understood. In (3.6), \( \Gamma^*_{j_1} \) is the Christoffel symbol of the second kind corresponding to the undeformed material coordinates.

The stress power developed in element \( \Theta_e \) is

\[
\nu_{(e)} = \text{tr}[\sigma \mathbf{y}'] = (\nabla \psi_N)^T \sigma (I + (\nabla \varphi_N \otimes u^N)) \cdot \dot{u}^N
\]  

(3.7a)

or, if the \( X^i \) are cartesian in \( C_o \),

\[
\nu_{(e)} = \sigma^{1j} \psi_{N,1} (\delta_{jk} + \psi_{N,i} u^N_k) \dot{U}^N_j
\]  

(3.7b)

The velocity field is

\[
\nu_{(e)} = \dot{u}_{(e)} = \psi_N(x) \dot{u}^N
\]  

(3.8)

Thus, introducing (3.4), (3.6), (3.7), and (3.8) into (2.26) and (2.27), we obtain

\[
\left\{ m_{NM} \ddot{u}^N + \int_{\Theta_e} \sigma^{ij} \frac{\partial y_{ij}}{\partial u^N} \, d\mathbf{u} - p_N \right\} \cdot \dot{u}^N = 0
\]  

(3.9)

and

\[
\left\{ \int_{\Theta_e} [\rho(T_o + \varphi_N T^N) \dot{\varphi}_N + q \cdot \nabla \varphi_N] \, d\mathbf{u} - q_N - \sigma_N \right\} T^N = 0
\]  

(3.10)

where \( m_{NM} \) is the consistent mass matrix for the element, \( p_N \) is the generalized force at node \( N \), \( q_N \) is the generalized normal heat flux at node \( N \), \( \sigma_N \) is the generalized dissipation at node \( N \), and \( \Theta_e \) is the volume of \( \Theta_e \).
\[
\begin{align*}
\mathbf{m}_{NH} &= \int_{V_e} \rho \psi_N \psi_N d\mathbf{u}, \\
\mathbf{p}_N &= \int_{V_e} \rho b \psi_N d\mathbf{u} + \int_{A_e} S \psi_N dA \\
q_N &= \int_{V_e} \rho h \phi_N d\mathbf{u} + \int_{A_e} q \cdot \nu \phi_N dA \\
\mathbf{\sigma}_N &= \int_{V_e} \sigma \phi_N dA \\
\end{align*}
\] (3.11)

If \( \hat{N} \) and \( T^N \) are each linearly independent sets, (3.9) and (3.10) must hold for arbitrary nodal velocities and temperatures. Consequently, the quantities in braces in these equations must vanish. This leads to the general equations of motion of a finite element

\[
\mathbf{m}_{NH} + \int_{V_e} \mathbf{\sigma}_N \frac{\partial \mathbf{u}_N}{\partial t} d\mathbf{u} = \mathbf{p}_N(t) \\
\] (3.12)

and the general equations of heat conduction of a finite element

\[
\int_{V_e} [\rho(T_0 + \phi_n T^N) \dot{\phi}_N + q \cdot \nabla \phi_N] d\mathbf{u} = q_N + \mathbf{\sigma}_N \\
\] (3.13)

To apply these models to specific problems, we must furnish appropriate constitutive equations for \( \mathbf{\sigma}_N \), \( \mathbf{\eta}_N \), and \( q \) (and \( \mathbf{\sigma}_N \)).

We remark that (3.12) is the finite-element analogue of Cauchy's first law of motion (2.16)\(_a\), while (3.13) is the analogue of (2.25)\(_a\). Equation (3.12) insures that the principle of balance of linear momentum is satisfied in an average sense over \( \mathbf{\Theta}_e \), and (3.13) insures that energy is conserved in an average sense over \( \mathbf{\Theta}_e \). Notice that (3.12) and (3.13) can also be obtained by multiplying (2.16)\(_a\) by \( \psi_N \) and (2.25)\(_a\) by \( \phi_N \).
integrating over the volume of the element, and using the Green-Gauss theorem. Then the procedure leading to (3.12) and (3.13) is interpreted as Galerkin's method.

It is also informative to consider the case in which all quantities are referred to initial material volumes $u_0$ and areas $A_0$ in $C_0$ and $g_{ij} = \delta_{ij}$. Then $u_0$, $A_0$, $\sigma^{ij}$, $b$, $S$, and $\rho$ in (3.12) and (3.13) are replaced, respectively, by $u_0$, $A_0$, $t^{ij}$, $b_0$, $S_0$, and $\rho_0$; (3.12) then becomes

$$m_N \ddot{u}_N^k + \int \left[ t^{ij} \frac{\partial \psi_{ij}}{\partial u} \right] dV = p_N$$

(3.14)

and

$$t^{ij} \frac{\partial \psi_{ij}}{\partial u} = t^{ij} \psi_{ij} \left( \delta_{jk} + \psi_{kj} u_N^k \right)$$

(3.15)

4. SIMPLE MATERIALS

A detailed discussion of constitutive theory is to be covered in another lecture in this series; summary accounts are also available in a number of books and monographs [1, 8, 9]. Rather than to consider a variety of classes of materials, we shall concentrate on one which is sufficiently broad to include most of the classical theories as special cases: the theory of thermomechanically simple materials [10]. A material is said to be simple if its response at particle $X$ at time $t$ is determined by the history of the local deformation at $X$ and the history of the temperature at $X$ at time $t$. Mathematically, a simple material is described by a collection of four constitutive equations which define the free energy, stress, entropy, and heat flux as functionals of the total history
of deformation and temperature and generally the current value of the
temperature gradient \( g(t) = \vartheta(t) \). If we use the strain tensor \( \gamma(X,t) \)
as a measure of the deformation, and if we decompose the total histories
\( \gamma^t(s), \theta^t(s) \), into past histories \( \gamma^t(s), \theta^t(s) \) and current values \( \gamma^t(0), \theta^t(0) \), then a simple material is described by the four functionals,

\[
\begin{align*}
\psi &= \psi \left[ \gamma^t_r, \theta^t_r; \gamma, \theta, g \right] \\
\sigma &= \sigma \left[ \gamma^t_r, \theta^t_r; \gamma, \theta, g \right] \\
\eta &= \eta \left[ \gamma^t_r, \theta^t_r; \gamma, \theta, g \right] \\
q &= q \left[ \gamma^t_r, \theta^t_r; \gamma, \theta, g \right]
\end{align*}
\]

Here the dependence of \( \psi, \sigma, \eta, q, \gamma, \theta, \) and \( g \) on \( X \) and \( t \) is understood.

The functionals in (4.1) must obey the principle of objectivity and they
must be consistent with the Clausius-Duhem inequality (2.21). We see that
this inequality can be written in the alternate form

\[
w - \rho \dot{\psi} - \rho \eta \dot{\theta} + \frac{1}{\theta} g \cdot q \geq 0 \tag{4.2}
\]

Thus, to test if (4.1) satisfies (4.2), we must compute the time-rate-of
change of \( \dot{\psi} \). Since \( \dot{\psi} \) is a functional of \( \gamma, \theta \), we have

\[
\begin{align*}
\dot{\psi} &= \delta_{\gamma} \psi \left[ -\gamma^t_r \right] + \delta_{\theta} \psi \left[ -\theta^t_r \right] + D_{\gamma} \psi \left[ -\gamma \right] \cdot \dot{\gamma} + D_{\theta} \psi \left[ -\theta \right] \cdot \dot{\theta} \\
&\quad + \partial_{\gamma} \psi \left[ -\gamma \right] \cdot \dot{\gamma} \\
&\quad + \partial_{\theta} \psi \left[ -\theta \right] \cdot \dot{\theta}
\end{align*}
\]

where, for simplicity in notation, we have used \([-\gamma^t_r, \theta^t_r; \gamma, \theta]\) and
the vertical stroke indicates that the functional is linear in the argument
following the stroke. The operators $\delta_\gamma$, $\delta_\theta$, $D_\gamma$, $D_\theta$, $\delta_\phi$ are Frechet and partial differential operators defined by

$$
\delta_\gamma \psi [-|h|] = \lim_{\alpha \to 0} \frac{\partial}{\partial \alpha} \{ \psi [\gamma_t^t, \theta_t^t, \gamma, \theta, g] \}
$$

$$
\delta_\theta \psi [-|\theta_t|] = \lim_{\alpha \to 0} \frac{\partial}{\partial \alpha} \{ \psi [\gamma_t^t, \theta_t^t, \gamma + \alpha \theta_t, \theta, g] \}
$$

$$
D_\gamma \psi [-] \cdot h = \lim_{\alpha \to 0} \frac{\partial}{\partial \alpha} \{ \psi [\gamma_t^t, \theta_t^t; \gamma, \theta, g] \}
$$

$$
D_\theta \psi [-] \cdot \beta = \lim_{\alpha \to 0} \frac{\partial}{\partial \alpha} \{ \psi [\gamma_t^t, \theta_t^t; \gamma, \theta + \alpha \beta_t, g] \}
$$

$$
\delta_\phi \psi [-] \cdot \xi = \lim_{\alpha \to 0} \frac{\partial}{\partial \alpha} \{ \psi [\gamma_t^t, \theta_t^t; \gamma, \theta, g + \alpha \xi] \}
$$

