Mechanical action of a continuous body does not manifest itself in mechanical effects alone. When the external forces acting on a body perform work, the body gets hotter. Conversely, supplying heat to a body results in motion. If the process is reversed, not all of the mechanical energy supplied is generally recoverable. The process is then an irreversible one; and, as such, its description may fall well outside the realm of both classical mechanics and classical thermodynamics. Indeed, although the interconvertibility of mechanical work and heat was recognized in the nineteenth century by Joule, works on thermodynamics have, until recent years, treated the bulk of thermodynamic processes both as reversible and as completely uncoupled with the mechanical behavior. Likewise, classical solid and fluid mechanics, as a whole, ignore the fact that viscous behavior, coupled heat conduction, and plasticity belong to the realm of irreversible thermodynamics; and many phenomena associated with continuous bodies that have traditionally been treated with reasonable accuracy as reversible, cease to be so when finite deformations are taken into account (for example, finite deformations of metals).

Although the development of a rational theory of thermodynamics of continuous media has lagged the development of the more familiar special theories by over a century, a significant step toward a general thermodynamics of continuous media was made in 1964 by Coleman [1], who presented a consistent and rational theory applicable to simple materials with memory. More recently, additional refinements of Coleman’s theory have been discussed by Laws [2], Truesdell and Noll.
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[3], Truesdell [4, 5], and Muller [6], among others. Although much experimental work remains to be done, it may be said that sufficient basis for the theory is now available to apply it to a wide class of problems involving nonlinear behavior of continuous media.

Applications of modern theories of irreversible thermodynamics to realistic physical problems very quickly exceed the limits of classical methods of analysis. The mathematical analysis of the thermomechanical behavior of continuous bodies of arbitrary shape and general boundary and initial conditions, complicated by the presence of finite deformations and time-dependent material properties, constitutes one of the most difficult classes of problems in applied physics. It appears that only through the use of modern numerical techniques is there hope of obtaining quantitative solutions to problems of this type.

Toward this end, several applications of the finite element concept have been made to the development of discrete models of nonlinear behavior of continua. Following the pioneering paper of Turner, Clough, Martin, and Topp [7], applications of the method to simple, geometrically nonlinear problems were discussed by Turner, Dill, Martin, and Melosh [8], among others. Summary accounts of applications to geometrically nonlinear problems involving infinitesimal strains were given by Martin [9, 10], Argyris [11, 12], Zienkiewicz [13], Przemieniecki [14], and Marcal [15], among others; and several solutions to problems in classical thermoelasticity and elasto-plasticity have been presented (for example, by Gallagher, Padlog, and Bijlaard [16], Akyuz and Merwin [17], Wilson [18], Felippa [19], Pope [20], Marcal and King [21], Zienkiewicz, Valliappan, and King [22], and Yamada, Yoshimura, and Sakurai [23]). Finite element formulations of problems of finite deformation of elastic and thermoelastic solids have been given by Oden et al [24–30] and Becker [31]. Applications to nonlinear viscoelasticity [32–34], heat conduction and coupled thermoelasticity [35–37], and potential flow [38–40] have also been presented. There does not appear to be available attempts at developing discrete models of general thermomechanical behavior of both solids and fluids.

In the present paper, we consider the problem of constructing general finite element models of finite deformation and irreversible thermodynamics of nonlinear continua. A major objective of the developments presented here is generality, in the sense that the resulting formulations represent discrete models of a wide range of problems in both solid and fluid mechanics. To achieve this, both spatial and material descriptions of the local displacement, velocity, temperature, and density fields are introduced: the former depicting the finite element as a subregion of
three-dimensional euclidean space through which the continuum flows, and the latter depicting the element as a material collection of particles specified in some reference configuration. Since most of the primary dependent variables are kinematic in nature, it is shown that the spatial forms of the finite element equations are complicated by the presence of nonlinear convective terms. Conversely, the material description is complicated by the fact that forces and heat fluxes are applied on a material surface in the deformed element.

Once the basic kinematic equations for finite elements have been developed, we examine the essential aspects of continuum thermodynamics. By considering balances of energy for typical finite elements, we obtain spatial and material forms of the general equations of motion and heat conduction of finite elements. In the case of compressible fluids, we show that the equations of motion must be supplemented by finite-element analogues of the spatial form of the equation of continuity, the mass density \( \rho \) being an additional unknown in the problem.

We then examine finite element formulations of a number of special cases that can be obtained from the general finite element equations. These include thermomechanically simple materials with memory, finite coupled thermoelasticity, finite elasticity, dynamic coupled thermoelasticity, classical elasticity, transient heat conduction, coupled thermoviscoelasticity, and compressible and incompressible Stokesian fluids. The latter equations represent finite-element models of the Navier-Stokes equations of fluid dynamics.

We conclude the investigation by presenting incremental forms of the finite element equations for the special case of finite displacements of elastic solids. We show that these exhibit the well-known initial stress and displacement matrices used in the stability and nonlinear analysis of elastic and elastoplastic structures.

**KINEMATIC PRELIMINARIES**

We consider the motion of a continuous body. To identify the configuration of the body at a given time, we assign to its particles the labels \( X^i (i = 1,2,3) \). The quantities \( X^i \) are referred to as intrinsic or convected coordinates, and we interpret them as being etched onto the body and to move with it as the body deforms. The motion of the body is a continuous one-parameter family of configurations \( C_t \), and the parameter \( t \) is associated with time. We shall refer to the configuration \( C_0 \) corresponding to \( t = 0 \) as the reference configuration. The configuration at \( t = \varepsilon \) is denoted \( C_\varepsilon \) and is termed the current configuration.
For simplicity, we shall assume that, when the body occupies its reference configuration $C_0$, the intrinsic particle labels $X^I$ are rectangular cartesian. However, at all other times $\tau > 0$ the coordinates lines $X^I$ will be, in general, curvilinear. When the body is in its reference configuration, we also establish a spatial reference frame $x_1$ which is rectangular cartesian and which, again for simplicity, coincides with $X^I$ at $\tau = 0$. We regard the frame $x_1$ as inertial; that is, it is absolutely fixed in space (alternately, we may give the $x_1$ any time-dependent rigid translation relative to $C_0$). It is important to realize that the numbers $X^I$ identify a particle, whereas the numbers $x_1$ identify a place in the three-dimensional space through which the body moves.

