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GENERAL MIXED FINITE ELEMENT METHODS OF NONLINEAR CONTINUA

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Abstract. General mixed finite element models of the nonlinear thermomechanical response of dissipative media are constructed. A number of existing specialized finite element models are derived from the general formulation presented in this paper.

1. Introduction

Most of the early applications of the concept of finite elements to problems in solid mechanics dealt with approximations of the displacement field which were designed so as to minimize the total potential energy of a discrete model of an elastic body. Such approximations generally lead to an upper bound for the potential energy of the body, and their construction involves ideas that, in large measure, have been available for over a century. In the early 1960's, a number of alternatives to what may be termed the "standard approach" to finite element formulations began to appear. These involved the use of models based on complementary energy ideas and the approximation of stresses rather than displacements [1,2,3]. Here lower bounds to the energy may be obtained, and the generation of stiffness matrices required the inversion of local flexibility matrices that arise naturally in the analysis. If one abandons the...
requirement that the finite element approximation provide bounds of one
form or another on the energy of the exact solution, then the use of so-
called mixed variational principles for constructing finite element
models suggests itself. Here simultaneous approximation of both displace-
ments and stresses can be generated via, say, the Hellinger-Reissner
principle (see, for example, [4,5,6]). The entire area of so-called
mixed and hybrid finite element formulations for linearly elastic bodies
has been explored in some detail by Pian and Tong [7-10]; an up-to-date
summary of the ideas with additional references to related work can be
found in [10].

If we look beyond linear elasticity to the general thermomechanical
theories of nonlinear continua, we see many other field variables in
addition to the displacements and stresses. In purely thermal phenomena,
for example, the equations governing heat conduction and the conservation
of energy involve the absolute temperature, temperature gradients, the
specific entropy, and components of heat flux. In more general processes,
these quantities appear coupled with those variables related to mechani-
cal response. The most natural choice of dependent variables in the
general case is the local displacement field and temperature field in
each finite element (e.g. [11]), since the use of measures of motion
and of changes of the degree of hotness are suggested by the so-called
axiom of causality [12]. So far as we know, no attempts appear to have
been made at constructing any sort of mixed model appropriate for the
general thermodynamical theory of continuous media; indeed, attempts at
the development of relatively modest hybrid models for large quasi-static
deformations of linearly elastic bodies have been confined to linearized
incremental formulations and are limited to infinitesimal strains [13].

In the present paper, we develop general mixed finite element models of the nonlinear thermomechanical response of dissipative media, in which independent approximations of the displacements, strains, stresses, internal dissipation, heat flux, temperature, temperature gradients, entropy, and, in the case of continuum plasticity, the inelastic strain tensor or the internal state (hidden variable) tensor are constructed. The result is a complete discrete analogue of the entire collection of local field equations for thermomechanical behavior of continuous media; e.g., linear momentum, the strain-displacement equations, constitutive equations, heat conduction, etc. While the resulting collection of equations appear to be too complex for immediate application, their generality makes it possible to view the idea of mixed models form in rather unifying manner. Indeed, by assuming that certain of the local field equations are identically satisfied, a priori, we introduce corresponding dependences among various fields and thereby reduce the list of dependent variables. In this way, we are able to produce a variety of specialized mixed models. For example, Nickell's variational model of linear thermoviscoelasticity problem [14] is shown to be derivable as a special case from our general model. Certain of the mixed and hybrid models of Pian and Tong [7-10] are, of course, also encompassed by our model.

The vehicle for our approximation is the variational principle developed by Oden [15] and Oden and Bhandari [16,17] together with their theory of "thermoplastically simple" materials [18]. These ideas are
reviewed briefly in Sections 2 and 3 of the paper. In Section 4 we develop general mixed finite element models for nonlinear continua and we provide interpretations of certain results. Finally, in Section 5 we examine a number of special cases obtained from the general formulation.

2. Nonlinear Thermoviscoplasticity

A detailed discussion of the constitutive theory of thermoviscoplasticity is given in [18]; consequently, here we shall only review briefly the field equations governing the behavior of thermoplastically simple materials.

Consider a material body \( \mathcal{B} \) occupying a reference configuration \( \mathcal{C}_0 \) in three-dimensional euclidean space. Following the notation in [11], we establish a fixed spatial frame of reference \( z_i (i = 1, 2, 3) \) which, for simplicity, is rectangular cartesian. While in \( \mathcal{C}_0 \), a material particle \( x \) is labeled \( x_i (i = 1, 2, 3) \) and, for convenience, we take \( x_i = z_i \) at time \( \tau = 0 \). The motion of \( \mathcal{B} \) is then the one-parameter family of configurations defined by the mapping \( z_i = \chi_i (x, t) \) and \( u_i (x, t) = \chi_i (x, t) - x_i \) are the cartesian components of displacement relative to \( \mathcal{C}_0 \).

The strain-displacement and thermal-gradient relations are given by the following expressions

\[
\begin{align*}
\gamma_{ij} & = \frac{1}{2} (u_{i,j} + u_{j,i} + u_{i}, u_{j,j}) \\
\theta_i & = \theta_{i,1}
\end{align*}
\]
where $\gamma(x,t), u(x,t), g(x,t)$ and $\theta(x,t)$ denote respectively, the strain tensor, the displacement vector, thermal gradient and the absolute temperature. The comma denotes partial differentiation with respect to the material coordinates $x_j$ which we assumed to be rectangular cartesian in the reference configuration.

The local forms of the laws of linear momentum, angular momentum and the energy are

\[
\begin{align*}
\left[\sigma^{ij}(\delta_{n,j} + u_{n,j})\right]_{,i} + pF = \rho \ddot{u}_i ; \quad \sigma^{ij} = \sigma^{ji} \quad (2.2a,b) \\
\sigma^{ij} \dot{\gamma}_{ij} + \rho \dot{h} + q^i_{,i} = \rho \dot{\varepsilon} \quad (2.2c)
\end{align*}
\]

wherein all quantities are referred to the reference configuration; $\sigma(x,t)$ being the second Piola-Kirchhoff stress tensor, $F(x,t)$ the body force vector, $\rho(x,t)$ the mass density of the solid, and $\epsilon(x,t)$ the internal energy density. The superposed dot indicates the partial differentiation with respect to time and $\delta_{i,j}$ denotes Kronecker delta.

Equation (2.2c) can be written in an alternate form by introducing the Helmholtz free energy density $\varphi = \epsilon - S\theta$ and the internal dissipation $\dot{\gamma} = \sigma^{ij} \dot{\gamma}_{ij} - \rho(\dot{\varphi} + S\dot{\theta})$ into (2.2c), i.e.

\[
q^i_{,i} + \rho \dot{h} + \dot{\gamma} = \rho T_0 S
\quad (2.2d)
\]

In writing (2.2d), for further convenience, we have replaced $\theta S$ by $T_0 S$.