(4.4)

Introducing (4.1) and (4.3) into (4.2) and arguing that the result must hold for arbitrary rates $\gamma_t$, $\theta_t$, $\xi_t$, we find that

$$
\delta_{\xi} \psi [-] = 0
$$

$$
\xi [-] = \rho D_\gamma \psi [-]
$$

$$
N_s [-] = -D_\theta \psi [-] \quad \sigma = -\rho [\delta_{\gamma} \psi [-|\gamma_t^t|] + \delta_{\theta} \psi [-|\theta_t^t|]}
$$

(4.5)

That is, the functional $\psi [-]_s$ is independent of $g$, and the constitutive functionals $\xi [-]_s$ and $N_s [-]$ for stress and entropy are determined by the free energy functional. Moreover, by Frechet differentiation, we can also determine the internal dissipation $\sigma$ from $\psi [-]_s$.

Special Forms of the Constitutive Equations. In trying to obtain specific forms of the constitutive functionals for simple materials, it is natural
to first consider possible expansions of the free energy functional.

Let \( \Omega^t(s) \), \( \Omega^t(s) \) be the zero histories of \( y \) and \( q \), and suppose that \( \psi \bigg|_{s=0}^\infty \) is Frechet differentiable to admit a Taylor expansion about \( \Omega^t(s) \), \( \Omega^t(s) \) of the form

\[
\begin{align*}
\psi \left[ \frac{\Omega^t}{t} + y^t, 0^t + \theta^t; 0 + y, 0 + \theta \right] &= \psi \left[ \tilde{\Omega}^t \right] + \delta_{s=0} \psi \left[ \tilde{\Omega}^t \right] \\
&+ \delta_{s=0} \psi \left[ \tilde{\Omega}^t \right] + D_{s=0} \psi \left[ \tilde{\Omega}^t \right] y + D_{s=0} \psi \left[ \tilde{\Omega}^t \right] \theta \\
&+ \delta_{s=0} \psi \left[ \tilde{\Omega}^t \right] y + D_{s=0} \psi \left[ \tilde{\Omega}^t \right] \theta \\
&+ \delta_{s=0} \psi \left[ \tilde{\Omega}^t \right] y + \delta_{s=0} \psi \left[ \tilde{\Omega}^t \right] \theta \\
&+ \delta_{s=0} \psi \left[ \tilde{\Omega}^t \right] y + \delta_{s=0} \psi \left[ \tilde{\Omega}^t \right] \theta \\
&+ \delta_{s=0} \psi \left[ \tilde{\Omega}^t \right] y + \delta_{s=0} \psi \left[ \tilde{\Omega}^t \right] \theta + ... \\
(4.6)
\end{align*}
\]

where \( \tilde{\Omega}^t = (\Omega^t, 0^t; 0, 0) \). Then, by truncating this series at various points, we obtain constitutive functions for various simple materials.

For example, Achenbach, Vogel, and Herrmann [11] proposed the free energy functional obtained from (4.6) by retaining cubic terms in \( y, \theta \), and their histories:

\[
\begin{align*}
\tilde{\psi} &= A_0 + A_1 y_1 T^2 + A_3 y_3 T + A_4 y_4 T + A_5 y_5 T^3 + B_1 T + B_2 T^2 + B_3 T^3 \\
(4.7)
\end{align*}
\]

Here \( \tilde{\psi} \) is the free-energy functional per unit undeformed volume, \( T = \theta - T_0 \) is the change in temperature from a uniform constant reference temperature \( T_0 \), and
\[ A_0 = \sum_{s=0}^{\infty} \frac{1}{2} \delta_r^2 \psi \left[ -|T_r^t, T_r^i| \right] + \sum_{s=0}^{\infty} \delta_r \phi \left[ -|T_r^t, T_r^i| \right] \]
\[ + \frac{1}{2} \delta_r^2 \phi \left[ -|T_r^t, T_r^i| \right] + \frac{1}{6} \delta_r^3 \phi \left[ -|T_r^t, T_r^i, T_r^i| \right] \]
\[ + \frac{1}{2} \delta_r^2 \phi \left[ -|T_r^t, T_r^i| \right] + \frac{1}{2} \delta_r \phi \left[ -|T_r^t, T_r^i, T_r^i| \right] \]
\[ (4.8a) \]

\[ A_1 = \sum_{s=0}^{\infty} \frac{1}{2} \delta_r^2 \psi \left[ -|T_r^t, T_r^i| \right] + \sum_{s=0}^{\infty} \delta_r \phi \left[ -|T_r^t, T_r^i| \right] \]
\[ + \delta_r \phi \left[ -|T_r^t, T_r^i| \right] + \frac{1}{2} \delta_r \phi \left[ -|T_r^t, T_r^i| \right] \]
\[ (4.8b) \]

\[ A_2 = \frac{1}{2} \sum_{s=0}^{\infty} \delta_r^2 \psi \left[ -|T_r^t, T_r^i| \right] + \sum_{s=0}^{\infty} \delta_r \phi \left[ -|T_r^t, T_r^i| \right] \]
\[ + \delta_r \phi \left[ -|T_r^t, T_r^i| \right] + \frac{1}{2} \delta_r \phi \left[ -|T_r^t, T_r^i| \right] \]
\[ (4.8c) \]

\[ A_3 = \frac{1}{2} \sum_{s=0}^{\infty} \delta_r^2 \psi \left[ -|T_r^t, T_r^i| \right] + \sum_{s=0}^{\infty} \delta_r \phi \left[ -|T_r^t, T_r^i| \right] \]
\[ + \delta_r \phi \left[ -|T_r^t, T_r^i| \right] + \frac{1}{2} \delta_r \phi \left[ -|T_r^t, T_r^i| \right] \]
\[ (4.8d, e) \]

\[ A_4 = \frac{1}{2} \sum_{s=0}^{\infty} \delta_r^2 \psi \left[ -|T_r^t, T_r^i| \right] + \sum_{s=0}^{\infty} \delta_r \phi \left[ -|T_r^t, T_r^i| \right] \]
\[ + \delta_r \phi \left[ -|T_r^t, T_r^i| \right] + \frac{1}{2} \delta_r \phi \left[ -|T_r^t, T_r^i| \right] \]
\[ (4.8f) \]

\[ B_1 = \sum_{s=0}^{\infty} \frac{1}{2} \delta_r^2 \psi \left[ -|T_r^t, T_r^i| \right] + \sum_{s=0}^{\infty} \delta_r \phi \left[ -|T_r^t, T_r^i| \right] \]
\[ + \delta_r \phi \left[ -|T_r^t, T_r^i| \right] + \frac{1}{2} \delta_r \phi \left[ -|T_r^t, T_r^i| \right] \]
\[ (4.8g) \]

\[ B_2 = \frac{1}{2} \sum_{s=0}^{\infty} \delta_r^2 \psi \left[ -|T_r^t, T_r^i| \right] + \sum_{s=0}^{\infty} \delta_r \phi \left[ -|T_r^t, T_r^i| \right] \]
\[ + \delta_r \phi \left[ -|T_r^t, T_r^i| \right] + \frac{1}{2} \delta_r \phi \left[ -|T_r^t, T_r^i| \right] \]
\[ (4.8h) \]

\[ B_3 = \frac{1}{6} \sum_{s=0}^{\infty} \delta_r^2 \phi \left[ -|T_r^t, T_r^i| \right] \]
\[ (4.8i) \]

wherein \( \psi \left[ -|..| \right] \equiv \psi \left[ 0_r^t, 0_r^i; 0_r, 0_r |..| \right] \), \( \delta_r = \delta_{0^2r} \equiv \delta_0 \), and \( D_r = D_{0^2r} \equiv D_0 \).

