FINITE ELEMENT APPROXIMATIONS OF A

CLASS OF NONLINEAR OPERATORS

J. T. Oden

Society for Industrial and Applied Mathematics

SIAM Fall Meeting

Boston, Massachusetts

October, 1970
1. Introduction

It is often the case with approximate methods of mathematical analysis that their practical value is established by numerous examples long before their theoretical basis or their limitations are completely understood. The finite element method is no exception. Introduced in an early form by Courant in 1943 [1] and extended to computer-oriented analysis of complex elastic structures by Turner et al in 1956 [2], the method has since found successful application to a wide range of problems of mathematical physics (e.g. [3,4,5,6,7]). Indeed, recent studies indicate that the concept is sufficiently general to handle, for example, the most general problems of nonlinear elasticity [8,9], nonlinear continuum mechanics [10,11], and the kinetic theory of rarefied gases [12]. On the other hand, available criteria for convergence or for the selection of acceptable local finite element approximations appear to be valid only for a rather narrow class of operators.

Since the majority of applications of the finite-element method have dealt with boundary value problems in linear elasticity, it is natural that most convergence studies have been confined to linear elliptic operators. The investigations of Melosh [13], McIay [14], Johnson and
McLay [15], and Pian and Tong [16] are representative. Completeness and convergence of positive bounded-below, linear operators were examined by Key [17] and Arantes e Oliveria [18,19], and certain special cases were discussed by Göel [20], Zlamal [21], Zenisek and Zlamal [22], Tabandeh [23]. One-dimensional nonlinear problems were studied by Ciarlet, Schultz, and Varga [24]. A survey article was given by Felippa and Clough [5].

In the present paper, the application of the finite element method to a class of nonlinear operator equations is considered. Previous conclusions drawn from studies of linear positive-definite operators are extended to nonlinear positive-definite, potential operators. The application of the method to nonlinear, non-positive definite operators is also demonstrated. It is shown that the sufficient criteria established for the selection of local base functions in the case of linear positive-definite operators also are sufficient for a class of positive-definite nonlinear potential operators.

2. Discrete Approximations

Finite-Dimensional Subspaces. Consider a Hilbert space $\mathcal{H}$ consisting of square integrable functions $u(\mathbf{x})$, $v(\mathbf{x})$, ..., defined on some region $\Omega$ of n-dimensional Euclidean space. The inner-product of two elements $u,v \in \mathcal{H}$ is denoted $\langle u,v \rangle$ and the natural metric is $d(u,v) = \|u - v\|$, where $\|u\| = \langle u,u \rangle^{1/2}$.

Let $\mathcal{H}$ denote a G-dimensional subspace of $\mathcal{H}$ spanned by a set $\{\varphi_\alpha(\mathbf{x})\}$ of G linearly independent functions. The identification of $\{\varphi_\alpha(\mathbf{x})\}$ defines a projection $\Pi: \mathcal{H} \rightarrow \mathcal{H}$, and every $u \in \mathcal{H}$ can be written as the sum of its projection $\hat{u} = \Pi u$ and some complement $\hat{u}$; i.e., $u = \hat{u} + \hat{u}$. Every element $v$ of $\mathcal{H}$ is, therefore, of the form

$$v = a^1 \varphi_1 + a^2 \varphi_2 + \ldots + a^G \varphi_G = a^\alpha \varphi_\alpha$$

(2.1)
where the $a^\alpha$ are scalar coefficients and the repeated index $\alpha$ is summed from 1 to $G$.

The best approximation to $u$ in $\mathcal{H}$ is taken to be the function $\bar{u}$ which minimizes $d^2(u,v)$. To obtain $\bar{u}$, we introduce the $G \times G$ Gram matrix

$$ C_{\alpha\beta} = \langle \varphi^\alpha, \varphi^\beta \rangle $$

Then, if $C^{\alpha\beta}$ denotes the inverse of $C_{\alpha\beta}$, we can obtain a set of biorthogonal conjugate base functions $\{\varphi^\alpha\}$, having the property $\langle \varphi^\alpha, \varphi^\beta \rangle = \delta^\alpha_\beta$, by setting [25,26]

$$ \varphi^\alpha = C^{\alpha\beta} \varphi^\beta $$

Then

$$ \Im u = \bar{u} = u^\alpha \varphi^\alpha = u_{\alpha} \varphi^\alpha $$

where

$$ u^\alpha = \langle u, \varphi^\alpha \rangle \quad u_{\alpha} = \langle u, \varphi^\alpha \rangle $$

and $\bar{u}$ minimizes $d(u,v)$. Then $\langle \hat{u}, \varphi^\alpha \rangle = \langle \hat{u}, \varphi^\alpha \rangle = \langle u-\bar{u}, \varphi^\alpha \rangle = 0$.