To complete the description of the behavior of a thermoplastically simple solid, we need only add to the basic kinematical and physical
law: the constitutive equations which characterize the material of
which the solid is composed. In [18], a thermoplastically simple material
is characterized by a set of five constitutive equations depicting the
free energy $\Phi$, the stress $\sigma$, the entropy $S$, the inelastic strain-rate
$\dot{\eta}$ and the heat flux $q$ as functionals of the histories of strain $\gamma$, the
inelastic strain $\eta$, absolute temperature $\Theta$ and the current value of the
temperature gradient $\theta$.

This theory presupposes a sufficiently smooth transition from an
"elastic state to an "inelastic" state to allow Frechet differentiation
of certain constitutive functionals. Without this assumption, which
we maintain is acceptable, the elaborate machinery of Owens [19,20]
must be employed. Happily, our smooth theory yields as special cases
Coleman's thermodynamics of simple materials [21] as well as the
general theory of visco-plastic materials of Valanis [22,23].

In [18], thermodynamical restrictions, in the sense of Coleman [21],
show that this class of materials can be characterized by only three
constitutive functionals, one describing the free energy $\Phi$ which is
independent of $\xi$ and the other two being heat flux $q$ and the inelastic
strain-rate $\dot{\eta}$. It is also shown that the constitutive equations for
stress and entropy are derivable from the free energy alone and the
internal dissipation is obtained from the constitutive equations of $\Phi$
and $\dot{\eta}$. Then the set of constitutive equations is given by

$$\sigma = \partial_{\gamma_{s=0}}^{\Phi} [\Gamma ; \Gamma] = \mathcal{F}_{s=0} [\Gamma ; \Gamma]$$  \hspace{1cm} (2.3a)
\[ S = -\partial_0 \Phi \left[ \Gamma^t_r; \Gamma_s \right] = \Phi \left[ \Gamma^t_r; \Gamma_s \right] \quad (2.3b) \]

\[ \hat{\sigma} = -\rho \left[ \partial_0 \Phi \left[ \Gamma^t_r; \Gamma_s \right] \cdot \dot{\Gamma}^t_r + \delta_{\Gamma_s} \Phi \left[ \Gamma^t_r; \Gamma_s \right] \right] = \Phi \left[ \Gamma^t_r; \Gamma_s \right] \quad (2.3c) \]

and

\[ \frac{d}{ds} \left[ \Gamma^t_r; \Gamma_s \right] = \frac{d}{ds} \left[ \Gamma^t_r; \Gamma_s \right] \quad (2.3d) \]

\[ \dot{\eta} = \frac{\partial}{\partial s} \left[ \Gamma^t_r; \Gamma_s \right] \quad (2.3e) \]

wherein (2.3) we have used the notation \( \Gamma = (\gamma, \eta, \theta) \) and the total histories are decomposed into the 'past histories' and the 'current values' of the respective argument functions, e.g. the total history \( \Gamma^t(s) \quad (0 \leq s \leq \infty) \) can be represented by a pair \( \left[ \Gamma^t_r(s), \Gamma^t_o(s) \right] \) where \( \Gamma^t_r(s) = \Gamma^t(s) \quad [s \in (0, \infty)] \).

The symbolism \( \partial_{\gamma}, \partial_{\theta} \) denotes partial differentiation with respect to \( \gamma, \theta \) whereas \( \delta_{\eta} \) and \( \delta_{\Gamma} \) denotes partial Frechet differentiation associated with variations of the past histories of \( \eta \) and \( \Gamma \); the vertical stroke in the argument of \( \delta_{\Gamma} \Phi \) in (2.3c) indicates linearity in the quantity which follows (i.e., \( \Gamma^t_r \)). The internal dissipation \( \hat{\sigma}(x,t) \) of (2.3c) is not, of course, a dependent constitutive variable in the strict sense; the functional \( \Phi[\cdot;\cdot] \) is, in general, uniquely determined from \( \Phi \) and \( \eta^t \) via the Clausius-Duhem inequality.

To the system of field equations, we adjoin initial and boundary conditions. The displacement and traction boundary conditions are

\[ u_s = \hat{u}_s \quad \text{on} \quad \partial \Omega \times (-\infty, \infty) \quad (2.4a) \]

\[ n_{t} \cdot (\delta_{\eta} + u_s) = \hat{T} \quad \text{on} \quad \partial \Omega \times (-\infty, \infty) \quad (2.4b) \]
The temperature and heat flux boundary conditions are

\[
\theta = \hat{\theta} \quad \text{on } \partial \Omega_\theta \times (-\infty, \infty) \quad (2.4c)
\]

\[
Q = n_1 q_1^\prime = \hat{Q} \quad \text{on } \partial \Omega_q \times (-\infty, \infty) \quad (2.4d)
\]

For simplicity and without loss of generality, we take the associated initial conditions to be

\[
\sigma(x,0) = y(x,0) = \eta(x,0) = 0 \quad (2.4e)
\]

\[
u(x,0) = \dot{u}(x,0) = 0 \; ; \; \theta(x,0) = T_0 \quad (2.4f,g)
\]

Since the solution of the initial-boundary value problems requires that we incorporate the initial conditions explicitly into the field equations, it is convenient to take the Laplace transform of (2.2a), (2.2d) and (2.3e) and then, taking the inverse transform, we write

\[
g^*[(\delta_{n,1} + u_{1,n})u_{1,n}] + \rho f_n = \rho u_n \quad (2.5a)
\]

\[
g^*q_{1,n} + \dot{H} + \Sigma = \rho T^\prime \hat{S} \quad (2.5b)
\]

\[
g^*\eta_{1,n} = \mathcal{L}^{-1}[\frac{\Gamma_r}{\Gamma s} g] \quad (2.5c)
\]

wherein \( g(t) = t, \; g'(t) = 1, \) and

\[
f_n = [g^*F_n](x,t) \quad (2.6a)
\]

\[
H + \Sigma = g^*[\rho \dot{h} + \hat{\delta}](x,t) \quad (2.6b)
\]

The symbol (*) appearing in (2.5) denotes the convolution integral which is defined as

\[
u^*v = \int_0^t u(t-\tau)v(\tau)d\tau \quad (2.7)
\]
where the dependence of \( u \) and \( v \) on \( x \) is understood. Thus the system of equations (2.1), (2.3a) - (2.3d), (2.5), together with the boundary conditions (2.4) completely characterize a thermoplastically simple material.

3. A General Variational Principle for Nonlinear Thermoviscoplasticity

A generalization of Vainberg's theorem [24] to nonlinear initial and boundary value problems in continuum mechanics has been presented and used by Oden in several papers [15-17]. An outline of the method with its application to nonlinear thermoviscoplasticity is given below; for details of the method see references [15-17].