More specifically, we may introduce the strain- and temperature-dependent integral coefficients.
Here $R_{11}, R_{12}, \ldots, G^{ijsn}, M^{ij}, \ldots, L^{ijsn}, \ldots, F_1$ are material kernels and * denotes a generalized convolution operator which, with an appropriate change-of-variables, indicates integrals of the form

$$R_{11}^{*y*Y'} = \int_{0}^{t} \int_{0}^{t} R_{11}^{ijsn}(t-s_1, t-s_2) \gamma_{ij}(s_1) \frac{dy_{sn}(s_2)}{ds_2} \, ds_1 \, ds_2 \quad (4.10a)$$

$$R_{12}^{*y*Y'*T'} = \int_{0}^{t} \int_{0}^{t} \int_{0}^{t} R_{12}^{ijsn}(t-s_1, t-s_2, t-s_3) \gamma_{ij}(s_1) \frac{dy_{sn}(s_2)}{ds_2} \frac{dT(s_3)}{ds_3} \, ds_1 \, ds_2 \, ds_3 \quad (4.10b)$$

$$M^{ij} = \int_{0}^{t} \frac{dM^{ij}(t-s)}{d(t-s)} \, T(s) \, ds \quad (4.10c)$$

etc. The lower limit of integration is assumed to be the time at which the deformation is initiated.
For the constitutive equation for heat flux, we might use the law proposed by Christensen and Naghdi [12]

\[ Q^1 = \int_0^{\infty} \kappa^1(t-s) \frac{\partial T_{1,1}(s)}{\partial s} \, ds \] (4.11)

where \( \kappa^1(t-s) \) is a thermal conductivity kernel. Alternately, we may use the generalized Fourier law

\[ Q^1 = \int_{s=0}^{\infty} \kappa^1[y_r^t, T_r^t, y_r, T] \, ds \] (4.12)

where \( \kappa^1[-] \) is a functional of the indicated arguments. If \( \kappa^1[-] \) is replaced by a constant \( \kappa^1 \), (4.12) reduces to the classical Fourier law of heat conduction.

**Thermorheologically Simple Materials.** Experimental evidence obtained from tests on a large class of viscoelastic materials has led to the identification of an important subclass of materials with memory, commonly referred to as thermorheologically simple materials. This classification arose from the observation that, among certain amorphous high polymers which approximately obey established linear and nonlinear viscoelastic laws at uniform temperature, are a group which exhibit a simple property with a change of temperature: namely, a translational shift of various material properties when plotted against the logarithm of time at different uniform temperatures. The shift phenomenon is the basic characteristic of all thermorheologically simple materials and makes it possible to establish an equivalence relation between temperature and \( \ln t \). Now in the case of a thermorheologically simple material, it is possible to write the relaxation moduli at uniform temperature \( \Theta \), e.g., \( J(t-s) \), as a function of \( \ln t \), denoted \( E_\Theta(\ln t) \). Then the shift property is apparent
if
\[ E_{\theta}(\ln t) = E_{\theta_0}(\ln t + f(\theta)) \]

where \( f(\theta) \) is a shift function relative to \( \theta_0 \) such that \( df/dt > 0 \) and \( f(\theta_0) = 0 \). With this property, the relaxation modulus curve will shift toward shorter times with an increase in \( \theta \). By introducing a shift factor,

\[ b(\theta) = \exp f(\theta) \] (4.13)

Then
\[ J_{\theta}(t) = E_{\theta}(\ln t) = E_{\theta_0}[g_n(tb(\theta))] = J_{\theta_0}(\xi) \]

where \( \xi \) is a reduced time given by
\[ \xi = tb(\theta) \] (4.14)

If it is possible to invert (4.14) so as to obtain \( t = g(\xi) \), then a constitutive equation for stress can be obtained from the form

\[ \hat{\sigma}_{11} = \int_{0}^{\xi} J(\xi-\xi') \frac{\partial \gamma^{'}_{11}}{\partial \xi'} \, d\xi' + \frac{1}{3} \delta_{11} \int_{0}^{\xi} K(\xi-\xi') \frac{\partial \gamma}{\partial \xi} \, d\xi' \] (4.15)

wherein \( \gamma'_{11} \) and \( \gamma \) are the deviatoric and dilatational components of \( \gamma_{11} \) and \( J(t-s) \) and \( K(t-s) \) are relaxation moduli, and we have used the transformations

\[ \hat{\sigma}_{11}(X, \xi) = \sigma_{11}(X, g(\xi)) \quad \hat{\gamma}_{11}(X, \xi) = \gamma_{11}(X, g(\xi)) \]

That is, when transformed to a reduced time, the constitutive equations for a thermorheologically simple material assume the same form as the constitutive law for the isothermal case.

While the above development applies to the special case of linear response of a body at various uniform temperatures, the basic ideas can
be extended to finite deformations of bodies subjected to transient, non-homogeneous temperatures. For example, if it is assumed that the relaxation properties at a particle \( \mathbf{x} \) depend only on the current temperature \( \theta(\mathbf{x}, t) \) at \( \mathbf{x} \), it may be postulated that an increment \( \Delta \xi \) in reduced time is related to an increment \( \Delta t \) in real time according to \( \Delta \xi = b[\theta(\mathbf{x}, t)] \Delta t \). Then, instead of (4.14), we have

\[
\xi = \int_0^t b[\theta(\mathbf{x}, \xi')] d\xi'
\]  

(4.16)

Clearly, (4.16) reduces to (4.15) in the case of uniform, constant temperatures.

As a representative example of a constitutive functional for the free energy of a thermorheologically simple material, consider the functional proposed by Cost [13]:

\[
\rho \dot{\varphi} = \varphi_0 + \int_0^\xi D_{ij}(\xi-\xi') \frac{\partial \dot{\gamma}_{ij}}{\partial \xi'} d\xi' - \int_0^\xi f(\xi-\xi') \frac{\partial \dot{\theta}}{\partial \xi'} d\xi'
\]

\[
+ \frac{1}{6} \delta_{ij} \delta_{mn} \int_0^\xi \int_0^\xi \left[ 3K(\xi-\xi', \xi-\xi'') - 2\zeta(\xi-\xi', \xi-\xi'') \right] \frac{\partial \dot{\gamma}_{ij}}{\partial \xi'} \frac{\partial \dot{\gamma}_{mn}}{\partial \xi'} d\xi' d\xi''
\]

\[
+ \frac{1}{4} \left( \delta_{ij} \delta_{mn} + \delta_{ij} \delta_{mn} \right) \int_0^\xi \int_0^\xi 2\zeta(\xi-\xi', \xi-\xi'') \frac{\partial \dot{\gamma}_{ij}}{\partial \xi'} \frac{\partial \dot{\gamma}_{mn}}{\partial \xi'} d\xi' d\xi''
\]

\[
- \delta_{ij} \int_0^\xi \int_0^\xi 3\lambda K(\xi-\xi', \xi-\xi'') \frac{\partial \dot{\gamma}_{ij}}{\partial \xi'} \frac{\partial \dot{\theta}}{\partial \xi'} d\xi' d\xi''
\]

\[
- \frac{1}{2} \int_0^\xi \int_0^\xi m(\xi-\xi', \xi-\xi'') \frac{\partial \dot{\theta}}{\partial \xi'} \frac{\partial \dot{\theta}}{\partial \xi'} d\xi' d\xi''
\]  

(4.17)
Here all quantities have been transformed to reduced time; i.e., $\hat{\gamma}_{1j}(x, \xi) = \gamma_{1j}(x, g(\xi))$, etc. ($t = g(\xi)$). Also $\xi = f(t)$, as given by (3.8), $\xi' = f(t')$ and $\xi'' = f(t'')$ are dummy variables in reduced time, $\varphi_0$ is a constant, $D_{1j}(\cdot)$, $K(\cdot, \cdot), G(\cdot, \cdot)$, and $m(\cdot, \cdot)$ are material kernels, and $\alpha$ is the coefficient of thermal expansion.

Integrating (4.19) by parts and introducing the resulting equation into (4.5) yields the following constitutive equations for stress, entropy, and internal dissipation:

$$\sigma_{1j} = \int_0^\xi 2G(\xi-\xi') \frac{\partial \hat{\gamma}_{1j}}{\partial \xi'} \, d\xi' + \frac{1}{3} \delta_{1j} \int_0^\xi [3K(\xi-\xi') - 2G(\xi-\xi')] \frac{\partial \hat{\theta}}{\partial \xi'} \, d\xi' - \frac{1}{2} \int_0^\xi 3K(\xi-\xi') \frac{\partial \hat{\theta}}{\partial \xi'} \, d\xi'$$

(4.18)

$$\rho \hat{\eta} = f(0) + 3\alpha K(0) + m(0) \hat{\theta}(\xi) \int_0^\xi \frac{\partial}{\partial \xi}, [3\alpha K(\xi')] \]$$

(4.19)

$$\sigma = -\frac{1}{6} \int_0^\xi \int_0^\xi \left[ 3K(\xi-\xi', \xi-\xi'') - 2G(\xi-\xi', \xi-\xi'') \right] \frac{\partial \hat{\gamma}_{1j}}{\partial \xi'} \frac{\partial \hat{\gamma}_{1j}}{\partial \xi''} \, d\xi' \, d\xi''$$

$$- \frac{1}{2} \int_0^\xi \int_0^\xi \left[ 2G(\xi-\xi', \xi-\xi'') \right] \frac{\partial \hat{\gamma}_{1j}}{\partial \xi'} \frac{\partial \hat{\gamma}_{1j}}{\partial \xi''} \, d\xi' \, d\xi''$$

$$+ \int_0^\xi \int_0^\xi \left[ \alpha 3K(\xi-\xi', \xi-\xi'') \right] \frac{\partial \hat{\theta}}{\partial \xi'} \frac{\partial \hat{\theta}}{\partial \xi''} \, d\xi' \, d\xi''$$
In these equations $3K(\xi') = 3K(0,\xi')$, $m(\xi') = m(0,\xi') = 3\alpha^2 \cdot 3K(0,\xi')$.