Mathematically, we can describe the motion of a particle $X^I$ by equations of the form

$$x_1 = x_1 (X^1, X^2, X^3, t) \quad (1)$$

Equation (1) defines the position of a particle at time $t$ in terms of its position in the reference configuration. The components of displacement $u_1$ of a particle originally at $X^I$ are given by

$$u_1 = x_1 - X^I \quad (2)$$

The components of deformation gradient, denoted $F_{ij}$ are defined by

$$F_{ij} = \frac{\partial x_i}{\partial x_j} = u_{1,i} + \delta_{i,j} \quad (3)$$

where the comma denotes partial differentiation with respect to $X^J$ (i.e., $u_{1,i} \equiv \partial u_1 / \partial X^J$) and $\delta_{i,j}$ is the Kronecker delta.

A measure of the degree of deformation at a point in the body is provided by the square of the material element of arc $ds^2$ in the deformed body;

$$ds^2 = G_{ij} dX^i dX^j \quad (4)$$

where $G_{ij}$ is Green's deformation tensor:

$$G_{ij} = F_{im} F_{nj} \quad (5)$$

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For our purposes, it is often more convenient to use the Green-Saint Venant strain tensor

\[ \gamma_{i,j} = \frac{1}{2} (G_{i,j} - \delta_{i,j}) = \frac{1}{2} (u_{i,j} + u_{j,i} + u_{k,i} u_{k,j}) \]  

(6)

Since the same label \( x^i \) identifies a given particle at all times, forms of measures of the rates of deformation are particularly simple when the present description of motion is used. Components of velocity are

\[ v_1 = v_1(x) = \frac{\partial u_1}{\partial t} = \dot{u}_1 \]

(7)

and rates-of-strain are given by

\[ \dot{\gamma}_{i,j} = \frac{\partial \gamma_{i,j}}{\partial t} = \frac{1}{2} (v_{i,j} + v_{j,i} + u_{k,i} v_{k,j} + v_{k,j} u_{k,j}) \]

(8)

Higher-order strain rates are obtained by repeated partial differentiations with respect to time.

**Eulerian Forms**

In the study of the motion of fluids, it is usually meaningless to trace the motion of a specific particle. Motion is then described by establishing a fixed reference frame (the \( x^i \)) and viewing the motion of the continuum through points in three-dimensional space. Then we use the Almansi-Cauchy strain sensor

\[ e_{i,j} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right) \]  

(9)

Since we now write all quantities as functions of \( x^i \) rather than \( x^1 \), time rates become more complicated:

\[ v_i = v_i(x) = \frac{\partial u_i}{\partial t} + v_a \frac{\partial u_i}{\partial x_a} \]

(10)

\[ \dot{e}_{i,j} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - e_{k,i} \frac{\partial v_k}{\partial x_j} - e_{k,j} \frac{\partial v_k}{\partial x_i} \right) \]  

(11)

etc.
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MOTION OF A FINITE ELEMENT

We shall now consider a discrete model of the continuum which consists of a collection of a finite number of finite elements connected appropriately together at various nodal points. Since the process of connecting elements together to form such a nodal is well documented and is based on purely topological properties of the model, we shall not elaborate on it here. It suffices to point out that two distinct types of finite element model can be developed for continuous media: the material model and the spatial model. The material model is viewed as an actual collection of physical bodies connected continuously together at nodal points. The nodal points themselves are material particles X_i. In the spatial model, the space through which the media moves is regarded as a connected set of subregions, each representing a finite element. Nodal points then represent places X_i in the three-dimensional euclidean space. In either case, a finite element is regarded as a subdomain of a field quantity: in the material description, the domain is a specified collection of particles; and, in the spatial case, the domain is a sub-region of E^3. It follows that the material model is used for material descriptions of motion and the spatial model is used for spatial descriptions. Moreover, in either case, typical finite elements can be isolated from the collection and appropriate fields can be described locally over an element, independently of the ultimate location of the element in the connected model. This latter property has been referred to as the fundamental property of finite element models [41].

MATERIAL MODEL

We consider a typical finite element e of a continuum, which, for our present purposes, we regard as a subdomain of the displacement field u_1 (and later, the temperature field T). A finite number N_e of material particles are identified in the element while in its reference configuration. These are called nodes and are further labeled X_N where N = 1, 2, ..., N_e. The values of the displacement field at the nodes are then

\[ u_{N_1} = u_1(X_N) \]  \hspace{1cm} (12)

Following the usual procedure, we now approximate the displacement field locally over the element by functions of the form

\[ u_1 \approx \psi^N(X) u_{N_1} \]  \hspace{1cm} (13)
where here (and henceforth) the repeated nodal index is summed from 1 to \( N_e \). The interpolation functions \( \psi^N(x) \) have the property that

\[
\psi^N(x) = \sum_{N=1}^{N_e} \psi^N(x) = 1
\]  

(14)

Moreover, \( \psi^N(x) \geq 0 \) for all particles \( x \) belonging to the element, and (13) is designed so that the local fields \( u_1 \) are uniquely defined over the element in terms of the nodal values \( u_{N_1} \). It is understood in (13) that \( u_{N_1} \) are functions of time.

With the displacement field approximated, it is now a simple matter to calculate all other quantities needed to define the motion of the element

\[
x_1 = x^1 + \psi^N(x) u_{N_1}
\]

\[
F_{i,j} = \delta_{i,j} + \psi^N_{i,j} u_{N_1} \\
v_1 = \frac{\partial u_1}{\partial t} = \psi^N u_{N_1}
\]

\[
2 \gamma_{i,j} = \psi^N_{i,j} u_{N_1} + \psi^N_{j,i} u_{N_1} + \psi^N_{i,j} \psi^N_{j,k} u_{N_k} u_{M_k}
\]

\[
2 \dot{\gamma}_{i,j} = \psi^N_{i,j} \dot{u}_{N_1} + \psi^N_{j,i} \dot{u}_{N_1} + \psi^N_{i,j} \psi^N_{j,k} (\dot{u}_{N_k} u_{M_k} + u_{N_k} \dot{u}_{M_k})
\]  

(15)

and so forth.