Our principal aim here is to construct a general variational principle associated with the nonlinear theory of thermoviscoplasticity defined by (2.1), (2.3a) - (2.3d), (2.4) and (2.5). The boundary-value problem described by (2.1) - (2.5) can be written in the form

\[
\hat{\mathbf{G}}(\Lambda) = \mathbf{Q}(\Lambda) - \bar{\Gamma} = 0
\]  

(3.1)

where \( \mathbf{Q} \) is a nonlinear operator defined on some dense set \( \Omega \subseteq \mathcal{Y} \), into \( \mathcal{Y}' \), \( \mathcal{Y} \) and \( \mathcal{Y}' \) being real Banach spaces. \( \Lambda = \Lambda(x,t) \) is an element of \( \Omega \), the domain of \( \mathbf{Q} \), and \( \bar{\Gamma} = \bar{\Gamma}(x,t) \in \mathcal{Y} \). The domain of \( \mathbf{Q} \) consists of ordered nine-tuples \( \Lambda = (u, \psi, \sigma, \dot{\sigma}, \dot{q}, q, \theta, S, \tau) \) whose components are functions of position \( x \) and time \( t \).

The operator \( \mathbf{Q}(\Lambda) \) defined by the formula

\[
\langle \mathbf{Q}(\Lambda), \Lambda \rangle = \lim_{\alpha \to 0} \frac{1}{\alpha} [K(\Lambda + \alpha \Lambda) - K(\Lambda)]
\]  

(3.2)

where \( K(\Lambda) \) is a functional on \( \Omega \), \( \Lambda \in \mathcal{Y} \), and \( \langle \cdot, \cdot \rangle \) is an appropriate bilinear map from \( \mathcal{Y}' \otimes \mathcal{Y}^* \) into \( E \) is called the gradient of \( K \) at \( \Lambda \) and...
is denoted \( \mathcal{O}(\Lambda) = \text{grad } K(\Lambda) \). The operator \( \mathcal{O}(\Lambda) \) is potential on \( \Omega \), if there exists a functional \( K(\Lambda) \) such that grad \( K(\Lambda) = \mathcal{O}(\Lambda) \) for \( \Lambda \in \Omega \). It is shown by Vainberg [24] that if \( \mathcal{O}(\Lambda) \) is a potential operator then there exists a unique functional \( K(\Lambda) \) whose gradient is \( \mathcal{O}(\Lambda) \) which is given by the direct integral

\[
K(\Lambda) = \int_0^1 \langle \mathcal{O}(\Lambda_s) + s(\Lambda - \Lambda_0), (\Lambda - \Lambda_0) \rangle ds + K_0 \quad (3.3)
\]

where \( K(\Lambda_0) = K_0 \) is a constant and \( s \) a real parameter.

Following the procedure adopted in [15-17], we introduce the bilinear map of Gurtin [25]

\[
\langle u, v \rangle = \int_{\Omega} [u^* v] d\mathbf{g} \quad (3.4)
\]

where \( u^* v \) is defined in (2.7). Then, substituting (3.1) into (3.3), we obtain the functional

\[
K(\Lambda) = \frac{1}{2} \int \left[ u^* \sigma_{nu} - 2g^* u^* \sigma_{n}^1 - g^* u^* (\sigma_{1}^1 u_{n},_1) - 2u^* \psi_{n} - 2g^* \nu_{ij}^1 \sigma_{ij}^1 \right] d\mathbf{g} \\
+ 2g^* \lambda_{ij} * \mathcal{J}^1_1 [\Gamma_1; \Gamma] - g^* \delta \hat{\sigma} - 2g^* \hat{\sigma} \left[ \Gamma_1; \Gamma \right] * \hat{\sigma} - g^* s \delta s \\
+ 2g^* s \delta \left[ \Gamma_1; \Gamma \right] - 2g^* e_1^1 g^1 + 2g^* \eta_{ij}^1 [\Gamma_1; \Gamma] * g^1 - 2g^* (g^* q_{ij}^1 + H + \Sigma) \\
- \rho \sigma \delta \eta \eta - g^* \lambda_{ij}^1 \lambda_{ij}^1 + 2g^* \mathcal{J}^1_1 [\Gamma_1; \Gamma] * \eta_{ij}^1] d\mathbf{g} + \frac{1}{2} \int [g^* \lambda_{ij} \sigma_{ij}^1 (u_{n},_1) \\
+ 2\lambda_{ij}^1] - 2\mathcal{J}^1_1 [u_{n}] d\mathbf{g} + \int [g^* \lambda_{ij}] d\mathbf{g} + \int [g^* e_1^1 \delta \hat{\theta} Q] d\mathbf{g} \\
+ \int [g^* \lambda_{ij}^1 (Q - \hat{\sigma}) Q] d\mathbf{g} \quad (3.5)
\]
arriving (3.5) from (3.3), we have set $K_0 = 0$, $A_0 = 0$. It is now a
d' matter to verify that the functional $K(\Lambda)$ assumes a stationary
value whenever (3.1) is satisfied. In other words, equations (2.1) -
(2.5) are the Euler equations of the functional $K(\Lambda)$ of (3.5). All nine
fields $u$, $y$, $\sigma$, $\hat{\sigma}$, $q$, $\hat{q}$, $\theta$, $S$ and $\bar{\eta}$ are varied independently.

4. General Mixed Finite-Element Models

We shall now examine general mixed finite-element models of nonlinear
continua obtained by the obvious Ritz-procedure suggested by (3.5).
Following the philosophy of finite-elements, we begin by representing
the body $\Omega$ as a collection $\bigcup_{e=1}^{R} \Omega_e$ of material finite-elements $\Omega_e$, connected
continuously together at finite number $G$ of nodal points $\mathbf{x}^\Delta$, $\Delta = 1, 2, \ldots, G$.
In finite-element analysis each element is temporarily considered to
be disconnected from the entire collection and once the properties
of individual elements are established, then the process of connecting
elements together involves only the topology of model and is accomplished
by simple transformations. Since this has been discussed at great length
elsewhere [11], we shall confine our attention to a typical element $\Omega_e$
of $\Omega$.

Consider, then, a typical element $\Omega_e$, in which $N_e$ nodal points
$x_e^1, x_e^2, \ldots, x_e^{N_e}$ have been identified. Let $\Lambda_e = \{u_e, y_e, \sigma_e, \hat{\sigma}_e, q_e, \hat{q}_e, \theta_e, S_e, \bar{\eta}_e\}$ denote the restriction of $\Lambda = \{u, y, \sigma, \hat{\sigma}, q, \hat{q}, \theta, S, \bar{\eta}\}$
to $\Omega_e$ and let $\Lambda_e^N(t)$ denote their values at node $x_e = x_e^N$. Then we intro-
duce the following local approximations of $\Lambda_e$:

\begin{align*}
  u_e(x,t) &= u_e^N(t) \psi_e^1(x) & g_e(x,t) &= g_e^N(t) \psi_e^2(x) \\
  y_e(x,t) &= y_e^N(t) \psi_e^3(x) & \theta_e(x,t) &= \theta_e + \theta_e^N(t) \psi_e^4(x) \\
\end{align*}
\[ \psi_{n}^{(m)}(x) = \delta_{n}^{m} \quad ; \quad \psi_{n}^{(0)}(x) = 0 \quad x \notin \Theta_{e} \]

\[ \sum_{N=1}^{N_{e}} \psi_{n}^{(m)}(x) = 1 \quad ; \quad \psi_{n}^{(m)}(x) > 0 \quad x \in \Theta_{e} \]  

(4.2)

and the repeated index \( N \) in (4.1) is summed from 1 to \( N_{e} \).