The material kernels $3K(\cdot)$ and $2G(\cdot)$ are relaxation bulk and shear moduli.

**Thermoelastic Materials.** If the free energy depends only upon the current values of $\gamma$ and $\theta$, then

$$\psi = \psi(\gamma, \theta)$$

and (4.21) and (4.5) reduce to

$$\sigma = \rho D \psi(\gamma, \theta) \quad \text{or} \quad \sigma_{ij} = \rho \frac{\partial \psi(\gamma, \theta)}{\partial \gamma_{ij}}$$

$$\eta = -D_\theta \psi(\gamma, \theta) \quad \text{or} \quad \eta = - \frac{\partial \psi(\gamma, \theta)}{\partial \gamma_{ij}}$$

Materials obeying constitutive laws such as these are called thermoelastic materials.

5. **FINITE ELEMENT MODELS OF SIMPLE MATERIALS**

In view of (3.6), the components of the history of the strain tensor, the temperature histories, and the temperature gradient for a typical finite element $B$ are

$$2\gamma_{i,j}(s) = g_{i,j} \cdot u^t_{i,j}(s) + g_{j,i} \cdot u^t_{j,i}(s) + u^t_{i,i}(s) \cdot w^t_{i,j}(s)$$

$$u^t_{i,j}(s) = g^r(\psi_{i,j}^r, \delta^i_{j} - \psi_{i,j}^r) u^w(s)$$

$$\theta^t(s) = T_0 + \varphi_N T^{N(t)}(s)$$

$$g = \nu \theta = \varphi_N T^{N(t)} g^t$$
Here $\Gamma_{1^r_i}$ are the Christoffel symbols of the second kind for the system $X^i$ while in $C_o$, $u_{1^r_i}^{N(s)}(s)$ and $T_{1^r_i}^{N(u)}(s)$ are the total histories of the nodal displacement components and temperatures. Thus, the arguments of the functionals (4.5) can be written in terms of the histories $u_{1^r_i}^{N(s)}(s)$ and $T_{1^r_i}^{N(u)}(s)$:

$$
\sigma = \rho D \int_{s=0}^{\infty} \psi [-] = \int_{s=0}^{\infty} \frac{1}{s} [u_{1^r_i}^{N(s)}, T_{1^r_i}^{N(u)}; u_{1^r_i}^{N(s)}, T_{1^r_i}^{N(u)}] \\
\eta = -P_0 \int_{s=0}^{\infty} \psi [-] = \int_{s=0}^{\infty} [u_{1^r_i}^{N(s)}, T_{1^r_i}^{N(u)}; u_{1^r_i}^{N(s)}, T_{1^r_i}^{N(u)}] \\
\phi = \frac{Q}{s} \int_{s=0}^{\infty} [-] = \int_{s=0}^{\infty} [u_{1^r_i}^{N(s)}, T_{1^r_i}^{N(u)}; u_{1^r_i}^{N(s)}, T_{1^r_i}^{N(u)}] 
$$

Likewise, the internal dissipation for the element is of the form

$$
\sigma = \int_{s=0}^{\infty} D [u_{1^r_i}^{N(s)}, T_{1^r_i}^{N(u)}; u_{1^r_i}^{N(s)}, T_{1^r_i}^{N(u)}] 
$$

Thus, the equations of motion and heat conduction for a finite element of a thermomechanically simple material are

$$
m_{1^r_i} \ddot{u}_{1^r_i} + \int_{s=0}^{\infty} \sigma_{1^r_i}^{ij} [-] \frac{\partial \psi_{1^r_i}^{ij}}{\partial u_{1^r_i}^{N}} du = P_{1^r_i} \\
\int_{u_{1^r_i}} \left[ \rho (T_0 + \varphi u_T) \frac{d}{dt} q_1 [-] + \varphi_1 \frac{\partial q_1}{\partial s} [-] \right] du \\
= q_N + \int \varphi_{1^r_i} D [-] T_{1^r_i}^{N(u)} du 
$$

where, for simplicity, we denote by $[-]$ the arguments listed in (5.2).

In order to obtain specific finite-element models, it now becomes a simple matter of introducing the appropriate constitutive equations into (5.2) and (5.3) to determine the forms of the integrals in (5.4) and (5.5).
Thermorheologically Simple Elements. The reduction of (5.4) and (5.5) for the case of thermorheologically simple materials requires special considerations. To outline briefly the procedure, we consider the case in which $g_{ij} = \delta_{ij}$, that is, the material coordinates $X^1$ are assumed to be initially cartesian. The following discussion is essentially a reproduction of that given by Oden and Armstrong [2].

Consider a class of thermorheologically simple materials for which it is possible to transform all field quantities to corresponding quantities in reduced time; i.e., if $f(X,t)$ is a function of real time $t$, a function $g(\xi) = t$ exists such that we can write

$$\dot{f}(X,\xi) = f(X,g(\xi))$$

Then it is easily shown that

$$\dot{f}(X,t) = \dot{\dot{f}}(X,\xi)\beta[\beta(\xi,\xi)]$$

where $\dot{f} = df/d\xi$ and $\beta[\ ]$ is the shift factor at $X$ at reduced time $\xi$ corresponding to $t$.

If $2\gamma_{ij} = u_{ij} + u_{ji}$, $\rho_0 \dot{\theta} \approx \rho T_0 \dot{\theta}$, and the constitutive equations are of the form (4.18), (4.19), and (4.20), then it can be shown [1,2,14] that the equations of motion and heat conduction for an element are of the form

$$\begin{align*}
\frac{d}{d\xi} \frac{\partial U^m}{\partial X_k} + u^m_n \frac{\partial U^m}{\partial X_k} + a^i_{NM} & \int_0^\xi 2G(\xi-\xi') \frac{\partial u^N}{\partial \xi'} d\xi' + \int_0^\xi 3K(\xi-\xi') [3K(\xi-\xi') - 2G(\xi-\xi')] \frac{\partial T}{\partial \xi'} d\xi' = \bar{p}_N \\
- 2G(\xi-\xi') \frac{\partial u^N}{\partial \xi'} d\xi' & = \int_0^\xi 3K(\xi-\xi') \frac{\partial T}{\partial \xi'} d\xi' = \bar{p}_N 
\end{align*}$$

(5.6)
\begin{align}
\mathcal{H}^{(1)}_{MN} &+ \mathcal{H}^{(2)}_{MN} = \mathcal{H}^{(3)}_{MN} + \int_0^\xi \! \frac{\partial}{\partial \xi} \left[ m(\xi-\xi') \right] \frac{\partial \mathcal{H}^{(1)}_{MN} }{\partial \xi} \, d\xi' + \mathcal{H}^{(4)}_{MN} \int_0^\xi \! \frac{\partial}{\partial \xi} \left[ 3K(\xi-\xi') \right] \frac{\partial \mathcal{H}^{(1)}_{MN} }{\partial \xi} \, d\xi' \\
&+ k_{MN} \int_0^\xi \! \left( \kappa(\xi-\xi') \frac{\partial \mathcal{H}^{(1)}_{MN} }{\partial \xi} \right) \, d\xi' + \int_0^\xi \! \frac{\partial}{\partial \xi} \left[ 3K(\xi-\xi', \xi-\xi'') - 2G(\xi-\xi', \xi-\xi'') \right] \frac{\partial \mathcal{H}^{(1)}_{MN} }{\partial \xi} \, d\xi' \\
&\times \frac{\partial \mathcal{H}_{i\ell}^{(1)}}{\partial \xi} \frac{\partial \mathcal{H}_{j\ell}^{(1)}}{\partial \xi} \, d\xi' \, d\xi'' + \int_0^\xi \! \left[ 2G(\xi-\xi', \xi-\xi'') \right] \frac{\partial \mathcal{H}^{(1)}_{MN} }{\partial \xi} \frac{\partial \mathcal{H}_{i\ell}^{(1)}}{\partial \xi} \frac{\partial \mathcal{H}_{j\ell}^{(1)}}{\partial \xi} \, d\xi' \, d\xi'' \\
&- \int_0^\xi \! \left[ 3K(\xi-\xi', \xi-\xi'') \right] \frac{\partial \mathcal{H}^{(1)}_{MN} }{\partial \xi} \frac{\partial \mathcal{H}_{i\ell}^{(1)}}{\partial \xi} \frac{\partial \mathcal{H}_{j\ell}^{(1)}}{\partial \xi} \, d\xi' \, d\xi'' \\
&- \int_0^\xi \! \left[ m(\xi-\xi', \xi-\xi'') \right] \frac{\partial \mathcal{H}^{(1)}_{MN} }{\partial \xi} \frac{\partial \mathcal{H}_{i\ell}^{(1)}}{\partial \xi} \frac{\partial \mathcal{H}_{j\ell}^{(1)}}{\partial \xi} \, d\xi' \, d\xi'' = \mathcal{Q}_M 
\end{align}