3. Finite-Element Approximations

Various methods of approximation differ in the way the base functions $\varphi^\alpha(x)$ are constructed. In finite-element approximations, the region $\mathcal{R}$ is approximated by a region $\overline{\mathcal{R}}$ in which a finite number $G$ of (global) nodal points $x_1, x_2, \ldots, x^G$ are identified. $\overline{\mathcal{R}}$ is regarded as being composed of a finite number $E$ of subregions $r_1, r_2, \ldots, r_E$, called finite elements, which meet at various global nodal points (Fig. 1). Among the fundamental properties of finite-element approximations [26,27] are: (1) elements are considered disjoint ($r_i \cap r_j = \emptyset$, $i \neq j$) for the purpose of approximating a
function locally over each element; (2) the local approximation $\bar{u}_{e}(x)$ pertaining to element $r_{e}(i.e., the restriction of $u$ to $r_{e}$) is uniquely defined by its values at a finite number $N_{e}$ of (local) nodal points $x^{N}$_{e}; (3) the final global approximation of $u$ is obtained by connecting all elements together to form $\bar{u}$ and matching values of local approximations (and possibly values of various partial derivatives) of adjacent elements at common nodal points.

For example, to obtain a finite-element approximation of a given function $u(\xi)$, $\xi \in \Omega$, we consider first its restriction $u_{e}$ to a typical isolated element $r_{e}$ containing $N_{e}$ nodal points. We then introduce a system of local interpolation functions $\psi_{N}^{(e)}(x)$ which have the properties

$$\psi_{N}^{(e)}(x) = 0 \quad x \notin r_{e} \quad \psi_{N}^{(e)}(x_{N}) = \delta_{N}^{e} \quad (3.1)$$

where $M, N = 1, 2, \ldots, N_{e}$ and $x_{N}$ is a local nodal point. Then, if $u_{e}=u_{e}(x_{N})$, we obtain for the local approximation

$$\bar{u}_{e}(x) = u_{e}(x_{N}) \psi_{N}^{(e)} \quad (3.2)$$

where the repeated index is summed from 1 to $N_{e}$. To obtain the global approximation of $u$, set

$$u_{e}^{N}(x) = \Omega_{e}^{N} \psi_{N}^{(e)} \quad (3.3)$$

where

$$(e)_{N} = \begin{cases} 1 & \text{if node } N \text{ of element } e \text{ is incident on node } \alpha \text{ of the connected model } \Omega \\ 0 & \text{otherwise} \end{cases} \quad (3.4)$$

Then, the global approximation $\bar{u}(x)$ of $u(\xi)$ is $u_{\alpha}^{\varphi}(\xi)$, as in (2.4), where, in this case (except for a set of measure zero),

$$\varphi_{\alpha}(\xi) = \sum_{e=1}^{E} (e)_{N} \psi_{N}^{(e)}(x) \quad (3.5)$$

4
4. Linear Operators

Given a finite-element model \( \mathcal{R} \) of \( \mathcal{R} \), we refer to a new model \( \hat{\mathcal{R}} \) obtained by increasing the number of elements and global nodes as a \textit{refinement} of \( \mathcal{R} \) if the class of the elements (i.e., the basic geometry and local interpolation functions) is preserved and if every \( \mathbf{x} \in \mathcal{R} \) also \( \in \hat{\mathcal{R}} \). If every node and interelement boundary of \( \mathcal{R} \) is a node and inter-element boundary of \( \hat{\mathcal{R}} \), then \( \hat{\mathcal{R}} \) is a \textit{regular} refinement. The maximum distance \( \delta(e) \) between two points \( \mathbf{x}(e), \mathbf{z}(e) \in \mathcal{R} \) is called the diameter of the finite element. The quantity \( \delta = \max \{ \delta(1), \delta(2), \ldots, \delta(E) \} \) is referred to as the \textit{index} of the finite-element model \( \mathcal{R} \). A refinement \( \hat{\mathcal{R}} \) with index \( \delta \) of a model \( \mathcal{R} \) with index \( \delta \) is \textit{uniform} if \( \delta \leq \delta \).

