Finite Deflections of a Nonlinearly Elastic Bar

The problem of finite deflections of a nonlinearly elastic bar is investigated as an extension of the classical theory of the elastica to include material nonlinearities. A moment-curvature relation in the form of a hyperbolic tangent law is introduced to simulate that of a class of elastoplastic materials. The problem of finite deflections of a clamped-end bar subjected to an axial force is given special attention, and numerical solutions to the resulting system of nonlinear differential equations are obtained. Tables of results for various values of the parameters defining the material are provided and solutions are compared with those of the classical theory of the elastica.

Introduction

In the two centuries that have elapsed since Euler first examined the problem of the elastica, problems involving finite deflections of thin bars have attracted the attention of numerous investigators. An account of several applications of the theory of the elastica with references to most of the previous work can be found in the monograph by Frisch-Fay [1]. In the present paper we investigate a problem of finite deflections of a bar constructed of a nonlinearly elastic material characterized by a moment-curvature law similar to that exhibited by a class of elastoplastic materials.

Consider a straight prismatic bar of length L which undergoes finite deflections and rotations but small strains. Assuming that plane cross sections remain plane during the deformation, the local bending moment at a typical section can be expressed as a function of the change-in-curvature at the section:

\[ M = F' \left( \frac{d\theta}{ds} \right) \]  

(1)

Here \( \theta \) is the slope of the bar, \( s \) is the arc length along the deformed axis, and \( M \) is the bending moment. The specific form of \( F(\cdot) \) depends, of course, on the material properties of the bar.

In the following, we assume that the moment-curvature relation for the bar under consideration is of the form

\[ M = (D_0 - D_1) \varphi_0 \tan h \left( \frac{\varphi}{\varphi_0} \right) + D_2 \varphi \]  

(2)

where \( D_0 \) and \( D_1 \) are characteristic flexural rigidities, as indicated in Fig. 1, \( \varphi_0 \) is a material constant corresponding to a characteristic change-in-curvature, and

\[ \varphi = \frac{d\theta}{ds} \]  

(3)

This relation is of a form similar to the one-dimensional stress-strain relation proposed by Prager [2] for an elastoplastic material and used by Havner [3] in his analysis of nonstrain-hardening materials.

Cantilevered Bar

As a special case of (2), we consider a clamped-end column loaded axially by a force \( P \) as indicated in Fig. 2. In this case

\[ M = -Pv \]  

(4)

where \( v \) is the transverse deflection of the bar. Noting that \( \sin \theta \) = \( \frac{v}{(1-c)v} \), we arrive at the following set of four first-order nonlinear differential equations:

\[ \frac{dv}{ds} = \sin \theta \]  

(5)

\[ \frac{d\theta}{ds} = \varphi \]  

(6)

\[ \frac{d\varphi}{ds} = \frac{-P \sin \theta}{D_0(1 - (D_2 \varphi - P_0) + (D_0 - D_2)D_2 \varphi)} \]  

(7)

It is clear that in the case \( D_0 = D_1 \), (5) and (6) reduce to

\[ \frac{d\theta}{ds} = \frac{P}{D_0} \sin \theta = 0 \]  

(8)

which we recognize as the basic differential equation governing the comparable problem in the classical theory of the elastica [4].

To facilitate the solution of these equations, we introduce the dimensionless quantities

\[ u = v/L, \quad z = s/L, \quad \epsilon = D_0/D_2, \quad p = \varphi L, \quad \psi = D_0 \]  

(9)

where \( P_0 = \pi D_0/L^2 \) is the classical Euler buckling load. Introducing (7) into (5) and augmenting the system with the additional condition that the load parameter \( \lambda \) is independent of \( z \), we arrive at the following set of four first-order nonlinear differential equations,
Fig. 2 Nonlinearly elastic clamped-end column

\[
\frac{du}{dz} = \sin \theta,
\]

\[
\frac{d\theta}{dz} = \psi,
\]

\[
\frac{d\psi}{dz} = \frac{(\pi^2/4)\lambda \sin \theta}{1 - \gamma(\pi^2\lambda u/4 - \epsilon \psi)^4},
\]

\[
\frac{d\lambda}{dz} = 0.
\]

Solutions are subject to the boundary conditions

\[
u(0) = 0, \quad \psi(0) = 0, \quad \theta(0) = \alpha, \quad \theta(1) = 0^\circ
\]

where \(\alpha\) is the end slope as indicated in Fig. 2. Thus the end slope is to be specified and the load required to maintain a given slope characterized by the unknown load parameter \(\lambda\) is computed. A similar procedure is usually used in the classical problem of the elastica \[4\].