where

\begin{align}
m^{(1)}_{NM} &= \int_{\Omega_0} \rho b^2 \psi N \psi_d \, du \\
m^{(2)}_{NM} &= \int_{\Omega_0} \rho b \psi N \psi_d \, du \\
a^{(1)}_{NM} &= \frac{1}{2} \int_{\Omega_0} \left( b \left[ \psi M, \psi N \right] + \psi M, \psi N \right) \, du \\
b^{(1)}_{NM} &= \frac{1}{3} \int_{\Omega_0} \left( b \psi M, \psi N \right) \, du \\
c^{(1)}_{NM} &= \alpha \int_{\Omega_0} \psi N \, du
\end{align}

and

\begin{align}
h^{(1)}_{MN} &= T_0 m(0) \int_{\Omega_0} \psi M \psi N \, du \\
k_{MN} &= \int_{\Omega_0} \psi M, \psi N, \, du \\
h^{(2)}_{MN} &= 2 \alpha T_0 K(0) \int_{\Omega_0} \psi M \psi N, \, du \\
h^{(3)}_{MN} &= 3 \alpha T_0 K(0) \int_{\Omega_0} \psi M \psi N, \, du \\
\mathcal{H}_{i\ell}^{(1)} &= \frac{1}{6} \int_{\Omega_0} \psi M \psi N, \psi L, \, du
\end{align}
Also
\[
H_{MN}^{(a)} = \frac{1}{m(0)} H_{MN}^{(1)}, \quad H_{MN}^{(b)} = \alpha T_o k_{MN}, \quad d_{MN}^{(a)} = \frac{2}{3} (d_{MN}^{(a)} n d_{MN}^{(a)} + d_{MN}^{(a)} i)
\]
\[
d_{MN}^{(a)} = \frac{1}{2} \int \varphi_1 \varphi_2 \varphi_3 d\nu, \quad d_{MN}^{(a)} = \int \alpha \varphi_1 \varphi_2 \varphi_3 d\nu \quad (5.8c)
\]

To further specialize (5.6) and (5.7), Oden and Armstrong [2] assumed that each material kernel was representable in the form of a decaying Prony series; e.g.
\[
K(\xi) = \sum_{i=1}^{n} K_i \exp(-\xi/\nu_i)
\]
where \(K_i\) and \(\nu_i\) are experimentally-determined constants, \(\xi \geq 0\), and \(\nu_i > 0\). Kernels of the type \(G(\xi, \xi')\) are assumed to have the property \(G(\xi, \xi', \xi''') = G(2\xi, \xi', \xi''')\) so that,
\[
G(\xi, \xi', \xi'') = \sum_{i=1}^{n} G_i \exp \left[\left(\frac{-2\xi + \xi' + \xi''}{\lambda_i}\right)\right]
\]
(5.9)
\[
\frac{\partial G}{\partial \xi}(\xi, \xi', \xi''') = \sum_{i=1}^{n} \frac{G_i}{\lambda_i} \exp \left[\left(\frac{-2\xi + \xi' + \xi''}{\lambda_i}\right)\right]
\]
etc., \(\lambda_i\) being constants > 0. Then, using Simpson’s rule for approximating integration between reduced times \(\xi_i\) and \(\xi_i+1\), we represent certain integrals according to
\[
\int_{0}^{\xi} 2G(\xi, \xi') \frac{\partial G}{\partial \xi'} d\xi' = \sum_{i=1}^{r} G_i e^{-\xi/\lambda_i} \sum_{m=1}^{r} \int_{\xi_{i-1}}^{\xi_{i+1}} e^{-\xi'/\lambda_i} \frac{\partial G}{\partial \xi'} d\xi'
\]
(5.10a)
\[
\int_{\xi_m}^{\xi_{m+1}} \frac{\partial \hat{u}^N}{\partial \xi'} \exp(\xi'/\lambda_1) \, d\xi' \approx \frac{h_B}{6} \left\{ \frac{\partial \hat{u}^N}{\partial \xi} (\xi_m) \exp(\xi_m/\lambda_1) + 4 \frac{\partial \hat{u}^N}{\partial \xi} (\xi_m+h_B/2) \right. \\
\left. \times \exp \left[ (\xi_m+h_B/2)/\lambda_1 \right] + \frac{\partial \hat{u}^N}{\partial \xi} (\xi_m+h_B) \exp[(\xi_m+h_B)/\lambda_1] \right\} (5.10b)
\]

Finally, incorporating such approximations into (5.6) and (5.7), we obtain for the equations of motion and heat conduction,

\[
\hat{p}_{Nj} = m_{NN} \hat{u}_{Nj} + a_{NN}^{(2)} \hat{u}_{Nj}^2 + (a_{NN} - b_{NN}) \sum_k G_k J_k^N(u, \lambda) \\
+ b_{NN} \sum_k K_k J_k^N(u, \nu) - c_{NN} \sum_k K_k J_k^N(T, \nu) \\ + \sum_k K_k J_k^N(u, \nu - \nu_k) - \sum_k G_k J_k^N(u, \lambda) + \sum_k E_k J_k^N(T, \nu)
\]

\[
\hat{q}_N = h_{NN}^1 \hat{u}_{NN}^1 + h_{NN}^2 \hat{u}_{NN}^2 - 3a^2 h_{NN}^3 \sum_k \frac{K_k}{\nu_k} J_k^N(T, \nu) - h_{NN}^4 \sum_k \frac{K_k}{\nu_k} J_k^N(u, \nu) \\
- 2(b_{NN}^1)^1 \sum_k \frac{K_k}{\nu_k} J_k^N(u, \nu) J_k^L(u, \nu) + 2(a_{NN}^1)^1 \sum_k \frac{G_k}{\lambda_k} J_k^N(u, \lambda) J_k^L(u, \lambda) \\
+ 2(b_{NN}^2)^1 \sum_k \frac{K_k}{\nu_k} J_k^N(u, \nu) J_k^L(T, \nu) + 6a_{NN}^2 \sum_k \frac{K_k}{\nu_k} J_k^N(T, \nu) J_k^L(T, \nu) \\
+ k_{NN} \sum_k E_k J_k^N(T, \nu)
\]

in which

\[
J_{Nj}^N(u, \lambda) = \overline{H}(\hat{u}_{Nj}^N, \lambda) \exp[-(\xi_m+h)/\lambda_1] + \overline{G}(\hat{u}_{Nj}^N, \lambda_1)
\]

(5.12a)

etc., and

\[
\overline{H}(\hat{u}_{Nj}^N, \lambda_1) = \sum_{m=1}^{r-1} \left[ \frac{1}{12} (\hat{u}_{Nj}^N)^{t+1} - \hat{u}_{Nj}^N \right] \exp(\xi_m/\lambda_1) + 2 \left[ (\hat{u}_{Nj}^N)^{t+1} - \hat{u}_{Nj}^N \right] \exp[(\xi_m+h)/\lambda_1] \\
\times \exp[(\xi_m+h_m/2)/\lambda_1] + \frac{1}{12} (3\hat{u}_{Nj}^N)^{t+1} - 4\hat{u}_{Nj}^N + \hat{u}_{Nj}^{N-1} \exp[(\xi_m+h)/\lambda_1]]
\]

(5.12b)
\[
\overline{CG}(\eta_j, \lambda_t) = \frac{1}{12} (\hat{G}_{j(r+1)} - \hat{G}_{j(r-1)}) \exp\left(-\varepsilon_r / \lambda_t \right) + \frac{2}{3} (\hat{G}_{j(r+1)} - \hat{G}_{j(r-1)}) \exp\left[-(\varepsilon_r + \varepsilon_{r+1}) / \lambda_t \right]
\]

(5.12c)

In (5.12a) we have separated the histories \(\overline{HG}(\hat{G}_j, \lambda_t)\) from the current values \(\overline{CG}(\hat{G}_j, \lambda_t)\), that is, the value of the integral at the current time \(\varepsilon_r + h\) is obtained by adding to the accumulated sum at \(\varepsilon_r\), the contribution between \(\varepsilon_r\) and \(\varepsilon_r + h\). By notation \(\hat{G}_{j(e)}\), \(\hat{G}_{j(e-1)}\) etc. we mean \(\hat{G}_j(\varepsilon_e)\), \(\hat{G}_j(\varepsilon_{e-1})\), etc.