An important question in finite-element analysis is: what conditions must be imposed on the local interpolation functions \( \psi_N(x) \) in order to insure that \( d(u, \bar{u}) \to 0 \) through a sequence of uniform, regular, refinements? That is, what conditions must \( \psi_N(x) \) satisfy in order that \( \phi(u) \), as given by (2.10), be complete with respect to the norm \( ||u|| = \langle u, u \rangle \)? While sufficient conditions for completeness in the mean can be obtained, a more fundamental question involves approximations of solutions to linear boundary value problems. Consider the equation

\[ \mathcal{L}u = f \]  

(4.1)

where \( \mathcal{L} \) is a linear operator on a set \( \mathcal{D} \) dense in \( \mathcal{H} \) and \( f \) is prescribed.

Mikhlin \([28, 29]\) has shown the following:

- the solution to (4.1) is a unique solution (if it exists) if \( \mathcal{L} \) is positive definite.
- If \( \mathcal{L} \) is positive definite and \( u^\star \) is the solution of (1), then the functional

\[
J(u) = \langle \mathcal{L}u, u \rangle - 2 \langle u, f \rangle
\]  

(4.2)

assumes its minimal value when \( u = u^\star \) (\( \mathcal{H} \) being real).
Conversely, if there exists a $u^* \in \mathcal{B}$ such that $u^*$ minimizes $J(u)$, then $u^*$ satisfies (4.1).

However, it may happen that no element exists in $\mathcal{B}$ which minimizes $J(u)$. It is then possible to extend $\mathcal{B}$ so as to contain a solution if and only if $\mathcal{L}$ is positive-bounded below; that is, if there exists a real number $\gamma$ such that $\langle \mathcal{L}u, u \rangle \geq \gamma^2 \langle u, u \rangle$. The extended domain $\mathcal{B}_\gamma$ belongs to a new space $\mathcal{H}_\gamma$, dense in $\mathcal{H}$, which is defined as the completion of the inner-product space obtained by associating with each pair $u, v \in \mathcal{B}$ the inner product $[u, v] = \langle \mathcal{L}u, v \rangle$. Then $\|u\|_\mathcal{L} = [u, u]^{1/2}$ and $d_\mathcal{L}(u, v) = \|u - v\|_\gamma$ are called the energy norm and distance in energy. The extended functional can then be expressed as

$$J(u) = \|u - u^*\|_\gamma^2 - \|u^*\|_\gamma^2 \tag{4.3}$$

If $\{u^n\}$ is a minimizing sequence in $\mathcal{H}_\gamma$ for the functional $J(u)$; i.e.,

$$\lim_{n \to \infty} J(u^n) = J(u^*) = \|u^*\|_\gamma^2$$

then $\{u^n\}$ converges in energy to $u^*$. If $\mathcal{L}$ is positive-bounded below, convergence in energy implies mean convergence; but the converse is not true.

Considering now a Ritz-approximation of (4.1), let $\{\varphi_\alpha\}$ be complete in $\mathcal{H}_\gamma$ and $u^{(G)}$ be an approximate solution $u^{(G)} = a_\alpha \varphi_\alpha$ obtained by determining $a_\alpha$ so as to minimize $J(u)$ in $\mathcal{H}$; i.e.

$$[\varphi_\alpha, \varphi_\beta]_\mathcal{B} = \langle \varphi_\alpha, \varphi_\beta \rangle \tag{4.4}$$