**SPATIAL MODEL**

For the spatial model, we consider a bounded subdomain of \( \mathbb{E}^3 \) in which we identify a number \( N_e \) of places called nodes and labeled \( x^1 \). Although it is a simple matter to construct spatial approximations of the displacement field, it is, for our purposes, more natural to begin with approximations of the velocity field \( v_1 \):

\[
v_1 \approx \psi^N(x) v_{N_1}
\]  

(16)

where \( v_{N_1} = v_1(x_N) \) are the values of the local velocity field at the nodal places \( x^1 \) and the interpolation functions \( \psi^N(x) \) have the same properties described in (14) except that they are cast in terms of the spatial coordinates \( x^1 \).
Other kinematic variables are computed as follows:

\[ a_i = \psi^N v_{N1} + \psi^N \frac{\partial \psi^N}{\partial x_i} v_{M2} v_{N1} \]

\[ \dot{e}_{ij} = \frac{1}{2} \left( \frac{\partial \psi^N}{\partial x_i} v_{Nj} + \frac{\partial \psi^N}{\partial x_j} v_{N1} - e_{ij} \frac{\partial \psi^N}{\partial x_j} v_{Nk} - e_{jk} \frac{\partial \psi^N}{\partial x_i} v_{Nk} \right) \]  

(17)

etc., where \( a_i \) is the \( i \)th component of the acceleration. Notice that the convective terms (e.g., \( v_m \frac{\partial v_1}{\partial x_m} \), etc.) complicate the eulerian description of the motion of a finite element.

TEMPERATURE FIELDS

In the study of thermomechanical phenomena, finite element models of the temperature field are also needed. Let \( \theta \) denote the absolute temperature and \( T_0 \) denote a uniform reference temperature that is independent of time. Then \( \theta \) is written as the sum of \( T_0 \) and \( T \), where \( T \) is the change in temperature.

For the local approximation of \( T \) over a finite element, we have for a material model:

\[ T \approx \Psi^N(x) T_N \]  

(18)

or, for a spatial model,

\[ T \approx \Psi^N(x) T_N \]  

(19)

where the functions \( \Psi^N(x) \) are identical to those described earlier and \( T_N = T(X_N) \) or \( T(X_N) \) are the nodal temperatures. Then

\[ \theta = T_0 + \Psi^N T_N \]  

(20)

Time rates of change of temperature are then, with \( T_N = \frac{dT_N}{dt} \),

\[ \dot{T} = \Psi^N(X) \dot{T}_N \]  

(21a)

or

\[ \dot{T} = \Psi^N(X) \dot{T}_N + \Psi^N v_M \frac{\partial \Psi^N}{\partial x_m} T_N \]  

(21b)

for the material and spatial finite element models, respectively.
In the following developments, we shall largely confine our attention
to material descriptions of the deformation of a continuous body. Spa-
tial forms will then be stated without detailed derivations.

The thermomechanical behavior of an arbitrary continuum is govern-
ed by five physical laws: (1) the conservation mass, (2) balance of
linear momentum and (3) angular momentum, (4) the conservation of
energy, and (5) the Clausius-Duhem inequality. For material descrip-
tions, mass and angular momentum are usually (but not always [42])
assumed to be satisfied a priori for a finite element. Linear momentum
is balanced globally for a finite element, but is balanced only in an
average sense throughout an element. Local and global forms of the
law of conservation of energy provide a natural means for developing
the equations governing the motion of a typical element, while the
Clausius-Duhem inequality provides bounds on the entropy produc-
in an element as well as restrictions on the nature of the constitutive
equations describing the material of which the element is composed.

Consider a material volume $\mathcal{V}$ of mass density $\rho$
bounded by a surface of area $A$. The global form of the law of conservation of energy for
this volume is, considering only thermomechanical behavior,

$$
\dot{\kappa} + \dot{U} = \Omega + Q
$$

(22)

where $\kappa$ is the kinetic energy, $U$ the internal energy, $\Omega$ the mechanical
power, and $Q$ the heat energy.

$$
\kappa = \frac{1}{2} \int_{\mathcal{V}} \rho \dot{\mathbf{u}}_1 \cdot \dot{\mathbf{u}}_1 \, d\mathbf{u} \\
U = \int_{\mathcal{V}} \rho \varepsilon \, d\mathbf{u}
$$

$$
U = \int_{\mathcal{V}} \rho F_1 \dot{\mathbf{u}}_1 \, d\mathbf{u} + \int_{A} \mathbf{S}^\top \dot{\mathbf{u}}_1 \, dA
$$

$$
Q = \int_{A} q^\top \mathbf{n}_1 \, dA + \int_{\mathcal{V}} \rho h \, d\mathbf{u}
$$

(23)

Here we have referred all quantities to the reference configuration $C_0$:
$\rho$ is the mass per unit volume $\mathcal{V}$ in the reference configuration, $\varepsilon$ is the
internal energy density, \( F_1 \), are the body forces per unit mass in the "undeformed" body, and \( h \) is the heat supply per unit mass in \( C_0 \). The quantities \( S^1 \) and \( q^1 \) are components of surface traction and heat flux per unit initial area referred to convected coordinates \( X^1 \) in the current configuration. These are unavoidably available only after some motion of the body takes place and are, therefore, functions of the displacement gradients \( u_{1,1} \). We explore this problem more thoroughly later.

If mass is conserved and the principle of balance of linear momentum is assumed to hold, then, provided certain continuity requirements are satisfied, it can be shown that the local form of the law of conservation of energy is

\[
\rho \dot{e} = \sigma^{1,j} \gamma_{1,j} + q^1_{;1} + \rho h
\]  

(24)

where \( \sigma^{1,j} \) are the contravariant components of stress per unit of deformed area referred to the intrinsic coordinates \( X^1 \) and the semi-colon denotes covariant differentiation with respect to the \( X^1 \).

Upon introducing (24) into (22) and making use of the Green-Gauss theorem, we obtain the alternate global form

\[
\int_{\Omega} \rho \dddot{u}_1 \, du + \int_{\Omega} \sigma^{1,j} \gamma_{1,j} \, du = \Omega
\]  

(25)

ENTROPY, FREE ENERGY, DISSIPATION

The entropy content of a body is defined by introducing the specific entropy \( \hat{n} \). The total entropy production \( \Gamma \) is then

\[
\Gamma = \int_{\Omega} \rho \dddot{\hat{n}} \, du - \int_{A} \frac{\dot{q}}{\theta} \cdot n \, dA - \int_{\Omega} \rho \dot{h} \, du
\]  

(26)

where \( \theta \) is the absolute temperature. According to the Clausius-Duhem inequality, \( \Gamma \geq 0 \), or locally,

\[
\rho \dot{\hat{n}} - q^1_{;1} - \rho h + \frac{1}{\theta^2} q^1_\theta \geq 0
\]  

(27)