We could, of course, use higher order representations for \( u_{e} \), \( y_{e} \), \( a_{e} \), etc. in which nodal values of various spatial derivatives of these functions also appear [11]. For example, let \( Z_{3}^{3} \) denote the space of ordered triples of non-negative integers; i.e. if \( \alpha \in Z_{3}^{3} \), then \( \alpha = (\alpha_{1}, \alpha_{2}, \alpha_{3}) \), each \( \alpha_{i} \) being a non-negative integer. Using the multi-index notation,

\[ |\alpha| = \alpha_{1} + \alpha_{2} + \alpha_{3} ; \quad D_{\alpha} = \frac{\partial |\alpha|}{\partial x_{1}^{\alpha_{1}} \partial x_{2}^{\alpha_{2}} \partial x_{3}^{\alpha_{3}}} \]  

(4.3)

we define a local interpolation function \( \psi_{n}^{(m)}(x) \) of degree \( q \) according to

\[ D_{\beta} \psi_{n}^{(m)}(x) = \delta_{n}^{m} \delta_{\beta_{1}}^{\alpha_{1}} \delta_{\beta_{2}}^{\alpha_{2}} \delta_{\beta_{3}}^{\alpha_{3}} \quad ; \quad |\alpha| , |\beta| \leq q \]  

(4.4)

where \( M, N = 1, 2, \ldots, N_{e} \), \( \alpha, \beta \in Z_{3}^{3} \) and \( \psi_{n}^{(m)}(x) = 0 \) if \( x \notin \Theta_{e} \). Properties (4.4) ensure that the local representation of a function \( u_{e} \) over an element \( \Theta_{e} \) is a linear combination of the values of all derivatives of \( u_{e} \) of order \( |\alpha| \leq q \) at each node \( x_{e}^{m} \in \Theta_{e} \). That is, if
then (4.4) implies that
\[ u_n(x,t) = \begin{cases} \sum_{|\alpha| \leq q} a^{n}_{\alpha}(t) \varphi^{(q)}_{N}(x) & \text{if } x \in \Omega_n \\ 0 & \text{if } x \notin \Omega_n \end{cases} \] 

(4.5)

Thus a higher order representation of degree \( q \) of the local displacement field \( u_n(x,t) \) takes the form
\[ u_n(x,t) = \sum_{N=1}^{N} \sum_{|\alpha| \leq q} u^{n}_{\alpha}(t) \varphi^{(q)}_{N}(x) \]

(4.7)

and in a similar manner higher order representations for \( y_n(x,t) \), \( \sigma_n(x,t) \) etc. can be constructed. The nodal quantities \( u^{n}_{\alpha}(t), y^{n}_{\alpha}(t), \ldots, \)
\( \tau^{n}_{\alpha}(t) \) are now functions of time, so that we can also construct a finite-
element approximation of these quantities over the "time element" (see
[11, p.153]).

Let \( K_n(A) \) denote the value of \( K_n(A) \) at time \( t=T \). We partition the
interval \([0,T]\) into partitions \( \Pi = [t_r, t_{r+1}] \) and represent each nodal
function \( u^{n}_{\alpha}(t), y^{n}_{\alpha}(t), \ldots, \tau^{n}_{\alpha}(t) \) by some appropriate difference
scheme in the time domain; e.g.
\[ u^{n}_{\alpha}(t) = \sum_{\beta=r}^{r+1} u^{n}_{\alpha}(t_{\beta}) \varphi^{(k)}_{\beta}(t) \]

(4.8)

where, for convenience, we have dropped the element label 'e'; \( t \in [t_r, t_{r+1}] \)
and \( \varphi^{(k)}_{\beta}(t) \) are the interpolation functions in time of degree \( k \), defined
so that
\[ \frac{d^m}{dt^n} \varphi^{(k)}_{\beta}(t) = \delta^{m}_{n} \delta_{\alpha \beta} \quad m,n = 0,1,2,\ldots,k-1 \]
\[ \alpha,\beta = 1,2 \]
and
\[ u^{n}_{N} = \frac{\partial^{n}}{\partial t^{n}} \Omega_{\alpha} u(x^{n}, t_{B}) \quad |\alpha| \leq q, \quad N = 1, 2, \ldots, N_{e} \tag{4.10} \]

Then, using the approximations of the form (4.7) and (4.10), local approximations of \( A_{\alpha} = \{ u_{\alpha}, \gamma_{\alpha}, \sigma_{\alpha}, q_{\alpha}, g_{\alpha}, \theta_{\alpha}, S_{\alpha}, \eta_{\alpha} \} \) are discretized into the form

\[
\begin{align*}
    u_{\alpha}(x, t) &= \sum_{|\alpha| \leq q} \sum_{n=0}^{k-1} \sum_{\beta=1}^{2} \sum_{N=1}^{N_{e}} u^{n}_{\alpha \beta} \psi_{T}^{n}(t) \psi_{N}^{\alpha}(x) \\
    \gamma_{\alpha}(x, t) &= \sum_{|\alpha| \leq q} \sum_{n=0}^{k-1} \sum_{\beta=1}^{2} \sum_{N=1}^{N_{e}} \gamma^{n}_{\alpha \beta} \psi_{T}^{n}(t) \psi_{N}^{\alpha}(x) \\
    \sigma_{\alpha}(x, t) &= \sum_{|\alpha| \leq q} \sum_{n=0}^{k-1} \sum_{\beta=1}^{2} \sum_{N=1}^{N_{e}} \sigma^{n}_{\alpha \beta} \psi_{T}^{n}(t) \psi_{N}^{\alpha}(x) \\
    \eta_{\alpha}(x, t) &= \sum_{|\alpha| \leq q} \sum_{n=0}^{k-1} \sum_{\beta=1}^{2} \sum_{N=1}^{N_{e}} \eta^{n}_{\alpha \beta} \psi_{T}^{n}(t) \psi_{N}^{\alpha}(x) \tag{4.11}
\end{align*}
\]

In order to obtain the discretized form of the functional \( K(\Lambda) \) of (3.5), we need to substitute (4.11) into (3.5). This substitution obviously leads to an extremely lengthy expression. However, to indicate the form of the resulting equations describing the discrete model and, at the same time, to keep the algebra within the reasonable limits, we shall consider here representations only of the first order (i.e. the first order representation of the local displacement field \( u_{\alpha}(x, t) = u^{n}_{\alpha \beta} \psi_{T}^{n}(t) \psi_{N}^{\alpha}(x) \)).