Then

- $\{u^{(G)}\}$ is a minimizing sequence for $J(u)$
- $\lim_{G \to \infty} \|u^* - u^G\|_\gamma = \lim_{G \to \infty} \|u^* - u^G\|_\gamma = 0$ if $\mathcal{L}$ is positive bounded below.
- If $\mathcal{L}$ is positive definite, $u^G$ converges to $u^*$ in energy.
The following are usually given as sufficient criteria for completeness in energy of the finite-element base functions \( \varphi_\alpha \) corresponding to problems involving quadratic functionals which contain derivatives of order \( r \). Let \( \mathbf{u}^{(e)} \) denote a finite-element approximation corresponding to element \( e \):

\[
\mathbf{u}^{(e)} = \sum_\alpha c_\alpha \varphi_\alpha.
\]

Then it is sufficient for \( \varphi_\alpha \) to be complete in energy if

- \( \mathbf{u}^{(e)} \in C^{r-1}(\mathcal{W}) \), \( \mathbf{u}^{(e)} \) being uniquely defined on \( r_e \) by nodal values on the interelement boundaries, and
- All uniform states of \( \mathbf{u} \) and its partial derivatives of order \( r \) are included in \( \mathbf{u} \).

Neither condition is necessary for completeness, but they are generally easy to impose and do guarantee energy completeness.

5. Nonlinear Operators

To establish the manner in which the ideas discussed previously are extended to nonlinear operators, we first review a few definitions and theorems. Let \( \mathcal{O} \) denote a nonlinear operator on a Banach space \( \mathcal{V} \). Then [29]

- If at \( u \in \mathcal{V} \),

\[
\mathcal{O}(u + h) + \mathcal{O}(u) = \delta\mathcal{O}(u, h) + w(u, h)
\]

wherein \( \delta\mathcal{O}(u, h) \) is linear in \( h \) and \( h \in \mathcal{V} \), and

\[
\lim_{\|h\| \to 0} \frac{\|w(u, h)\|}{\|h\|} = 0
\]

then \( \delta\mathcal{O}(u, h) \) is the Frechet differential of \( \mathcal{O} \) at \( u \) and \( w(u, h) \) is the remainder.

- \( \mathcal{O}'(u) \) is the Frechet differential of \( \mathcal{O} \) at \( u \) if \( \mathcal{O}'(u)h = \delta\mathcal{O}(u, h) \).

Let \( K(u) \) be a functional defined on a set \( \Omega \subset \mathcal{V} \) and let \( \langle u, v \rangle \) denote an inner product defined on \( \mathcal{V} \) and its conjugate \( \mathcal{V}^* \). If \( K(u) \) is
Frechet differentiable on $\Omega$, the operator $\mathcal{G}$ defined by

$$
\langle \mathcal{G}(u), h \rangle = \lim_{\alpha \to 0} \frac{1}{\alpha} [K(u + \alpha h) - K(u)]
$$

is called the gradient of $K(u)$ and we write $\text{grad } K(u) = \mathcal{G}(u)$.

If there exists a functional $K(u)$ such that $\text{grad } K(u) = \mathcal{G}(u)$, then $\mathcal{G}(u)$ is said to be a potential operator. It can be shown that a necessary and sufficient condition for $\mathcal{G}(u)$ to be potential is that

$$
\langle \delta \mathcal{G}(u, h_1), h_2 \rangle = \langle \delta \mathcal{G}(u, h_2), h_1 \rangle.
$$

A point $u$ at which $\text{grad } K(u) = 0$, $0$ being the null element of $\mathcal{G}^*$, is called a critical point of $K(u)$.

Theorem (Vainberg [30]). Let $\mathcal{G}$ be an operator that is potential on a ball $\mathcal{N}(u_0, r)$. Then there is a unique functional $K(u)$ whose value at $u_0$ is $K_0$ and whose gradient is $\mathcal{G}$ which is given by

$$
K(u) = \int \langle (u_0 + s(u - u_0), u - u_0) \rangle ds + K_0
$$

(5.4)

It is easily shown that (5.4) is a special case of (4.2). Take

$$
\frac{1}{2} \mathcal{G}(u) = \mathcal{L}(u) - f, \quad \mathcal{G} = \mathcal{L}, \quad u_0 = 0.
$$