It is convenient to introduce the Helmholtz free energy density \( \varphi \) and the internal dissipation \( \sigma \):

\[
\varphi = -\frac{1}{\theta} \left( \frac{\partial \varphi}{\partial \theta} \right) - \frac{1}{2} \frac{\partial \varphi}{\partial \chi^1}
\]  

\[
\sigma = \frac{1}{\theta} \frac{\partial \varphi}{\partial \theta} + \frac{1}{2} \frac{\partial \varphi}{\partial \chi^1}
\]
FINITE DEFORMATION AND IRREVERSIBLE THERMODYNAMICS

\[ \phi = \varepsilon - \eta \theta \quad \quad \sigma = \sigma^{i,j} \gamma_{1,j} - \rho \left( \dot{\psi} + \eta \dot{\theta} \right) \]  
(28)

Then

\[ \sigma + \frac{1}{\theta} \left[ q^{i} \theta_{j} \right] \geq 0 \]  
(29)

and (24) can be written in the alternate forms

\[ \rho \dot{\psi} = \sigma^{i,j} \gamma_{1,j} - \rho \eta \dot{\theta} - \sigma \]  
(30)

\[ \rho \theta \dot{\theta} = q^{i} + \rho h + \sigma \]  
(31)

Another useful form of the energy balance is obtained from (31) after multiplying both sides by the temperature increment \( T \):

\[ \rho \theta \dot{\theta} = (q^{i} T)_{,j} - q^{i} T_{,j} + \rho T h + \sigma T \]  
(32)

EQUATIONS OF MOTION AND HEAT CONDUCTION OF A FINITE ELEMENT

MATERIAL FORMS

We now consider a typical finite element of the continuum on which the displacement, velocity, and temperature fields are given by (12), (15)a, and (20). Introducing (12) into (25) and simplifying, we obtain for the general balance of mechanical energy for the element [33, 41]

\[ 0 = u_{Nk} \left[ m^{NM} \dot{u}_{M} + \int_{V_{e}} \sigma^{i,j} \psi^{N}_{,j} \left( \delta_{k}^{i,j} + \psi_{,j} \psi_{M} \right) \right] - p_{k}^{N} \]  
(33)

where \( m^{NM} \) is the consistent mass matrix and \( p_{k}^{N} \) are the consistent generalized forces for the element:

\[ m^{NM} = \int_{V_{e}} \rho \psi^{N} \psi^{M} \right]  
(34)
ANALYSIS OF FLOW AND SPECIAL PROBLEMS

\[ p_k^N = \int_{V_0} \rho F_k \psi^N \, dV + \int_{A_0} S_k \psi^N \, dA \] (35)

Here \( V_0 \) and \( A_0 \) are the volume and the surface area of the element in the reference configuration \( C_0 \). The quantities \( \psi^N \) are functions of the material coordinates \( X^i \). Further, we are again reminded that (35) is, in one sense, only symbolic since the surface tractions \( S_k \) are themselves functions of the nodal displacements. We examine this aspect of the equations in the following section.

Since (33) represents a general law of balance of energy for the element, it must hold for arbitrary nodal velocities. Thus, the term within brackets must vanish, and we arrive at the general equations of motion of a finite element:

\[ m^{NM} \ddot{u}_{M_k} + \int_{V_0} \sigma^{ij} \psi^N_{,i} (\delta_{k,j} + \psi^N_{,j} u_{M_k}) \, dV = p_k^N \] (36)

Observe that (1) no restrictions have been placed on the order-of-magnitude of the deformations \(-(36)\) holds for finite deformations of the element \-(2) no restrictions have been placed on the material of which the element is composed \-the constitution of the material is reflected in the constitutive equation for stress which is, as yet, unspecified.

To obtain the corresponding equations of heat conduction, we turn to (18) and (20) which we introduce into (32). Upon simplifying, making use of the Green-Gauss theorem, and requiring that the result hold for arbitrary nodal temperatures, we obtain the general equation of heat conduction for a finite element of a continuum [30, 34]:

\[ \int_{V_0} \rho (T_0 + \psi^N T_M) \psi^N \, dV + \int_{V_0} q^i \psi^N_{,i} \, dV = q^N + \sigma^N \] (37)

where \( q^N \) and \( \sigma^N \) represent the generalized normal heat flux and the generalized nodal dissipation at node \( N \):
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\[ q^N = \int_{A^e} \psi^N q^1 n_1 \, dA + \int_{\Omega^e} \rho \psi^N \, d\mathbf{v} \]  \hspace{1cm} (38)

\[ \sigma^N = \int_{\Omega^e} \sigma \psi^N \, d\mathbf{v} \]  \hspace{1cm} (39)

Again, no restriction is placed on the magnitudes of various quantities appearing in (37) or on the nature of the material. Specific forms of (37) valid for specific material can be obtained when corresponding constitutive equations for \( T \), \( q^1 \), and \( \sigma \) are introduced. We also note that the heat flux \( q^1 \) in (38) is available to the observer in the current configuration (on a material surface). Consequently, the quantities \( q^N \) are, like the generalized forces \( p_k^N \), functions of the nodal displacements \( u_{N1} \).

SPATIAL FORMS

To obtain spatial forms of the general finite element equations for applications to problems in, say, fluid dynamics, it is necessary to regard the interpolation functions \( \psi^N \) as functions \( \psi^N \) of the spatial coordinates \( x_1 \). Convective terms must be added to time rates, but the tractions \( S \) and heat fluxes \( q_1 \) act now on spatial surfaces which are independent of the motion.

In the case of compressible fluids, the density \( \rho \) is represented by \( \psi^N (x) \rho_N \), \( \rho_N \) being the value of the density field at node \( N \). Then the equations of motion assume the highly nonlinear form

\[ a^{HN} \rho_M \dot{v}_Q + b^{HQR} \rho_M \nu_R \nu_{Q1} + \int_{\Omega^e} t_{11} \frac{\partial \psi^N}{\partial x_j} \, d\mathbf{v} = p^N_1 \]  \hspace{1cm} (40)

where \( t_{11} = F_{i}F_{i}^{j} \sigma^{MN} \) is the Cauchy stress tensor and

\[ a^{HN} = \int_{\Omega^e} \psi^M \psi^Q \psi^N \, d\mathbf{v} \quad b^{HQR} = \int_{\Omega^e} \psi^M \psi^Q \frac{\partial \psi^R}{\partial x_a} \psi^N \, d\mathbf{v} \]  \hspace{1cm} (41)
It is again emphasized that the $\psi^N$ are now functions of $x_1$ and that $u_{se}$ and, in computing $p_{1}^{N}$, $A_{s}$ are now spatial volumes and areas. Note also that the spatial equations of motion are cast in terms of nodal velocities rather than displacements, and that inclusion of inertial effects and an unknown density leads to terms which are nonlinear in the nodal velocities and densities. To (40) we must add the finite element analogue of the continuity equation $\partial \rho / \partial t + \partial (\rho v) / \partial x = 0$, which, by following a procedure similar to that used in deriving (40), is found to be

$$c^{MN}\dot{\rho}_M + d^{NMR}_K \rho_R v_{MK} = 0 \quad (42)$$

where $\dot{\rho}_M = d\rho_M/dt$.

$$c^{MN} = \int_{U_0} \psi^M \psi^N \, du \quad (43)$$

$$d^{NMR}_K = \int_{U_0} \frac{\partial}{\partial x_k} (\psi^N \psi^R) \, du \quad (44)$$

Here $n_k$ are the components of a unit vector normal to the bounding surface of the element.