6. TEST CASES - NUMERICAL RESULTS

We now consider several applications of the theory presented previously to the finite-element analysis of a number of representative problems.

Torsional Oscillations of a Thermoviscoelastic Cylinder. As a first example, we reproduce the results of Cost [13] on the finite-element analysis of a solid circular cylindrical rod of a thermorheologically simple material subjected to torsional oscillations about its longitudinal axis. The free end of the cylinder is subjected to tangential displacements \(\frac{1}{r} u = a_o \sin \omega t\), \(a_o\) being a given amplitude and \(\omega\) a given frequency, and \(r\) being the radial coordinate \((x^1, x^2, x^3) = (r, z, \theta)\). The only nonzero strain component is then \(\gamma_{\theta z} = a_o r \sin \omega t / L\), \(L\) being the length of the cylinder. With the strain so specified as a function of time, we can concentrate on the phenomena of heat generation in the solid. The axis and exterior surface of the rod are insulated, and the temperature at the end of the rod is set equal to zero. In this case, the internal dissipation
(4.20) assumes the form

\[ \sigma = \int_0^t \int_0^t \frac{\partial}{\partial t} \left[ 2G(2\xi - \xi' - \xi'') \frac{\partial \gamma_\theta}{\partial t}, \frac{\partial \gamma_\theta}{\partial t} \right] dt'dt' \]  

(6.1)

The kernal \( G(\xi, \xi') \) is assumed to be of the form (5.9).

We assume that the material is a polyurethene rubber for which the shift factor is such that

\[ \log_{10} b(\theta) = 0.6 + \frac{8.89(\theta - 89.6^\circ F)}{183.5 + (\theta - 89.6^\circ F)} \]

and

\[ G(t) = \sum_{i=1}^{8} G_i \exp(-t/\lambda_i) \]

where \( \lambda_1 = 1.0 \times 10^{-13} \), \( i = 1, 2, \ldots, 7 \), \( \lambda_8 = \infty \); \( G_1 = 1653 \) psi, \( G_2 = 4747 \) psi, \( G_3 = 5539 \) psi, \( G_4 = 4701 \) psi, \( G_5 = 3527 \) psi, \( G_6 = 858 \) psi, \( G_7 = 257 \) psi, and \( G_8 = 143 \) psi. These properties correspond to the Thiokol Chemical Co. rubber, Solithane 113 [13], for which \( \alpha = 1.46 \times 10^{-6}/^\circ F \); the coefficient of thermal conductivity is \( 8.56 \times 10^{-3} \) BTU/in.hr.\(^\circ F \), the specific heat \( c = 0.48 \) BTU/lb.m.\(^\circ F \), and \( \rho = 0.0361 \) lb.m./in.\(^3 \).

The cylinder and a finite-element model of it consisting of composite rectangular finite elements is shown in Fig. 1. In this case, a rectangular element is generated by connecting four triangles together. Thus, over each triangle we use local interpolation functions of the form

\[ \varphi_n(r, z) = a_n + b_n r + c_n z \quad n = 1, 2, 3 \]  

(6.2)

Introducing (6.1) and (6.2) and the thermal and mechanical properties mentioned into (3.10) [or an appropriately modified form of (5.11b)] leads to the system of differential equations governing heat conduction in the
Figure 1. Torsional oscillations of a thermoviscoelastic cylinder and a finite-element model of a quadrant of the cross-section of the cylinder.
element. These were integrated numerically, and the reduced time $\xi$ at 
$r$ at time $t_{n+1}$ was generated using the approximate quadrature formula

$$\xi(r,z,t_{n+1}) = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} b[\theta(r,z,s)]ds$$

$$\approx \sum_{i=1}^{n} \frac{1}{2} \{b[\theta(r,z,t_i)] + b[\theta(r,z,t_{i-1})]\}(t_i - t_{i-1}) \quad (6.3)$$

wherein the same time increment used in integrating the differential 
equations governing the finite element model is employed.

In the case of convective heat transfer boundary conditions, the 
heat flux $q_i$ at boundary is given by

$$q_i = f_i(T^* - T_o^*) \quad (6.4)$$

where $T^*$ is the temperature of the media surrounding the body, $T_o^*$ is the 
temperature across a surface boundary layer, and $f_i$ are film coefficients.

Then the corresponding generalized heat flux of (3.11) is of the form,

$$q_n = \int_{A_o} \psi n f_1 (\psi T^* - T_o^*)dA \quad (6.5)$$

Considering again the cylinder of Fig. 1, we prescribe convection 
boundary conditions at the ends of the cylinder and the exterior of the 
cylinder. Cost obtained film coefficients of $f = 0.012$ BTU/hr.-in.$^2$-$^\circ$F by 
adjusting computed temperature distributions to fit those determined ex-
perimentally for the case in which the cylinder is rotated at 1000 cycles 
per minute but under no strain. Computed and experimental results are 
reproduced in Figs. 2 and 3. With the film coefficients so determined, 
then torsional oscillations of 1000 cycles per minute at 5.34 percent
Figure 2. Temperature distribution in radial direction in a thermoviscoelastic cylinder (after Cost [13]).
Figure 3. Temperature distribution in axial direction in a thermoviscoelastic cylinder (after Cost [13].
strain were induced. The computed and measured temperature distributions are shown in Fig. 4, together with the finite-element solution for the case of temperature-independent material properties. In the latter case, the predicted temperature generated is considerably higher than that measured or computed using the complete thermoviscoelastic model.

Transient, Thermomechanically Coupled Phenomenon. As another test case, we cite the recent results of Oden and Armstrong [2,14] on the transient, nonlinear response of a hollow cylinder subjected to axisymmetric mechanical and thermal boundary conditions. Again, a thermorheologically simple material with \( b(\theta) = 10^{0.9}, \alpha(T) = 9T/100 + T \, ^\circ\text{C} \) is considered. The material properties are: \( \rho = 10^{-3}\text{in.-lb/in.} \, ^\circ\text{C} \, \text{sec.}, \alpha = 10^{-5}/^\circ\text{C}, \, K(\xi) = 10^8 \exp (-\xi/10^{-5}) + 5 \times 10^6 \exp (-\xi/10^0) \, \text{psi}, \, C(\xi) = 1 \times 10^5 (-\xi/10^8) + 0.1 \times 10^5 \exp (-\xi/10^0) \, \text{psi}, \) and, using the hereditary-type heat conduction law, \( \kappa(\xi) = 0.1 \exp (-\xi/10^{-5}) + 0.01 \exp (-\xi/10^0) \, \text{in.-lb./in.} \, ^\circ\text{C} \, \text{sec.} \) The outer radius of the cylinder is 11.0 in., and the inner radius is 10.0 in.

The complete nonlinear system (5.6) and (5.7) [(5.11)] is used in this case for a system of twenty finite elements, each of 0.05 in. thickness. The variation (in reduced time) of various quantities appearing in the equations of motion and heat conduction we assumed to vary quadratically in reduced time. For example, if, for an arbitrary function of reduced time \( f(\xi) \equiv f_1, \, f(\xi - h) = f_{1-1}, \) and \( f(\xi + h) = f_{1+1}, \) \( h \) being the reduced time step, then quadratic interpolation in time yields the difference approximations

\[
\begin{align*}
\dot{f}_{1-1} &= (4f_1 - 3f_{1-1} - f_{1+1})/2h \\
\dot{f}_1 &= (f_{1+1} - f_{1-1})/2h \\
\dot{f}_{1+1} &= (-4f_1 + 3f_{1+1} + f_{1-1})/2h \\
\dot{f}_1 &= (f_{1+1} - 2f_1 + f_{1-1})/2h
\end{align*}
\]
Figure 4. Comparison of computed and measured temperatures in a thermorheologically simple cylinder under torsional oscillations at time $t = 30$ minutes, 5.34 percent strain (after Cost [13]).
Thus, for each $h$, we encounter a system of nonlinear equations in $f_{i+1}$. The Newton-Raphson method was employed for solving these nonlinear equations, its starting values being obtained from difference approximations of the initial conditions on $\dot{u}^N$ and $\dot{T}^N$. All quantities (stresses, displacements, temperatures, etc.) are computed as functions of reduced time and then transformed to real time using $\Delta t = \Delta \xi / \hat{b}(\Theta(r, \xi))$; that is, the reduced time $t$ at step $r + 1$ is $t_r + h/\hat{b}(\Theta(r, \xi))$, where $h = \Delta \xi$ is the reduced time step. In this procedure, we must also account for the fact that the stress tensor is associated with a shift factor that involves the entire element temperature and not simply the temperature at a node.