To proceed further, we note that the generalized force vector at node \( N \) is \( \bar{f}^{N} = \bar{f}_{B}(t) \). Thus, if we approximate \( \bar{f}^{N}(t) \) in time by

\[
\bar{f}^{N}_{\alpha}(t) = \sum_{|\alpha| \leq q} \sum_{n=0}^{k-1} \sum_{\beta=1}^{2} \sum_{N=1}^{N_{e}} \bar{f}_{\alpha \beta}^{n} \psi_{T}^{n}(t) \psi_{N}^{\alpha}(x)
\]

...
\[ f^H(t) = \tilde{\psi}_{\alpha}(t) f^{H\alpha} \] (4.12)

then

\[ f(x,t) = f^{H\alpha} \tilde{\psi}_{\alpha}(t) \tilde{E}_N(x) \] (4.13)

Substitution of (4.13) together with the first order approximations for \( \Lambda_r \) into (3.5), leads to the following local approximations of \( K(\Lambda) \):

\[
K_r(\Lambda) = \frac{1}{2} \left[ m_{uv} u_{\alpha} v_{\alpha} u_{\beta} v_{\beta} - 2 \sum_{j=1}^{n} u_{\alpha} v_{\alpha} u_{\beta} v_{\beta} \right] - 2 \sum_{j=1}^{n} u_{\alpha} v_{\alpha} u_{\beta} v_{\beta} - 2 \sum_{j=1}^{n} u_{\alpha} v_{\alpha} u_{\beta} v_{\beta} + 2 \sum_{j=1}^{n} u_{\alpha} v_{\alpha} u_{\beta} v_{\beta}
\]

\[ + 2 \sum_{j=1}^{n} u_{\alpha} v_{\alpha} u_{\beta} v_{\beta} + \cdots + \sum_{j=1}^{n} u_{\alpha} v_{\alpha} u_{\beta} v_{\beta} \]

\[ + 2 \sum_{j=1}^{n} u_{\alpha} v_{\alpha} u_{\beta} v_{\beta} - 2 \sum_{j=1}^{n} u_{\alpha} v_{\alpha} u_{\beta} v_{\beta} + \cdots + \sum_{j=1}^{n} u_{\alpha} v_{\alpha} u_{\beta} v_{\beta} \]

\[ - 2 \left[ T_0 \left( E_{uv} q_{uv} \partial_0 + E_{uv} q_{uv} \partial_0 + E_{uv} q_{uv} \partial_0 - T_0 E_{uv} q_{uv} \partial_0 \right) \right] 
\]

\[ + \sum_{j=1}^{n} u_{\alpha} v_{\alpha} u_{\beta} v_{\beta} + \cdots + \sum_{j=1}^{n} u_{\alpha} v_{\alpha} u_{\beta} v_{\beta} \]

\[ + 2 \sum_{j=1}^{n} u_{\alpha} v_{\alpha} u_{\beta} v_{\beta} - 2 \sum_{j=1}^{n} u_{\alpha} v_{\alpha} u_{\beta} v_{\beta} + \cdots + \sum_{j=1}^{n} u_{\alpha} v_{\alpha} u_{\beta} v_{\beta} \]

\[ + 2 \sum_{j=1}^{n} u_{\alpha} v_{\alpha} u_{\beta} v_{\beta} - 2 \sum_{j=1}^{n} u_{\alpha} v_{\alpha} u_{\beta} v_{\beta} + \cdots + \sum_{j=1}^{n} u_{\alpha} v_{\alpha} u_{\beta} v_{\beta} \]

\[ + 2 \sum_{j=1}^{n} u_{\alpha} v_{\alpha} u_{\beta} v_{\beta} - 2 \sum_{j=1}^{n} u_{\alpha} v_{\alpha} u_{\beta} v_{\beta} + \cdots + \sum_{j=1}^{n} u_{\alpha} v_{\alpha} u_{\beta} v_{\beta} \]

\[ + 2 \sum_{j=1}^{n} u_{\alpha} v_{\alpha} u_{\beta} v_{\beta} - 2 \sum_{j=1}^{n} u_{\alpha} v_{\alpha} u_{\beta} v_{\beta} + \cdots + \sum_{j=1}^{n} u_{\alpha} v_{\alpha} u_{\beta} v_{\beta} \]

(4.14)

where the matrix arrays and the terms \( a_{\alpha}^0 \), \( b_{\alpha}^0 \), etc are given in tables 1 and 2, and the first order representation used in (4.14) are given in Appendix A.
Table 1

Matrix Arrays

\[
\begin{align*}
\bar{E}_{uv} &= \int \phi(x) \psi(x) \, dB; \\
\bar{T}_{uv} &= \int \psi(x) \phi(x) \, dB; \\
\bar{T}{\mu}_{uv} &= \int \phi(x) \psi(x) \psi_{\lambda}(x) \, dB; \\
\bar{T}{\nu}_{uv} &= \int \psi(x) \phi(x) \psi_{\lambda}(x) \, dB; \\
\bar{F}_{uv} &= \int \psi(x) F(x) \, dB,
\end{align*}
\]

\[
\begin{align*}
A_{uv} &= \int \psi(x) \psi(x) \, dB; \\
B_{uv} &= \int \psi(x) \psi(x) \, dB; \\
C_{uv} &= \int \psi(x) \psi(x) \, dB; \\
D_{uv} &= \int \psi(x) F(x) \, dB,
\end{align*}
\]

\[
\begin{align*}
J_{uv} &= \int x(x) x(x) \, dB; \\
E_{uv} &= \int \psi(x) \psi(x) \, dB; \\
F_{uv} &= \int \psi(x) \psi(x) \, dB; \\
G_{uv} &= \int \psi(x) F(x) \, dB,
\end{align*}
\]

\[
\begin{align*}
\bar{L}_{uv} &= \int \tau(x) F_{\mu}(x) \, dB; \\
\bar{L}_{uv} &= \int \phi(x) \psi(x) \, dB; \\
\bar{L}_{uv} &= \int \psi(x) \phi(x) \, dB; \\
\bar{L}_{uv} &= \int \psi(x) F(x) \, dB,
\end{align*}
\]

\[
\begin{align*}
I_{uv} &= \int n(x) \psi(x) \psi(x) \, ds; \\
H_{uv} &= \int n(x) \psi(x) \psi(x) \, ds; \\
B_{uv} &= \int n(x) \psi(x) \psi(x) \, ds; \\
B_{uv} &= \int n(x) \psi(x) \psi(x) \, ds;
\end{align*}
\]

\[
\begin{align*}
H_{uv} &= \int F(x) \tau(x) \, ds; \\
H_{uv} &= \int F(x) \tau(x) \, ds; \\
H_{uv} &= \int F(x) \tau(x) \, ds.
\end{align*}
\]
Table 2