For incompressible fluids, $\rho$ is known and the equation of a continuity is satisfied. Then, instead of (40), we use (39).

$$m^{MN}\dot{\psi}_{M1} + n^{NMR}_a v_{Ma} v_{R1}$$

$$+ \int_{U_0} (\delta_{ij} - \rho \delta_{ij}) \psi^N_{,j} \, du = p_N^1 \quad (45)$$

where $\delta_{ij}$ is the deviatoric stress, $p$ is the hydrostatic pressure, and $n^{NMR}_a$ is the convective mass matrix.

$$n^{NMR}_a = \int_{U_0} \rho \psi^N \psi^M \frac{\partial \psi^R}{\partial x_a} \, du \quad (46)$$
We must also require that the incompressibility condition be satisfied in an average sense over each element:

\[ \int_{\mathbf{u}_e} \frac{\partial \mathbf{v}_k}{\partial x_k} \, d\mathbf{u} = v_{N_k} \int_{\mathbf{u}_e} \frac{\partial \psi^N}{\partial x_k} \, d\mathbf{u} = 0 \quad (47) \]

The spatial equations of heat conduction for the element are identical in form to (37) except that \( \mathbf{u}_e \) and \( A_e \) are regarded as spatial volumes and areas, and corresponding interpretations are given to \( \gamma, \psi^1, \) and \( \sigma \). When specific forms of \( \gamma \) are introduced, the term containing \( \gamma \) will give rise to nonlinear convective terms in the nodal velocities and temperatures.

**GENERALIZED FORCES AND FLUXES FOR FINITE DEFORMATION**

We have remarked previously that for material descriptions of the motion of finite elements, the tractions \( \mathbf{S}^1 \) and heat fluxes \( \mathbf{q}^1 \) act on areas in the deformed body. Consequently, the generalized nodal forces and heat fluxes must be expressed as functions of the nodal displacements. We shall now examine certain forms of these functions.

Let \( \mathbf{n} \) denote a unit normal to the bounding surface \( dA_o \) of the element while it occupies its reference ("undeformed") configuration, and \( \mathbf{n} \) the unit normal to the same material area \( dA \) in the current configuration ("after deformation"). Further, let \( \mathbf{s}, \mathbf{q} \) and \( \mathbf{s}_a, \mathbf{q}_a \) denote the surface tractions and heat flux per unit undeformed and deformed area, respectively. Then the mechanical power \( \Omega_s \) and the heat \( Q_q \) developed by \( \mathbf{s} \) and \( \mathbf{q} \) are given by

\[ \Omega_s = \int_{A_0} \mathbf{S}^1 \mathbf{F} \psi^N \mathbf{u}_{N_r} \, dA_o \quad (48) \]

\[ Q_q = \int_{A_0} q^a \sqrt{G} \mathbf{n}_a \, dA_o \quad (49) \]
where $\hat{S}^1$ and $q^2$ are the contravariant components of $\hat{s}$ and $q$ referred to the convected coordinate lines $X^i$, $F^r_i$ are the deformation gradients defined in (3), and $G = |G_{1j}|$ is the determinant of the deformation tensor $G_{1j}$ of (5). Observing that for the finite element

$$F^r_i = (\delta^r_i + \psi^r_{1j} u^r_M)$$

and

$$G_{1j} = (\delta^r_i + \psi^N_{1j} u^r_M) (\delta^r_j + \psi^N_{1r} u^r_M)$$

and recalling (35) and (38), we find for the generalized nodal forces and normal heat fluxes

$$p_k^N = \int_{U_{0e}} \rho_0 F_k \psi^N \, du + \int_{A_{0e}} \hat{S}^1 (\delta^1_k + \psi^1_{1N} u^1_M) \psi^N \, dA$$

and

$$q^N = \int_{U_{0e}} \rho_0 h \psi^N \, du + \int_{A_{0e}} q^r \sqrt{G} \hat{n}_m \, dA$$

Here $F_k$ is the cartesian component of body force per unit mass in the reference configuration.

To specialize still further, consider the case in which a uniform pressure loading and a uniform heat flux of intensities $p$ and $q$ act on the deformed areas of the element. Then, ignoring body forces and internal heat sources for simplicity, (52) and (53) become

$$p_k^N = - \int_{A_{0e}} p \sqrt{G} \hat{n}_m (\delta^r_k + \psi^r_{1r} u^r_M) \psi^N \, dA$$

and

$$q^N = \int_{A_{0e}} q \sqrt{GG^r} \hat{n}_r \hat{n}_m \psi^N \, dA$$
where $G^{-1}$ is the inverse of the deformation tensor $G_r$.

Fortunately, simplified forms of these equations can be obtained for specific shapes of finite element boundaries. For simplicial approximations (i.e., cases in which the $N$ are linear in $X^I$), the generalized forces and heat fluxes assume significantly more manageable forms [25, 26]. We shall not elaborate on these in this paper.