We set the real time increment associated with nodal displacements and temperatures at a node $N$ in the connected model equal to $h/\hat{b}^N$, where $\hat{b}^N$ is the value of the shift factor at node $N$. In computing stresses, we set the real time increment in an element $e$ equal to $h/\hat{b}^e$, where $\hat{b}^e$ is an averaged shift factor for element $e$.

To initiate the Newton-Raphson iterations, initial conditions (e.g. $\dot{u}^N(\xi_1) = \dot{u}^N(\xi_1) = 0, \dot{T}^N(\xi_1) = \dot{T}^N(\xi_1) = 0$) are cast in central difference approximations. Then, if $h = \Delta \xi$ is the time step in reduced time, $\dot{u}^N$

$\ddot{\hat{u}}^N(\xi_1 + h) = 2\hat{u}^N(\xi_1 + h)/h$ and $\ddot{\hat{T}}^N(\xi_1 + h) = 2\hat{T}^N(\xi_1 + h)/h^2$; similar equations hold for $\dot{T}^N$ and $\dot{T}^N$. Taking

$$\ddot{\hat{u}}(\xi_1 + h) = \frac{1800 \varphi_m \hat{T}^N(\xi_1 + h)}{h[100 + \varphi_m \hat{T}^N(\xi_1 + h)][100 + \varphi_m \hat{T}^N(\xi_1 + h)\varphi_m \hat{T}^N(\xi_1 + h)\varphi_m \hat{T}^N(\xi_1 + h)]}$$

(6.7)

where $M, N, R = 1, 2$ and $\varphi_m = a_m + b_m (r_1 + r_2)/2$ (here we use the linear approximations $\varphi_m = a_m + b_m r$, with $\varphi_m$ computed in certain integrations using the average radius, $r = (r_1 + r_2)/2$), then we use as the shift factor

$$\hat{b} = 10 \varphi_m \hat{T}^N/[100 + \varphi_m \hat{T}^N], \quad b = \hat{b} \Delta n \cdot 10$$

(6.8)
Equations (6.6), (6.7), and (6.8) are then used in calculations over the second time step. Adequate data then exists for carrying the solution forward in (reduced) time. With these approximations, the mass matrices of (5.8a) become

$$m_{11}^N = 2\pi p_0 \left[ a_N r_1^2 - r_1^4 \right] + \left( a_M b_N + b_M a_N \right) \frac{r_2^2 - r_1^2}{3} + b_M b_N \frac{r_2^2 - r_1^2}{2}$$

$$m_{22}^N = \ln (10) \cdot \frac{1}{T} \cdot m_{11}^N \quad (6.9)$$

Here $r_1$ and $r_2$ are the radial coordinates of nodes 1 and 2 of the element $a_N$, $b_N$ are the nodal coefficients appearing in the local interpolation functions ($\phi_N = a_N + b_N r; N = 1, 2$). Specific forms of the remaining matrices and the equations of motion and heat conduction for a typical finite element are given in an Appendix.

As a first example, consider the case in which the model subjected to a step-pressure loading of 10 pounds-per-square-inch (psi) on the inner boundary. The outer boundary is stress-free, and the problem is considered to be isothermal. Figure 5 presents the radial displacements as a function of real time. The stress components in the radial and circumferential directions are presented in Figs. 6 and 7. It is seen that all stresses are initially compressive in nature. This is a significant difference from the conventional quasi-static solution which gives compressive stress only in the radial direction. The quasi-static solution, obtained from the finite-element program by deleting inertia terms, is compared with the exact solution of the quasi-static isothermal problem in Figs. 8 and 9. It is also noted in Figs. 6 and 7 that the dynamic circumferential and meridional stresses become tensile at later times. The solution was not carried past $10^{-3}$ seconds, but it is expected that these stress components would eventually reach values equal to the quasi-static condition.
Figure 5. Radial displacement profiles for dynamic isothermal case with a step load of 10 psi at inner boundary.
Figure 6. Radial stress profiles for dynamic isothermal case with a step load of 10 psi at inner boundary.
Figure 7. Circumferential stress profiles for dynamic isothermal case with a step load of 10 psi at inner boundary.
Figure 8. Radial stress profiles for quasi-static isothermal case with a step load of 10 psi at inner boundary.
Figure 9. Circumferential stress profiles for quasi-static isothermal case with a step load of 100 psi at inner boundary.
The final example to be considered is a dynamic, coupled nonisothermal problem with a step loading of 10 psi applied at the inner boundary while the thermal boundary conditions correspond to convective heat transfer. Film coefficients of 0.1 and 1.0 in. lb./in.² C·sec. respectively are assumed for the inner and outer boundaries of the cylinder. The heat supply from internal sources is assumed to be zero and the initial temperature of the cylinder and the surrounding medium are taken to be 300°C. Heat is thus generated through thermomechanical coupling. Since energy is dissipated in the system, the internal temperature increases during the transient part of the motion while heat is expelled at different rates at each boundary. The displacements are presented in Fig. 10 and the radial and circumferential stresses are shown in Figs. 11 and 12 compared with the isothermal solution. The meridional stress is indicated in Fig. 13. While stress and displacement profiles are obtained at different times for the isothermal and nonisothermal cases owing to the necessity of transforming all computed quantities to real time in the nonisothermal case, results do indicate a lag in the nonisothermal stress waves and a noticeable decrease in amplitude. Circumferential stresses are initially compressive but become tensile almost everywhere in the cylinder at $10^{-3}$ seconds. The circumferential stress wave requires approximately $2.5 \times 10^{-5}$ seconds to traverse the 1.0 inch thickness of the cylinder in the isothermal case and approximately $4.0 \times 10^{-5}$ seconds in the nonisothermal case. Displacement profiles are qualitatively the same for the isothermal and nonisothermal cases. The nonisothermal displacements are higher than those predicted by the isothermal analysis at the outer boundary, but they are lower on the inner boundary. This is due chiefly to the relative
**Figure 10.** Radial displacement profiles isothermal and nonisothermal
Figure 11. Nondimensional radial stress profiles in a thermorheologically simple cylinder-isothermal and nonisothermal cases.
Figure 12. Nondimensional circumferential stress profiles in a thermorheologically simple cylinder-isothermal and nonisothermal cases
Figure 13. Meridional stress profiles for dynamic nonisothermal case with a step load of 10 psi at inner boundary.
magnitudes of the film coefficients at these boundaries. The results indicated in all examples were computed using a reduced time increment of $10^{-6}$ seconds; rather rapid convergence of the Newton-Raphson scheme was observed at each time step.

The temperature distribution is shown in Fig. 14. Note that the mechanically loaded inner boundary is the first to experience an increase in temperature. It is seen that the temperature change reaches maximum values for the material parameters and boundary conditions considered, at approximately $4 \times 10^{-5}$ seconds. Values for times greater than $8 \times 10^{-5}$ were not obtained.
Figure 14. Radial temperature profiles for dynamic nonisothermal case with a step load of 10 psi at inner boundary.
7. REFERENCES


8. APPENDIX - EQUATIONS OF MOTION AND HEAT CONDUCTION OF CYLINDRICAL ELEMENTS

We present here explicit forms of the equations of motion and heat conduction for finite elements of a thermorheologically simple material, obeying (4.17) - (4.20). These are special cases of (5.6) and (5.7) for the case in which $\hat{\mathbf{u}}$ and $\hat{T}$ are assumed to vary linearly in the radial coordinate $r$ and quadratically in time. For an element joining nodes 1 and 2 of a finite element model, let

$$a_1 = \frac{r_2 - r_1}{L}, \quad a_2 = -\frac{r_1}{L}, \quad b_1 = -\frac{1}{L}, \quad b_2 = \frac{1}{L} \quad (A.1)$$

where $L = r_2 - r_1$. Then the equations of motion of a finite element are of the form

$$\hat{\mathbf{P}}_m = \hat{\mathbf{M}}_{mn} \hat{\mathbf{a}} \left[ \frac{3\hat{\mathbf{u}}_n (\xi^+_h) - 4\hat{\mathbf{u}}_n (\xi^-_h) + \hat{\mathbf{u}}_n (\xi^-_h)}{2h} \right]$$

$$+ \ln 10 \cdot \frac{\hat{\mathbf{T}}_m}{h} \left[ \sum_{j=1}^{m_n} \left( G_{i} \frac{\xi_j}{h} \cdot \overline{\mathbf{G}(\hat{\mathbf{u}}_n, \lambda_j)} + \overline{\mathbf{C} \mathbf{G}(\hat{\mathbf{u}}_n, \lambda_j)} \right) \right]$$