Then

$$
K(u) = 2 \int \langle \mathcal{L}u - f, u \rangle ds = 2 \langle \mathcal{L}u, u \rangle \int ds - 2 \langle f, u \rangle \int ds
$$

$$
= \langle \mathcal{L}u, u \rangle - 2 \langle f, u \rangle = J(u)
$$

It can be shown that at least one solution to the nonlinear equation

$$
\mathcal{G}(u) = 0
$$

(5.5)

exists if $\mathcal{G}$ is potential and $\delta \mathcal{G}(u, h)$ is positive bounded below with respect to $h$ (i.e., $\langle \delta \mathcal{G}(u, h), h \rangle \geq \gamma^2 ||h||^2$). If the second Frechet differential $\delta^2 K(u, h, h) \geq ||h|| \lambda ||h||$, where $\lim_{s \to \infty} \lambda(s) = \infty$ and $\lambda(s) > 0$, then $K(u)$ is a relative minimum at $u$. 

8
6. Convergence, Positive Nonlinear Operators

We extend the notions of convergence in energy to a class of nonlinear potential operators which are gradients of positive-definite functionals K(u). In the nonlinear operator equation (5.5) suppose

\( \varphi \) is a potential operator

\[ \text{grad } K(u^*) = \varphi(u^*) = 0, \ u^* \text{ exists and } \epsilon \ \text{Dom}(\varphi) \text{ and } \min K(u) = K(u^*) \]

Then the solution \( u^* \) of (6.1) minimizes \( K(u) \).

Suppose that \( K(u-w) \equiv d_k(u-w) \) satisfies metric axioms. Then, if

\[ \min_{u \in \mathcal{H}} K(u) = K(v) \quad (6.1) \]

we have

\[ d_k(u^*-v) \leq d_k(u^*-w) \quad (6.2) \]

for every \( w \in \mathcal{H} \), and

\[ K(u^*) \leq K(v) \leq K(w) \quad (6.3) \]

The procedure for testing convergence is straightforward: Let \( w \in \mathcal{H} \) be a finite-element approximation that coincides with \( u^* \) and its principal derivatives at the nodal points of each element. For a given class of elements, show that \( \{ q_a \} \) are complete in an appropriate Sobolev norm. Then show that \( d_k(u^*-w) \to 0 \) under a sequence of uniform refinements. Convergence of the Ritz approximation to \( u^* \) then follows from (6.4).

While such a procedure works for only positive-definite \( K(u) \), the general method of approximation applies to arbitrary \( K(u) \). We consider examples in the following section.

7. Example Problems

Triangular Element Approximations. As examples, we consider the following
triangular element approximations: T1 - the simplex elements, with \( u^{(e)} \) linear over \( e \); T2 - the complex element, six nodes, one at each vertex and one at the midpoint of each side, \( u^{(e)} \) quadratic over \( e \); T3 - the quintic triangle, \( u, u_x, u_y, u_{xx}, u_{xy}, u_{yy} \), specified at each vertex and the normal derivative \( \partial u/\partial n \) specified at the midpoint of each side (Fig. 2). Let \( \delta = \max(\delta_1, \delta_2, \ldots, \delta_e) \), \( \theta_e \) be the smallest angle of the triangle \( e \), and \( \theta = \min(\theta_1, \ldots, \theta_e) \). We consider only regular refinements for which \( \theta = \text{constant} \).

Let \( u \) be a given function in \( \mathcal{H} \), \( w \) be a finite element approximation in \( \mathcal{H} \), and \( \varphi = u - w \). For T1, let \( w(\mathcal{X}^n) = u(\mathcal{X}^n) \) and \( |u_{ij}| \leq K \). Then [17],

\[
|\varphi, i| \leq \frac{K}{\sin \theta / 2} \delta^2 ; 
|\varphi| \leq \frac{K_1}{\sin \theta / 2} \delta^2
\]

(7.1)

For T2, Zlamal [21] has shown that if \( |D^i \varphi| \leq M \), \( |i| = 3 \), then

\[
|\varphi, i| \leq \frac{2}{\sin \theta M^2} \quad |\varphi| \leq M^3
\]

(7.2)

whereas for T3, if \( |D^i \varphi| \leq M \), \( |i| = 6 \), then

\[
|\varphi, i| \leq \frac{C}{(\sin \alpha)^i} M^6 - |i| \quad |i| \leq 4
\]

(7.3)