MATERIALS WITH MEMORY

Finite element formulations of Coleman's [1] thermodynamics of thermomechanically simple materials lead to results which can be applied to a wide range of problems. For materials whose response depends upon the history of the deformation and temperature, the free energy $\varphi$ is given by a functional $\mathfrak{F}$ of the histories $G_{ij}(t-s)$, $\Theta(t-s)$ where $s = t - \tau \geq 0$ is a parameter:

$$\varphi = \mathfrak{F}_{s=0} \left[ G_{ij}(t-s), \Theta(t-s) \right]$$

(56)

Coleman has shown that for such materials the stress tensor and entropy are determined from $\varphi$ as follows:

$$\sigma^I = \mathfrak{X}^I_{s=0} \left[ G_{ij}(t-s), \Theta(t-s) \right] = \rho \mathcal{D}^I_{s=0} \mathfrak{F}_{s=0}$$

(57)

$$\eta = \mathfrak{S}_{s=0} \left[ G_{ij}(t-s), \Theta(t-s) \right] = -\mathcal{D}^\Theta_{s=0} \mathfrak{F}_{s=0}$$

(58)

where $\mathfrak{X}^I[ ]$ and $\mathfrak{S}[ ]$ are functionals of the indicated histories and $\mathcal{D}_{\sigma}$ and $\mathcal{D}_{\Theta}$ are Frechet differential operators. The heat flux is given by an independent functional $\mathfrak{H}[ ]$:

$$q^I = \mathfrak{H}_{s=0} \left[ G_{ij}(t-s), \Theta(t-s); \nabla \Theta(t) \right]$$

(59)

and the internal dissipation can be obtained directly from $\varphi$, $\sigma^I$, and $\eta$ by means of (28), and is itself a functional of the same histories:

$$\sigma = \mathfrak{S}_{s=0} \left[ G_{ij}(t-s), \Theta(t-s) \right]$$

(60)

Upon introducing (57)–(60) into (36) and (37) and integrating over the volume of the finite element, we eliminate any dependence on $X^I$. The equations of motion and heat conduction of a typical finite element then become
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\[ m_{NM} u_{MK} + \sum_{s=0}^{\infty} \left[ u_{RR}(t-s), T_R(t-s) \right] \]

\[ + u_{M1} \prod_{s=0}^{\infty} \left[ u_{RR}(t-s), T_R(t-s) \right] = p_k \quad (61) \]

\[ \int \rho_o (T_o + \psi^M T_M) \sum_{s=0}^{\infty} \left[ u_{RR}(t-s), T_R(t-s) \right] \, du \]

\[ \int \psi^O [u_{RR}(t-s), T_R(t-s)] \, du \]

\[ + \sum_{s=0}^{\infty} \left[ u_{RR}(t-s), T_R(t-s) \right] = q^N + \sum_{s=0}^{\infty} \]

\[ \int \psi^N [u_{RR}(t-s), T_R(t-s)] \, du \]

\[ \left[ \begin{array}{cc} \prod_{s=0}^{\infty} [ & ] = \int \psi^N, j \sum_{s=0}^{\infty} \{ \} \, du \right] \]

\[ \psi^M [u_{RR}(t-s), T_R(t-s)] = \int \psi^N, j \sum_{s=0}^{\infty} \{ \} \, du \]

\[ \sum_{s=0}^{\infty} \{ \} = \sum_{s=0}^{\infty} \{ \delta_{1s} + \psi^J u_{RN}(t-s) \} \]

\[ + \psi^M T_M(t-s) \}

\[ \left[ \begin{array}{cc} \prod_{s=0}^{\infty} [ & ] = \frac{d}{dt} \left[ G_{ij}(t-s), \theta(t-s) \right] \right] \]

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Thus, the behavior of the finite element model depends upon the histories of the nodal displacements and temperatures.

**SPECIAL CASES**

We now cite as examples a number of finite element formulations that are reducible as special cases of the general equations of motion and heat conduction derived previously.

**FINITE COUPLED THERMOELASTICITY**

In this case the free energy $\varphi$ is an ordinary function of the current deformation and temperature:

$$\varphi = \tilde{\varphi}(Y_1, T) \quad (64)$$

Then, according to (55) and (56), $\sigma^J = \partial \tilde{\varphi} / \partial Y_1$ and $\lambda = -\partial \tilde{\varphi} / \partial T$ and (36) and (37) are put in terms of (64) by direct substitution.

In the case of isotropic thermoelastic solids, $\varphi$ can be expressed as a function of $T$ and the principal invariants $I_1, I_2,$ and $I_3$ of $G_{1,1}$:

$$I_1 = 3 + 2Y_{1,1}$$

$$I_2 = 3 + 4Y_{1,1} + 2(Y_{1,1}Y_{2,2} - Y_{1,2}Y_{1,2})$$

$$I_3 = |\delta_{1,1} + 2Y_{1,1}| \quad (65)$$

Then the equations of motion and heat conduction of a finite element become (30)

$$m^N \ddot{u}_M + \frac{1}{2} \int_{\Omega_{0,1}} \frac{\partial \tilde{\varphi}(I_1, I_2, I_3, T)}{\partial I_{\alpha}} \left( \frac{\partial I_{\alpha}}{\partial Y_{1,1}} + \frac{\partial I_{\alpha}}{\partial Y_{2,2}} \right) d\Omega + \int_{\Omega_{0,1}} \psi^N (\delta_{1,1} + \dot{\psi}^N u_M) d\Omega = p^N \quad (66)$$
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\[- \int_{\Omega_0} \rho_0 (T_0 + \Psi^M T^N) \psi^N \frac{\partial^2 \Phi(I_1, I_2, I_3, T)}{\partial T \partial t} \, d\Omega \]

\[+ \chi^{MN} T_M = q^N \quad (67)\]

where \( \alpha, i, j = 1, 2, 3; M, N = 1, 2, \ldots, N_\alpha \). Here the internal dissipation \( \sigma \) is zero and \( \chi^{MN} \) is the thermal conductivity matrix

\[\chi^{MN} = \int_{\Omega_0} \hat{\chi}^{I J} \psi^M \psi^N \, d\Omega \quad (68)\]

where \( \hat{\chi}^{I J} \) is the thermal conductivity tensor. It has been assumed in (66) that the heat flux is given by the Fourier Law

\[q^I = \hat{\chi}^{I J} T_J \quad (69)\]

FINITE ELASTICITY

In the case of finite isothermal deformations of incompressible elastic solids, \( \Phi = W(I_1, I_2) \), where \( W(\cdot) \) is referred to as the strain energy density. Then the equations of motion become

\[m^{NM} \ddot{u}_M + \frac{1}{2} \int_{\Omega_0} \left[ \frac{\partial W}{\partial I_\nu} \left( \frac{\partial I_{\nu}}{\partial Y_{1 j}} + \frac{\partial I_{\nu}}{\partial Y_{1 j}} \right) + p \frac{\partial I_3}{\partial Y_{1 j}} \right] \]

\[\cdot \psi^N (\delta_{1k} + \psi^M u_M^k) \, d\Omega = p_k^N \quad (70)\]

where \( \nu = 1, 2 \) and \( p \) is a uniform hydrostatic pressure for the element. To determine \( p \), we must add to (70) the incompressibility condition

\[\int_{\Omega_0} (I_3 - 1) \, d\Omega = 0 \quad (71)\]
Solutions of static forms of the nonlinear equations (70) have been presented by Oden [28, 29], Oden and Sato [25, 27] and Oden and Kubitzka [26].