Coefficients

\[ \Phi^0_{\alpha \beta} = \Phi^0_{\alpha \beta} (t) \cdot q^0_{\beta} (t); \quad \Phi^1_{\alpha \beta} = g \cdot \Phi^0_{\alpha \beta} (t) \cdot q^0_{\beta} (t); \quad \Phi^2_{\alpha \beta} = g \cdot \Phi^0_{\alpha \beta} (t) \cdot q^0_{\beta} (t) \cdot q^0_{\beta} (t); \]

\[ \lambda^0_{\alpha} = g \cdot \lambda^0_{\alpha} (t); \quad \lambda^1_{\alpha} = \lambda^0_{\alpha} (t) \cdot q^0_{\beta} (t); \quad \lambda^2_{\alpha} = g \cdot \lambda^0_{\alpha} (t) \cdot q^0_{\beta} (t) \cdot q^0_{\beta} (t); \]

\[ \lambda^0_{\alpha} = g \cdot \lambda^0_{\alpha} (t); \quad \lambda^1_{\alpha} = \lambda^0_{\alpha} (t) \cdot q^0_{\beta} (t); \quad \lambda^2_{\alpha} = g \cdot \lambda^0_{\alpha} (t) \cdot q^0_{\beta} (t) \cdot q^0_{\beta} (t); \]

\[ \lambda^0_{\alpha} = g \cdot \lambda^0_{\alpha} (t); \quad \lambda^1_{\alpha} = \lambda^0_{\alpha} (t) \cdot q^0_{\beta} (t); \quad \lambda^2_{\alpha} = g \cdot \lambda^0_{\alpha} (t) \cdot q^0_{\beta} (t) \cdot q^0_{\beta} (t); \]

\[ h^0_{\alpha \beta} = g \cdot p^0_{\alpha \beta} (t) \cdot p^0_{\beta} (t); \quad h^1_{\alpha \beta} = g \cdot p^0_{\alpha \beta} (t) \cdot p^0_{\beta} (t); \quad h^2_{\alpha \beta} = g \cdot p^0_{\alpha \beta} (t); \quad h^3_{\alpha \beta} = g \cdot p^0_{\alpha \beta} (t); \]

\[ H^0_{\alpha \beta} = g \cdot h^0_{\alpha \beta} (t) \cdot h^0_{\beta} (t); \quad H^1_{\alpha \beta} = g \cdot h^0_{\alpha \beta} (t) \cdot h^0_{\beta} (t); \quad H^2_{\alpha \beta} = g \cdot h^0_{\alpha \beta} (t); \quad H^3_{\alpha \beta} = g \cdot h^0_{\alpha \beta} (t); \]

\[ J^0_{\alpha} = g \cdot J^0_{\alpha} (t); \quad J^1_{\alpha} = g \cdot J^0_{\alpha} (t) \cdot J^0_{\alpha} (t); \quad J^2_{\alpha} = g \cdot J^0_{\alpha} (t) \cdot J^0_{\alpha} (t); \quad J^3_{\alpha} = g \cdot J^0_{\alpha} (t) \cdot J^0_{\alpha} (t); \]

The constitutive functionals of stress, entropy, heat flux, internal dissipation and the inelastic strain-rate for the element \( \Theta \), appearing in (4.14) are obtained by using the procedure of Oden (see [11, p. 366]) and are defined as:

\[ \Xi^1_{N} [-] = \int_{\Theta} \Xi^1_{N} \left( x \right) \Xi^1_{N} \left[ \Gamma^N \right] d\Omega \]

\[ \Xi^0_{N} [-] = \int_{\Theta} \Xi^0_{N} \left( x \right) \Xi^0_{N} \left[ \Gamma^N \right] d\Omega \]

\[ \Xi^1_{N} [-] = \int_{\Theta} \Xi^1_{N} \left( x \right) \Xi^1_{N} \left[ \Gamma^N \right] d\Omega \]

\[ \Xi^0_{N} [-] = \int_{\Theta} \Xi^0_{N} \left( x \right) \Xi^0_{N} \left[ \Gamma^N \right] d\Omega \]
We then generate the complete system of equations governing the
discrete model by setting the first variation of the functional of
eq. (4.13) equal to zero, i.e. \[ \delta A = 0. \] Then

\[
\frac{\partial K_e}{\partial u_{\alpha}} = 0; \quad \frac{\partial K_e}{\partial \gamma_{ij}} = 0; \quad \frac{\partial K_e}{\partial \sigma_{\alpha}} = 0; \quad \frac{\partial K_e}{\partial \sigma_{\beta}} = 0;
\]

\[
\frac{\partial K_e}{\partial s_{\alpha}} = 0; \quad \frac{\partial K_e}{\partial \eta_{ij}} = 0; \quad \frac{\partial K_e}{\partial T_{\alpha}} = 0; \quad \frac{\partial K_e}{\partial q_{\alpha}} = 0; \quad \frac{\partial K_e}{\partial q_{\beta}} = 0 \quad (4.16)
\]

Equations (4.16) are the finite element analogues of the field
equations (2.1) - (2.5); e.g. equation (4.16) is the finite element
analogue of the local linear momentum equation

\[
\sum_{i=1}^{m} u_i \phi_{\alpha} = \int_{\Omega} \sigma_{\alpha} \phi_{\alpha} d\Omega - \int_{\Gamma} \sigma_{\alpha} \phi_{\alpha} d\Gamma + \int_{\Omega} \sigma_{\alpha} \phi_{\alpha} d\Omega = \sum_{i=1}^{m} u_i \phi_{\alpha} \quad (4.17a)
\]

and the traction boundary condition

\[
\sum_{i=1}^{m} u_i \phi_{\alpha} + \int_{\Omega} \sigma_{\alpha} \phi_{\alpha} d\Omega = B_{\alpha} \phi_{\alpha} = \sum_{i=1}^{m} u_i \phi_{\alpha} \quad (4.17b)
\]

and equations (4.17) - (4.17) correspond to the finite-element analogues
of the constitutive equations for,

the stress,

\[
A_{\alpha\beta} \dot{\sigma}_{\alpha\beta} = \sum_{s=0}^{\infty} \chi_{s=0}^{[s]} \phi_{\alpha} \quad (4.18)
\]

the internal dissipation,

\[
B_{\alpha\beta} \dot{b}_{\alpha\beta} = \sum_{s=0}^{\infty} \psi_{s=0}^{[s]} \phi_{\alpha} \quad (4.19)
\]

the entropy,
\[ C_{NM}^N \beta^B_{\alpha \beta} = \mathcal{Q}_N^N \left[ \beta^C_{\alpha} \right] \]
\[ (4.20) \]

the heat flux,
\[ D_{NM}^N \beta^B_{\alpha \beta} = \mathcal{Q}_N^N \left[ \beta^C_{\alpha} \right] \]
\[ (4.21) \]

and the inelastic strain rate,
\[ J_{NM}^N \beta^B_{\alpha \beta} = \mathcal{Q}_N^N \left[ \beta^C_{\alpha} \right] \]
\[ (4.22) \]

Equation (4.16) is the discrete analogue of the energy equation and the heat flux boundary condition i.e.
\[ (4.23a) \]
\[ (4.23b) \]