Now consider a functional \( K(u) \) of the form

\[
K(u) = \int \left\{ \sum_{m=1}^{r} \left( \sum_{i_1=1}^{n} \sum_{i_p=1}^{n} \left[ A^2 u^{3a} + A^{i_1}(u, i_1)^{2a} \\ \vdots A^{i_1 \cdots i_p}(u, i_1 \cdots i_p)^{2a} \right] \right) \right\} d\mathcal{R}
\]

(7.4)

where \( n \) is the dimension of \( \mathcal{R} \), \( u, i_1 \cdots i_p = \partial_p u/\partial x^{i_1} \cdots \partial x^{i_p} \), and \( \max\left| A^{i_1 \cdots i_p} \right| \leq K_{a_1 \cdots i_p} \) for all \( p \). In every case we assume \( u \) is differentiable as many times as required and that all \( p \)-partial derivatives are bounded on \( \mathcal{R} \). Then
\[ |y_{id}| \leq \frac{k}{\sin \theta/2} \delta \]

\[ |y_{ia}| \leq \frac{2}{\sin \theta} \delta^2 \]

\[ |D_{ij}^e| \leq \frac{k}{(\sin \theta)^{14}} \delta^{6-12i} \]

\[ |\tilde{u}| \leq 4 \]

Figure 2
\[ K(u) \leq \sum_{m}^{n} k_{\text{max}} |u|^2 + \sum_{i_1=1}^{h} \sum_{m}^{n} k_{i_1 \text{max}} |u_{i_1}|^2 \]

\[ + \cdots + \sum_{i_1}^{n} \sum_{i_2}^{n} \cdots \sum_{i_p}^{n} \sum_{m}^{r} k_{i_1 \cdots i_p \text{max}} |u_{i_1 \cdots i_p}|^2 \]  

(7.5)

Consequently, if \(|u_{i_1 \cdots i_p} - w_{i_1 \cdots i_p}| < C_p \delta\), and \(|u-w| < C_1 \delta^p\), then certainly \(K(u-w) \leq \delta^2\), \(B\) being a finite constant which can be determined in terms of \(k_{i_1 \cdots i_p}, r, n,\) and \(C_p\). Thus, as \(\delta \to 0\), \(d_k(u-w) \to 0\), which \(\Rightarrow d_k(u-v) \to 0\).

Consequently, the convergence of the finite-element approximation is proved.

**Special Cases.** To cite special examples, consider the functionals

\[ K_1(u) = \int_{\Omega} (u^2 + u_x^4 + u_y^4) d\Omega \]  

(7.6a)

\[ K_2(u) = \int_{\Omega} a(uu_x)^2 d\Omega \]  

(7.6b)

where \(a(uu_x)^2 > 0\) for all \((x,y) \in \Omega\). If a T1 approximation is used,

\[ K_1(u-w) \leq \delta^2 \left( \frac{K_1}{\sin \theta/2} \right)^2 + 2\delta \left( \frac{K_1}{\sin \theta/2} \right)^2 \]  

(7.7a)

\[ K_2(u-v) \leq \delta^2 K_2 a^2 K_1^2 \delta \sin \theta/2 \]  

(7.7b)

where \(\max|a| \leq A\). In either case, rapid convergence is obtained. For T2, we obtain terms \(O(\delta^8)\) in (7.7a) and \(O(\delta^2)\) in (7.7b). For T3, we obtain terms \(O(\delta^{24})\) and \(O(\delta^{30})\).

**Non-Positive K(u).** While our convergence arguments hold only for positive-definite \(K(u)\), we emphasize that the general procedure is still applicable if \(K(u)\) is not positive-definite. Consider the nonlinear boundary-value problem
\(2u\nabla^2 u + u_x^2 + u_y^2 = f(x,y)\) \hspace{1cm} (7.8)

\((x,y) \in \mathcal{R}\) and \(u = 0\) on \(\partial\mathcal{R}\). Defining

\[\mathcal{L}(u) = f - 2u\nabla^2 u - u_x^2 - u_y^2\]

we compute

\[\delta \mathcal{L}(u, h) = -2(\nabla u h_x - u_x h_x - u_y h_y)\] \hspace{1cm} (7.10)