**DYNAMIC COUPLED THERMOELASTICITY**

Consider the case of finite displacements but infinitesimal strains of an anisotropic thermoelastic solid for which the free energy is a quadratic form in the strains and temperature increments:

\[ \varphi = \frac{1}{2} E^{ijkl} \gamma_{ij} \gamma_{kl} + B^{ij} \gamma_{ij} T + \frac{c}{2T_0} T^2 \]  

(72)

Here \( E^{ijkl} \) and \( B^{ij} \) are thermoelastic constants and \( c \) is the specific heat at constant deformation. Then motion and heat conduction in an element are governed by

\[ m^{NM} \ddot{u}_{Mk} + \int_{\Omega_{oe}} E^{ijkl} \gamma_{ij} \gamma_{kl} (\delta_{ik} + \psi_i u_{Mk}) \, du \]

\[ + \int_{\Omega_{oe}} B^{ij} \psi_j (\delta_{ik} + \psi_i u_{Mk}) \, du \, T_R = p_k^N \]  

(73)

\[ - \int_{\Omega_{oe}} \rho_o (T_o + \psi^M T_M) \psi^N [B^{ij} \psi_j (\delta_{ik} + \psi_i u_{Mk})] \, du \]

\[ + \frac{c}{T_0} \psi^R \dot{T}_R \right) \, du + \chi^{MN} T_M = q^N \]  

(74)

where \( i, j, k, l, m = 1, 2, 3 \) and \( M, N, R, S = 1, 2, \ldots, N_e \). Again Fourier's law of heat conduction is assumed.

**CLASSICAL INFINITESIMAL ELASTICITY**

If displacements and strains are small and only isothermal deformations are considered, (73) reduces to

\[ m^{NM} \ddot{u}_{Mk} + K^{NM} \dot{u}_{Mk} = p_k^N \]  

(75)
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where

\[ K_{k}^{NM} = \int_{\Omega_{0}} E^{j\,k\,n} \delta_{\kappa}^{j} \psi_{i}^{N} \psi_{i}^{M} \, d\Omega \]  

(76)

The array \( K_{k}^{NM} \) is the well-known stiffness matrix for the element.

TRANSIENT HEAT CONDUCTION

Finite element formulations of the classical problem of heat conduction in a rigid solid are obtained by linearizing (74) and neglecting terms involving the nodal displacements:

\[ h_{NM} T_{M} + \kappa_{NM} T_{M} = q^{N} \]  

(77)

Here \( h_{NM} \) is the specific heat matrix:

\[ h_{NM} = \int_{\Omega_{0}} \rho_{c} c \psi_{i}^{N} \psi_{i}^{M} \, d\Omega \]  

(78)

Equations similar to (77) have been derived by several authors [43, 36, 37].

TRANSIENT COUPLED THERMOVISCOELASTICITY

The possible characterizations of thermoviscoelastic solids are practically unlimited, and appropriate finite element models can always be generated by introducing the appropriate constitutive equations into (36) and (37). For completeness, we cite here just one such application. This involves a thermoviscoelastic material of the Voigt type, described by Eringen [44], for which

\[ \sigma^{ij} = E^{i}\,n\,Y_{mn} + F^{i}\,n\,Y_{mn} + B^{i}\,T + H^{i}\,T_{n} \]  

(79)

\[ \eta = B^{i}\,Y_{ij} + \frac{c}{2T_{0}} T^{2} \]  

(80)
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\[ q_i = \kappa \dot{T}_{ij} + \Gamma_{ij} \dot{\gamma}_{ij} \]  \hspace{1cm} (81)

where \( E_{ij} \), \( F_{ij} \), \( H_{ij} \), \( B_{ij} \), \( \kappa_{ij} \), and \( \Gamma_{ij} \) are arrays of material constants. The motion and heat conduction in a finite element of such materials are governed by

\[ m^{NM} \ddot{u}_{Mk} + D^{NM}_{k} \dot{u}_{Mn} + K^{NM}_{k} \dot{u}_{Mn} + H^{NM}_{k} T_{M} = p_{k} \]  \hspace{1cm} (82)

\[ a^{NM}_{j} \dot{u}_{Mj} + c^{NM} \dot{T}_{M} + \kappa^{NM} T_{M} = q^{N} + \sigma^{N} \]  \hspace{1cm} (83)

where

\[ D^{NM}_{k} = \int_{V} F_{ij}^{ME} \psi_{ij}^{M} \psi_{ij}^{N} \delta_{ik} \ du \]  \hspace{1cm} (84)

\[ H^{NM}_{k} = \int_{V} \delta_{ik} \psi_{ij}^{M} (B_{ij}^{M} \psi_{ij}^{M} + H_{ij}^{M} \psi_{ij}^{M}) \ du \]  \hspace{1cm} (85)

\[ a^{NM}_{j} = \int_{V} \psi_{ij}^{M} (\rho_{0} T_{0} B_{ij}^{M} \psi_{ij}^{M} + \Gamma_{ij}^{M} \psi_{ij}^{M}) \ du \]  \hspace{1cm} (86)

\[ \sigma^{N} = \int_{V} \psi_{ij}^{N} (F_{ij}^{MN} \dot{\gamma}_{ij} \dot{T}_{MN} + H_{ij}^{MN} \dot{\gamma}_{ij} \dot{T}_{MN}) \ du \]  \hspace{1cm} (87)

COMPRESSIBLE STOKESIAN FLUIDS

To develop finite element models of general form for the case of compressible Stokesian fluids, we note that the stress tensor \( \tau_{ij} \) for such fluids is of the form

\[ \tau_{ij} = (\mu + \lambda \nu \frac{\partial v_{k}}{\partial x_{k}}) \delta_{ij} + 2\mu \nu \left( \frac{\partial v_{i}}{\partial x_{j}} + \frac{\partial v_{j}}{\partial x_{i}} \right) \]  \hspace{1cm} (88)
where $\Pi = \pi(p) = \partial \phi / \partial p^{-1}$ is the thermodynamic "pressure" defined by the equation of state of the fluids, and $\lambda$ and $\mu$ are the dilatational and shear viscosities. Introducing (16) into (88), incorporating the result into (40), and recalling that for compressible fluids (42) must also be satisfied, we obtain for the equations governing compressible flow through a finite element

$$a^{MN} \rho_M \dot{v}_{Q_1} + b^{MN} \rho_M v_{Q_1} v_{R_1}$$

$$+ \int_{V_0} \frac{\partial \psi^N}{\partial x_j} \left[ (\Pi + \lambda \frac{\partial \psi^M}{\partial x_k} v_{MK}) \delta_{1j} + 2\mu \left( \frac{\partial \psi^N}{\partial x_j} v_{M1} + \frac{\partial \psi^M}{\partial x_j} v_{M1} \right) \right] \text{d}u = p_1^N \tag{89}$$

$$c^{MN} \rho_M + d^{NMR} v_{MK} = 0 \tag{90}$$

where the arrays $a^{MN}, \ldots$, etc. are defined in (41), (43), (44), and $M, N, Q, R, s = 1, 2, \ldots, N_e$; $i, j, k, m = 1, 2, 3$. Equations (89) and (90) are the finite element analogues of the Navier-Stokes equations for compressible fluids.