To derive the finite-element analogues of the compatibility equation and the displacement boundary conditions (4.16) and the thermal temperature-gradient relation together with the temperature boundary conditions (4.16), we apply Green-Gauss theorem to (3.5) to obtain the alternate form of the functional \( K(\lambda) \):
\[ (4.24) \]
Once again, following the procedure used in equation (4.14), we obtain the discrete model of \( \tilde{K}(\Lambda) \) of (4.24). We then consider the variation of \( \tilde{K}(\Lambda) \) with respect to \( \sigma^{1MB} \) and \( q^{1MB} \) in the interior of the body \( \beta \), as well with respect to \( T^{\text{NO}} \) and \( Q^{\text{NO}} \) on the boundaries \( \partial \beta_u \) and \( \partial \beta_d \), respectively. This yields the discrete compatibility equation

\[
\sum_{\nu}^{(1)} \int_{\beta} I_{MH}^{1} u_{\nu}^{\text{NO}} \frac{\partial}{\partial \beta} + \int_{\beta} I_{NR}^{1} u_{\nu}^{\text{NO}} \frac{\partial}{\partial \beta} + \int_{\beta} I_{HR}^{1} u_{\nu}^{\text{NO}} \frac{\partial}{\partial \beta} = 2A_{NM} \gamma_{\text{NO}} \frac{\partial}{\partial \beta} (4.25a)
\]

and the displacement boundary conditions

\[
u_{\nu}^{\text{NO}} = \hat{\gamma}_{\nu}^{\text{NO}} (4.25b)
\]

The discrete forms of the temperature thermal-gradient relation and the temperature boundary conditions are

\[
\sum_{\nu}^{(1)} \int_{\beta} I_{M}^{1} T^{\text{NO}} \frac{\partial}{\partial \beta} = D_{\nu}^{\text{NO}} \hat{T}_{\nu}^{\text{NO}} (4.26a)
\]

\[
T^{\text{NO}} = \hat{T}^{\text{NO}} (4.26b)
\]

where

\[
I_{NM}^{1} = \int_{\beta} \psi_{\nu, 1}(x) \hat{\psi}_{\nu}(x) d\beta ; I_{NR}^{1} = \int_{\beta} \psi_{\nu, 1}(x) \psi_{\nu}(x) \hat{\psi}_{\nu}(x) d\beta
\]

\[
J_{NM}^{1} = \int_{\beta} \tau_{\nu, 1}(x) F_{\nu}(x) d\beta
\]

and \( A_{NM}, D_{\nu}, \hat{\gamma}_{\nu}^{\text{NO}}, \hat{T}_{\nu}^{\text{NO}} \) etc. are defined in tables 1 and 2.

Thus equations (4.17) - (4.23) and (4.25) - (4.27), are the complete finite-element analogues of the field equations (2.1) - (2.5).

5. Some Special Cases

There exists a number of important special cases that fall within the general framework of the theory developed thus far. In this section,
we shall discuss some of these cases which can be derived from the general formulation presented in Section 4.

We note that by setting \( \mathbf{I} = 0 \) and deleting the constitutive equation (2.5c) from the collection of field equations (2.1) - (2.5), equations (4.16) reduce to the finite-element analogues of the field equations governing the behavior of nonlinear thermoviscoelasticity. Furthermore, if only the approximations in displacement and temperature fields are considered, then we obtain discrete models of thermomechanical behavior of materials with memory presented by Oden and Ramírez [26].

Consider now the static thermoelastic case when terms involving the time rates and histories can be neglected. Then the functional \( K(\hat{\lambda}) \) of (3.5) reduces to

\[
K_1(\hat{\lambda}) = \frac{1}{2} \int \left[ 2 \gamma_1 \mathcal{F}^J(\gamma, \theta) - 2 \gamma_1 \sigma^J - \mathbf{s}^2 + 2 \mathbf{s} \mathbf{g}(\gamma, \theta) - 2 \mathbf{u}_n \sigma_n^{(i)} - 2 \mathbf{u}_n \rho f_n \right] \mathbf{d}S \\
- \mathbf{u}_n (\sigma_n^{(i)} u_{n,i}), \quad \mathbf{S}^2 + 2 \mathbf{s} \mathbf{g}(\gamma, \theta) - 2 \mathbf{q}_1 \mathbf{g}_1 + 2 \mathbf{g}_1 \mathcal{G}^J(\gamma, \theta, \mathbf{g}) \\
- 2 (\mathbf{q}_1 + \rho \mathbf{h} + \mathbf{G}) \mathbf{d}S + \frac{1}{2} \int \left[ \left[ n_1 \mathbf{r}^{(i)} (u_{n,i} + 2 \hat{\delta}_n) - 2 \mathbf{T}_n \right] \mathbf{u}_n \right] \mathbf{d}S \\
+ \int \mathbf{u}_n T_n dS + \int (\theta - \hat{\theta}) Q dS + \int Q \mathbf{d}S \\
(5.1)
\]

wherein \( \mathcal{F}^J(\cdot) \), \( \mathbf{s}(\cdot) \), \( \mathbf{g}(\cdot) \) and \( \mathcal{G}^J(\cdot) \) are now functions of the current values of their respective arguments. Then introducing the approximations (4.1) - (4.1) into (5.1) and noting that the nodal values \( u^n, \mathbf{u}_n, \ldots, S^n \) are independent of time, we obtain, analogous to (4.14), a function:
In (5.2), the functions \( \tilde{\mathbf{T}}_u^{(i)}(\cdot) \), \( \Theta_u(\cdot) \), \( \mathcal{Y}_u(\cdot) \) and \( \mathcal{S}_u^{(i)}(\cdot) \) for the element \( \Omega \) are defined in a similar manner as those in (4.15) except that, in this case, these are functions of the current values of \( \gamma \) and \( \Theta \) rather than the functionals of their histories. Thus, once again, the vanishing of the first variation of \( K_u \) in (5.2) yields the finite-element analogues of the field equations governing the behavior of nonlinear thermoelastic solid. We omit the details here.

Now consider the case of linearized thermoviscoelasticity in which the displacement field \( \mathbf{u}(x,t) \) and the temperature field \( \Theta(x,t) \) are kinematically and thermally admissible (i.e. the displacements \( \mathbf{u}(x,t) \) and temperature \( \Theta(x,t) \) satisfy the strain-displacement relation, temperature thermal-gradient relation and the constitutive equations). Then the functional of (3.5) yields

\[
\mathcal{H}[\mathbf{u}, \Theta] = \frac{1}{2} \int \left[ \mathbf{u} \cdot \mathbf{f}_u + 2g \mathbf{u} \cdot \mathbf{f}_u + \mathbf{g} \mathbf{u} \cdot \mathbf{f}_u \right] d\Omega - \int g \cdot \mathbf{f}_u \cdot \mathbf{d} \Theta - \int g \cdot \mathbf{f}_u \cdot \mathbf{d} \Theta - \int g \cdot \mathbf{f}_u \cdot \mathbf{d} \Theta
\]

(5.3)
n arriving at (5.3) from (3.5) we have set the internal dissipation function $\gamma$ equal to zero and $S^{1^j}_r = \int_{\mathcal{B}_e} \gamma^{1^j}_r[y^t_r, \Theta_t, \Sigma_t] \, d\Omega$, and $\bar{S} = \mathcal{S}_{\gamma^1}(y^t_r, \Theta_t, \Sigma_t)$ are linear in the strain-history $y^t_r$ and the temperature history $\Theta_t$. We note that the functional $\mathcal{H} \{ \cdot \}$ of (5.3) is identical with (IV.1) of [14].