If \(\mathcal{L}(h, g) = \int h g d\mathcal{R}\), we may show by successive integrations that \(\langle \delta \mathcal{L}(u, h), g \rangle = \langle \delta \mathcal{L}(u, g), h \rangle\) if \(h\) and \(g\) satisfy the homogeneous boundary conditions on \(\mathcal{R}\). Assuming this to be the case (i.e., limiting the domain of \(\delta \mathcal{L}(u, g)\) to functions with this property), it follows that \(\mathcal{L}(u)\) is potential. Thus, introducing (7.9) into (5.4) and performing the indicated integration, we obtain the non-positive-definite functional

\[K(u) = \int_{\mathcal{R}} \left[ u(u_x^2 + u_y^2) + fu \right] d\mathcal{R}\]

\hspace{1cm} (7.11)

clearly

\[\lim_{\alpha \to 0} \frac{1}{\alpha} [K(u + \alpha h) - K(u)] = \langle \mathcal{L}(u), h \rangle\]

\hspace{1cm} (7.12)

To solve (7.8) by the method of finite elements, we approximate \(\mathcal{R}\) by a collection of \(E\) finite elements connected together at \(G\) nodal points. If \(k(e)\) is the restriction of \(K(u)\) to \(e\), then, approximately,

\[K(u) = \sum_{e=1}^{E} k(e)(u)\]

\hspace{1cm} (7.13)

If \(\bar{u}(e) = u_0^N(\psi_N(e)(x,y) (N=1,2,\ldots,N_e)\) is the local approximation of \(u\) corresponding to \(r_e\), then
\[ k_N(u) = \sum_{N} \sum_{M} \sum_{P} a_{NMP}^{(e)} u_N^{(e)} u_M^{(e)} u_P^{(e)} + \sum_{N} f_N^{(e)} u_N^{(e)} \]  

(7.14)

where \( N, M, P = 1, 2, \ldots, N_e \) and

\[ a_{NMP}^{(e)} = \int_{\Omega_e} (\psi_N, x \psi_P, x + \psi_M, y \psi_P, y) d\mathbf{r}_e \]  

(7.15)

\[ f_N^{(e)} = \int_{\Omega_e} f_N^{(e)} d\mathbf{r}_e \]  

(7.16)

The connectivity of the model is established by

\[ u_N^{(e)} = \sum_{\Delta=1}^{G} \sum_{\Omega} u^\Delta_{\Omega} \]  

(7.16)

Thus, the global finite-element approximation of the functional is

\[ K(u) = \sum_{\Delta} \sum_{\Gamma} \sum_{\Omega} A_{\Delta\Gamma\Omega}^\Delta u^\Delta_{\Omega} u^\Gamma_{\Omega} + \sum_{\Delta} F_{\Delta} u^\Delta \]  

(7.17)

where

\[ A_{\Delta\Gamma\Omega}^\Delta = \sum_{e=1}^{E} \sum_{N} \sum_{M} \sum_{P} \sum_{\Omega} \sum_{\Omega} (e) N (e) N (e) P (e) a_{NMP}^{(e)} \]  

(7.18)

\[ F_{\Delta} = \sum_{e=1}^{E} \sum_{\Omega} \sum_{\Omega} (e) N f_{\Delta}^{(e)} \]  

(7.18)

Thus, we may see an approximate solution to (7.8) in the subspace \( \tilde{N} \) spanned by \( \psi^{(e)}(x,y) = \sum_{\Omega} \psi_N^{(e)}(x,y) \) by setting

\[ \frac{\partial K}{\partial u^\Delta} = \sum_{\Gamma} \sum_{\Lambda} A_{\Delta\Gamma\Lambda}^\Delta u^\Gamma_{\Omega} + F_{\Delta} = 0 \]  

(7.19)

By solving the system of \( G \) quadratic equations (7.19), we obtain the approximate solution \( \bar{u} = \sum_{\Delta} \bar{u}_{\Delta}^\Delta \) of (7.8) in \( \tilde{N} \).
Acknowledgement. The support of this work by the Air Force Office of Scientific Research through Contract F4462-69-C-0124 is gratefully acknowledged. Helpful discussions of this work with Professors H. J. Brauchli and J. Hoomani are also acknowledged.

8. References