**INCOMPRESSIBLE STOKESIAN FLUIDS**

In the case of incompressible Stokesian fluids, $\Pi$ becomes $p$, the hydrostatic pressure, the density is assumed to be known, and

$$\tau_{1j} = - p \delta_{1j} + \mu \left( \frac{\partial v_1}{\partial x_j} + \frac{\partial v_j}{\partial x_1} \right) \tag{91}$$

We have then, instead of (89) and (90),

$$m^{MN} \dot{v}_{M1} + n^{MN} v_{M1} v_{R_1} + \int_{V_0} \frac{\partial \psi^N}{\partial x_j} \left[ \mu \left( \frac{\partial \psi^N}{\partial x_j} v_{M1} + \frac{\partial \psi^M}{\partial x_j} v_{M1} \right) + \frac{\partial \psi^M}{\partial x_1} v_{MJ} \right) - p \delta_{1j} \right] \text{d}u = p_1^N \tag{92}$$

$$d^N_k v_{NK} = 0 \tag{93}$$
where, according to (47), \( d_s^n = \iint \frac{\partial \psi^N}{\partial x_k} \, du \). Equation (92) was also obtained by Oden and Somogyi [39], and finite element solutions of the problem of incompressible potential flows were considered by Martin [38].

**INCREMENTAL FORMS**

In applications of the finite element method to problems of instability and large deflections of elastic solids under infinitesimal strain but finite rotations and displacements, a popular procedure that is usually followed is to derive certain “initial stress” and “initial displacement” matrices to enable solutions to be obtained by solving a sequence of linear problems (e.g., see [10], [45], [46], [47]). We shall now demonstrate that all of these special stiffness matrices, including terms erroneously omitted in previous work, can be obtained by considering incremental forms of the nonlinear finite element equations for the special case of a linearly elastic solid.

Consider a material finite element in a state of finite deformation characterized by the quantities \( u^0_1, \sigma^1_0, F^0_1, \) and \( \hat{S}^0_1 \). Now consider a neighboring motion characterized by corresponding quantities \( u^*_1, \sigma^1_1, F^*_1, \) and \( \hat{S}^1_1 \) where

\[
\begin{align*}
{u}^*_1 &= {u}^0_1 + {u}^*_1 \\
{\sigma}^1_1 &= {\sigma}^1_0 + {\sigma}^1_1 \\
{F}^*_1 &= {F}^0_1 + {F}^*_1 \\
{\hat{S}^1}_1 &= \hat{S}^0_1 + \hat{S}^1_1
\end{align*}
\]

(94)

Here \( u^*_1, \sigma^1_1, F^*_1 \) and \( \hat{S}^1_1 \) represent small perturbations in the initial quantities. We assume that the stress increment is linear in the strain increments:

\[
{\sigma}^1_1 = E^{1\,j\,k\,n} \gamma_{kn} = E^{1\,j\,k\,n} \psi^N_{,k \, u^M_{,n}}
\]

(96)

Introducing (94)—(96) into (36), linearizing in the increments, and noting that the initial values must also satisfy the equations of motion, we obtain for incremental equations of motion for a finite element

\[
\begin{align*}
{m}^N_{NM} \ddot{u}^N_{MK} + \left[ {K}^N_{KN} + {K}^N_{Ku} + {K}^N_{K\sigma} \right] u^N_{MK} + R^N_{Ku} u^N_{MK} &= p^N_k
\end{align*}
\]

(97)

where \( {K}^N_{KN} \) is the ordinary stiffness matrix of (75) corresponding to the initial linearized problem, \( {K}^N_{K\sigma} \) is the "geometric" or "initial stress" stiffness matrix [10, 45, 47], \( {K}^N_{Ku} \) is the "initial displacement"
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matrix [47], $R_{NM}^{NM}$ is a load-correction stiffness matrix, apparently omitted in previous incremental formulations (e.g. [49], [19], with the exception of those which began with nonlinear forms of the finite element equations such as [25], [26], [29], [50], [51]):

$$K_{NM}^{NM} = \int_{\Omega} \sigma^{ij} \psi_i^N \delta_k^M \, d\Omega$$

(98)

$$K_{NM}^{NM} = \int_{\Omega} E^{ij} \psi_i^N \psi^M \delta_k^R \, d\Omega$$

(99)

$$R_{NM}^{NM} = \int_{\Omega} \hat{S}^{NM} \gamma^N \gamma_k^M \, d\Omega$$

(100)

The generalized nodal forces $p_k^N$ are now given by

$$p_k^N = \int_{\Omega} \rho \dot{F}_k \psi^N \, d\Omega + \int_{\Omega} \hat{S}^{ij} (\delta_{ik} + \psi_j^M \dot{u}_k^M) \psi^N \, d\Omega$$

(101)

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DISCUSSION

COMMENT (H. C. Martin)

This interesting paper strongly suggests that finite elements have much to offer the researcher interested in problems in continuum mechanics. The general theory can, through the introduction of the finite element method, be used in obtaining solutions to actual problems.

The question of non-conservative systems has come up previously in this Seminar. Since the general formulation of Dr. Oden includes the non-conservative cases, it would now be interesting to have his comments on this subject.

RESPONSE (J. T. Oden)

In my work, I have derived equations without neglecting nonlinearities
in nodal displacements. Then any one of a number of available methods, including incremental procedures, can be tried.

For nonconservative forces, the location of the material surface on which forces act is given in terms of the displacements of boundary nodes of the model. In most cases, however, the expressions for the deformed surface area and orientation are extremely complicated. In such cases, I suggest that a reasonable approximation can be obtained by fitting a lower-order polynomial through displaced nodes on the boundary. Discussions of some simple expressions obtained using linear approximations are given in References 25, 26, and 49 (for example) of the paper.