Now substituting the first order local approximations (4.11) for displacement field $u^e(x, t)$ and the temperature field $\Theta^e(x, t)$ into the functional $\mathcal{H} \{ \cdot \}$ of (5.3), we obtain the discretized form which is identical to that of [14]:

$$\mathcal{H}_r \{ u^e_r, \Theta^e_r \} = \frac{1}{2} \sum_{n, j \in \mathcal{N}} \int_{\mathcal{B}_e} \frac{1}{2} \tilde{\tau}^{1^j}_{n, j} \left[ u^N_t, T^N_t, u^N_t, T^N_t \right] \, d\Omega$$

$$+ \int_{\mathcal{B}_e} \left[ u^N_t, T^N_t \right] \mathcal{S}^0 \, d\Omega - \sum_{n, j \in \mathcal{N}} \int_{\mathcal{B}_e} \frac{1}{2} \tilde{\tau}^{1^j}_{n, j} \left[ u^N_t, T^N_t, u^N_t, T^N_t \right] \, d\Omega$$

$$- \sum_{n, j \in \mathcal{N}} \int_{\mathcal{B}_e} \frac{1}{2} \tilde{\tau}^{1^j}_{n, j} \left[ u^N_t, T^N_t, u^N_t, T^N_t \right] \, d\Omega$$

(5.4)

Here the functionals of stress, entropy and heat flux for the elements $\Theta^e_r$ are defined in a similar manner as those in (4.15); i.e.

$$\tilde{\tau}^{1^j}_{n, j} \{ \cdot \} = \int_{\mathcal{B}_e} \psi_n(x) \, \tilde{\tau}^{1^j}_{n, j} \left[ u^N_t, T^N_t, u^N_t, T^N_t \right] \, d\Omega$$

$$\Theta^e_r \{ \cdot \} = \int_{\mathcal{B}_e} \psi_n(x) \, \Theta^e_r \left[ u^N_t, T^N_t, u^N_t, T^N_t \right] \, d\Omega$$

$$\mathcal{S}^0 \{ \cdot \} = \int_{\mathcal{B}_e} \mathcal{S}^0 \left[ u^N_t, T^N_t, u^N_t, T^N_t \right] \, d\Omega$$

(5.5a, 5.5b, 5.5c)

The quantities $m_{nm}, \mathcal{P}_{nm}, \mathcal{S}^0_{\alpha\beta}, \mathcal{Q}^0_{\alpha}, \mathcal{S}^0_{\alpha\beta}$ etc. appearing in (5.4) are defined in tables (1) - (2).
As a final comment, we note that certain of the mixed or hybrid finite-element models of Pian and Tong [7-10], based on Hellinger-Reissner principle can also be obtained from the general formulations presented in Section 4 of this paper. The Hellinger-Reissner principle for finite deformations of elastic bodies follows immediately from (4.24) by ignoring all thermal effects, assuming each of the remaining mechanical constitutive equations are satisfied, ignoring rate terms in (2.1) - (2.5) (and, hence, avoiding the necessity of the operation $g^*)$, and setting

$$2Y_{ij}(\varepsilon_{ij}^f - \sigma_{ij}) = -2B(\sigma_{ij})$$

Here $B(\sigma_{ij})$ is the complementary energy density; i.e. if $A(Y_{ij})$ is the strain energy function,

$$B = \sigma_{ij}Y_{ij} - A$$

Incorporating all of these restrictions into (4.24), we obtain

$$\Pi_r = \int_{\Omega} \left[ -B(\sigma_{ij}) + \frac{1}{2} \sigma_{ij}^f(u_{i,j} + u_{j,i} + u_{s,i}u_{s,j}) - u_iF_i \right] d\Omega$$

$$- \int_{\partial\Omega} (u_i - \hat{u}_i) T_i dS - \int_{\partial\Omega} \sigma_{ij} \frac{\partial u_i}{\partial \theta_i} dS$$

(5.6)

Note that by neglecting the second order terms in (5.6), and then substituting the local approximations for $u_{ij}(\xi)$ and $\sigma_{ij}(\xi)$, we obtain the mixed models which are essentially the same as those of Pian and Tong.

A number of other alternate variational models can also be precipitated from the general principles presented herein. Since the general procedure for constructing these has been adequately demonstrated, we shall not elaborate further.
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The first order local approximations of $h_{\alpha}$ in (4.11) are discretized in the following form:

\[ u_{\alpha}(x,t) = u^{N\alpha}_{\psi}(x)\varphi^0_\alpha(t); \quad \varphi_{\alpha}(x,t) = \varphi^{N\alpha}_{\psi}(x)\varphi^0_\alpha(t) \]
\[ q_{\alpha}(x,t) = q^{N\alpha}_{\psi}(x)\varphi^0_\alpha(t); \quad \varphi_{\alpha}(x,t) = \varphi^{N\alpha}_{\psi}(x)\varphi^0_\alpha(t) \]
\[ g_{\alpha}(x,t) = g^{N\alpha}_{\psi}(x)\varphi^0_\alpha(t); \quad \varphi_{\alpha}(x,t) = \varphi^{N\alpha}_{\psi}(x)\varphi^0_\alpha(t) \]
\[ \theta_{\alpha}(x,t) = T_0 + T^{N\alpha}_{\psi}(x)\varphi^0_\alpha(t); \quad S_{\alpha}(x,t) = S^{N\alpha}_{\psi}(x)\varphi^0_\alpha(t) \]
\[ \Pi_{\alpha}(x,t) = \Pi^{N\alpha}_{\psi}(x)\varphi^0_\alpha(t) \]  \hspace{1cm} (A-1)

In addition to (A-1), we also approximate locally the prescribed functions for displacements $\hat{u}(x,t)$, tractions $\hat{T}(x,t)$, temperature $\hat{\theta}(x,t)$, heat flux $\hat{Q}(x,t)$ of (2.4) and the heat supply $h(x,t)$ by the following

\[ \hat{u}_{\alpha}(x,t) = \hat{u}^{N\alpha}_{\psi}(x)\varphi^0_\alpha(t); \quad \hat{T}_{\alpha}(x,t) = \hat{T}^{N\alpha}_{\psi}(x)\varphi^0_\alpha(t) \]
\[ \hat{\theta}_{\alpha}(x,t) = \hat{T}^{N\alpha}_{\psi}(x)\varphi^0_\alpha(t); \quad \hat{Q}_{\alpha}(x,t) = \hat{Q}^{N\alpha}_{\psi}(x)\varphi^0_\alpha(t) \]
\[ h(x,t) = h^{N\alpha}_{\psi}(x)\varphi^0_\alpha(t) \]  \hspace{1cm} (A-2)
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General mixed finite element models of the nonlinear thermomechanical response of dissipative media are constructed. A number of existing specialized finite element models are derived from the general formulation presented in this paper.
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