$$+ C_{1mn} \left[ \sum_{i=1}^{s} \left( K_{i} \frac{\xi_j}{h} \cdot \overline{\mathbf{K}(\hat{\mathbf{u}}_n, \nu_k)} + \overline{\mathbf{C} \mathbf{K}(\hat{\mathbf{u}}_n, \nu_k)} \right) \right]$$

$$+ C_{2mn} \left[ \sum_{k=1}^{s} \left( K_{k} \frac{\xi_j}{h} \cdot \overline{\mathbf{K}(\hat{\mathbf{T}}_n, \nu_k)} + \overline{\mathbf{C} \mathbf{K}(\hat{\mathbf{T}}_n, \nu_k)} \right) \right]$$

$$- C_{3mn} \left[ \sum_{k=1}^{s} \left( K_{k} \frac{\xi_j}{h} \cdot \overline{\mathbf{K}(\hat{\mathbf{T}}_n, \nu_k)} + \overline{\mathbf{C} \mathbf{K}(\hat{\mathbf{T}}_n, \nu_k)} \right) \right] \quad ; \quad m, n = 1, 2 \quad (A.2)
where \( \bar{H}_{MN} \) is the term in brackets in (6.9a) and

\[
\hat{H}(\xi) a_{\xi} \beta_k = \sum_{m=1}^{(\xi-1)} \left[ \frac{2e^2}{12} \left[ a_{\xi} (\xi + h) - a_{\xi} (\xi - h) \right] + \frac{2e}{3} \left[ a_{\xi} (\xi + h) - a_{\xi} (\xi - h) \right] + \frac{e}{12} \left[ 3a_{\xi} (\xi + h) - 4a_{\xi} (\xi - h) + a_{\xi} (\xi - h) \right] \right]
\]

(A.3a)

\[
\hat{C}(\xi) a_{\xi} \beta_k = \frac{e^2}{12} \left[ a_{\xi} (\xi + h) - a_{\xi} (\xi - h) \right] + \frac{2e}{3} \left[ a_{\xi} (\xi + h) - a_{\xi} (\xi - h) \right] + \frac{e}{12} \left[ 3a_{\xi} (\xi + h) - 4a_{\xi} (\xi - h) + a_{\xi} (\xi - h) \right]
\]

(A.3b)

The equations of heat conduction for an element are

\[
\hat{q}_m = 3a^2 \cdot T_0 \cdot D_{MN} \cdot \left\{ \sum_{k=1}^{s} K_k \left[ \frac{3\hat{T}_N (\xi_{+} h) - 4\hat{T}_N (\xi_{-} h) + \hat{T}_N (\xi_{-} h)}{2h} \right] \right\}
\]

\[
- \sum_{k=1}^{s} K_k \left[ \langle e^{-(\xi + h)/\nu_k} \cdot H_{\xi} (\xi_{+} h, \nu_k) \rangle \right]
\]

\[
+ \alpha \cdot T_0 \cdot D_{MN} \cdot \left\{ \sum_{k=1}^{s} K_k \left[ \frac{4\hat{U}_N (\xi_{-} h) + \hat{U}_N (\xi_{+} h) - 3\hat{U}_N (\xi_{+} h)}{2h} \right] \right\}
\]

\[
- \sum_{k=1}^{s} K_k \left[ \langle e^{-(\xi + h)/\nu_k} \cdot H_{\xi} (\xi_{+} h, \nu_k) \rangle \right]
\]

\[
+ C_N C_N \cdot (r^2 - r^2) \cdot \left\{ \sum_{p=1}^{r} \left[ 3 e^{-(\xi + h)/\nu_p} \cdot H_{\xi} (\xi_{+} h, \nu_p) + H_{\xi} (\xi_{-} h, \nu_p) \right] \right\}
\]

\[
+ D_{MN} \cdot \left\{ \sum_{k=1}^{s} K_k \left[ \langle e^{-(\xi + h)/\nu_k} \cdot H_{\xi} (\xi_{+} h, \nu_k) \rangle \right] \right\}
\]
\[ + \sum_{k=1}^{s} \left( \frac{K_k}{\nu_k} \cdot \left( e^{-\left(\xi g + h\right)/\nu_k} \cdot \overline{HK}(\hat{G}_N, \nu_k) \cdot \overline{HK}(\hat{G}_L, \nu_k) \right) + \overline{CK}(\hat{G}_N, \nu_k) \cdot \overline{CK}(\hat{G}_L, \nu_k) \right) \]

\[ + e^{-\left(\xi g + h\right)/\nu_k} \cdot \left( \overline{HK}(\hat{T}_N, \nu_k) \cdot \overline{HK}(\hat{T}_L, \nu_k) \right) \]

\[ + e^{-\left(\xi g + h\right)/\nu_k} \cdot \left( \overline{HK}(\hat{U}_N, \nu_k) \cdot \overline{HK}(\hat{U}_L, \nu_k) \right) \]

\[ + \overline{CK}(\hat{G}_N, \nu_k) \cdot \overline{CK}(\hat{T}_N, \nu_k) \quad \text{for} \quad K, L, M, N = 1, 2 \quad (A.4) \]

where \( \overline{HK}, \overline{CK}, \overline{HG}, \overline{CG}, \overline{HE}, \) and \( \overline{CE} \) are defined by (A.3) and

\[ C_{1M} = \frac{b}{3} \left[ a_m a_n \left( r_2^2 - r_1^2 \right) + (a_m b_n + b_m a_n) (r_2 - r_1) + 2b_m b_n \ln \frac{r_a}{r_1} \right] \]

\[ C_{2M} = \frac{b}{3} \left[ 2a_m a_n \left( r_2^2 - r_1^2 \right) + 2(a_m b_n + b_m a_n) (r_2 - r_1) + b_m b_n \ln \frac{r_2^2}{r_1^2} \right] \]

\[ C_{3M} = \frac{b}{3} \left[ 2a_m a_n \left( \frac{r_2^2 - r_1^2}{3} \right) + (a_m b_n + b_m a_n) (r_2^2 - r_1^2) + b_m b_n (r_2 - r_1) \right] \]

\[ D_{1M} = 3\alpha^2 T_0 \left[ a_m a_n \left( r_2^4 - r_1^4 \right) + (a_m b_n + b_m a_n) \left( \frac{r_2^2 - r_1^2}{2} \right) + b_m b_n \left( \frac{r_2^2 - r_1^2}{2} \right) \right] \]

\[ D_{2M} = \alpha T_0 \left[ 2a_m a_n \left( \frac{r_2^3 - r_1^3}{3} \right) + (a_m b_n + 2b_m a_n) \left( \frac{r_2^2 - r_1^2}{2} \right) + b_m b_n \left( r_2 - r_1 \right) \right] \]

\[ D_{3MNL} = \frac{1}{3} \left[ 4a_m a_n a_l \left( \frac{r_2^3 - r_1^3}{3} \right) + (a_m b_n b_l + a_m b_n a_l + 2b_m a_n a_l) (r_2^2 - r_1^2) \right. \]

\[ + (a_m b_n b_l + 2b_m a_n b_l + 2b_m b_n a_l) (r_2 - r_1) + b_m b_n b_l \ln \frac{r_a}{r_1} \]

\[ D_{4MNL} = \frac{1}{3} \left[ 2a_m a_n a_l \left( \frac{r_2^3 - r_1^3}{3} \right) + (a_m b_n b_l + a_m b_n a_l + 2b_m a_n a_l) \left( \frac{r_2^2 - r_1^2}{2} \right) \right. \]

\[ + (2a_m b_n b_l + b_m a_n b_l + b_m b_n a_l) (r_2 - r_1) + 2b_m b_n b_l \ln \frac{r_a}{r_1} \]
\[ D_{5, M N L} = 3 \alpha^2 \left[ a_m a_n a_l \left( \frac{r_5 - r_5}{5} \right) + (a_m a_n b_l + a_n a_m a_l + b_m a_n a_l) \left( \frac{r_4 - r_4}{4} \right) \right. \]
\[ \left. + (a_m b_n b_l + b_m a_n b_l + b_m b_n a_l) \left( \frac{r_3 - r_3}{3} \right) + b_m b_n b_l \left( \frac{r_2 - r_2}{2} \right) \right] \]
\[ D_{6, M N L} = 2 \alpha \left[ 2 a_m a_n a_l \left( \frac{r_5 - r_5}{4} \right) + (2 a_l a_n b_m + a_l b_n a_m + 2 b_l a_n a_m) \left( \frac{r_4 - r_4}{3} \right) \right. \]
\[ \left. + (a_m b_n b_l + 2 b_m a_n b_l + b_m b_n a_l) \left( \frac{r_3 - r_3}{2} \right) + b_m b_n b_l \left( r_2 - r_1 \right) \right] \]

(A.5)