**Numerical Solution**

For convenience, we denote \(u = q_1, \quad \theta = q_2, \quad \psi = q_3, \quad \lambda = q_4\), and rewrite (8) in the vector form

\[q' = f(q)\]

where \(q = [q_1, q_2, q_3, q_4]\), \(q' = dq/dz\), and

\[f_1 = \sin q_1, \quad f_2 = q_2, \quad f_3 = \frac{(\pi^2/4)q_1 \sin q_1}{1 - \gamma(\pi^2q_1q_1/4 - \epsilon q_4)^4}, \quad f_4 = 0\]

We wish to solve (10) in an iterative manner. Following Bellman and Kalaba \[5\] and Kalaba \[6\], we use the Newton-Kantorovich technique to establish the linear equations relating the \(n\)th and \((n + 1)\)th iterations:

\[q^{n+1} = f(q_n) + J(q_n)[q_n - q_n]\]

Here

\[J(q_n) = \frac{\delta f(q_n)}{\delta q} [q_n - q_n]\]

The vector \(q_n\) is the solution of (10) subject to the initial conditions that made (12) satisfy (9) on the \(n\)th iteration. The initial condition for the first iteration is easily estimated from the known solutions of the classical elastica.

Equation (12) is linear with varying coefficients. It is solved as a superposition of the two particular solutions

\[p' = f_n + J_n[p - q_n],\]

\[r' = f_n + J_n[r - q_n]\]

subject to the initial conditions.
Fig. 5 $P/P_{cl}$ versus end slope $\alpha$ for $p_0L = 0.40$

Fig. 6 $P/P_{cl}$ versus end slope $\alpha$ for $p_0L = 0.05$

Table 1 $D_1/D_0 = 0.75$

<table>
<thead>
<tr>
<th>$\alpha$ (degrees)</th>
<th>$0^\circ$</th>
<th>$1^\circ$</th>
<th>$2^\circ$</th>
<th>$4^\circ$</th>
<th>$6^\circ$</th>
<th>$8^\circ$</th>
<th>$10^\circ$</th>
<th>$20^\circ$</th>
<th>$40^\circ$</th>
<th>$60^\circ$</th>
<th>$80^\circ$</th>
<th>$100^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P/P_{cl}$</td>
<td>1.000</td>
<td>1.000</td>
<td>0.999</td>
<td>0.995</td>
<td>0.990</td>
<td>0.983</td>
<td>0.974</td>
<td>0.925</td>
<td>0.857</td>
<td>0.825</td>
<td>0.809</td>
<td>0.809</td>
</tr>
<tr>
<td>$x_\alpha/L$</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>0.998</td>
<td>0.997</td>
<td>0.995</td>
<td>0.992</td>
<td>0.969</td>
<td>0.878</td>
<td>0.734</td>
<td>0.550</td>
<td>0.337</td>
</tr>
<tr>
<td>$y_\alpha/L$</td>
<td>0.000</td>
<td>0.011</td>
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<td>0.045</td>
<td>0.067</td>
<td>0.089</td>
<td>0.111</td>
<td>0.143</td>
<td>0.186</td>
<td>0.222</td>
<td>0.260</td>
<td>0.279</td>
</tr>
<tr>
<td>$\varphi_{0L}$</td>
<td>0.40</td>
<td>0.20</td>
<td>0.10</td>
<td>0.05</td>
<td>0.025</td>
<td>0.011</td>
<td>0.003</td>
<td>0.001</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
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</tr>
</tbody>
</table>

Results

Solutions were obtained using the procedure described in the previous section for $\alpha = 1, 2, 4, 6, 8, 10, 20, 40, \ldots, 100$ deg and for various values of $\epsilon$ and $\rho$. After a few iterations, $\beta$ approaches zero and the process is considered to have converged. In the cases considered, if iterations were continued until $\beta < 10^{-4}$, $\gamma(1)$ was $< 10^{-4}$ in all instances. Integrations were also performed for the case in which $\epsilon = 1, \rho = 0$ ($\gamma = 0$), and results were obtained which agreed with those of the classical elastica.

For purposes of discussion it is convenient to compare the moment-curvature relation in Fig. 1 (and equation (2)) with that of an elastoplastic material in which no unloading is permitted.

Then the parameter $\varphi_{0L}$ may be regarded as measure of the...
relative “elastic zone” of the material and \( \epsilon = D_1/D_0 \) is an indication of the relative stiffness of the material after yielding. Thus, for a given \( D_0 \), the smaller the value of \( \phi_0 L \) the more narrow the elastic portion of the \( M - \phi \) curve characterizing the material. Likewise, the smaller the value of \( \epsilon \) the softer the material after initial yielding. A value of \( \epsilon \) of unity, for example, might be used for an elastic-perfectly plastic material.

Figs. 3 and 4 indicate the variation of the nondimensional load \( P/P_{cl} \), where \( P_{cl} \) is the end force predicted by the classical theory of the elastica, versus the end slope \( \alpha \) for values of \( \phi_0 L = 0.40 \) and 0.05, respectively. For the rather wide elastic zone case typified by \( \phi_0 L = 0.40 \), it is seen that a reduction in the load required to produce an end slope of 100 deg is over 60 percent for the realistic case in which \( \epsilon = 0.25 \). For most mild steel bars, however, the value of \( \phi_0 L \) ranges between 0.02-0.10. For \( \phi_0 L = 0.05 \), it is seen that only 27 percent of the load predicted by the classical theory of the elastica is required to produce an \( \alpha \) of 100 deg. Other values of \( P/P_{cl} \) versus \( \alpha \) for various values of \( \phi_0 L \) and \( \epsilon \) are given in Tables 1-3.

For infinitesimal changes in curvature, equations (5) reduce to the familiar differential equation governing the stability of prismatic columns. Thus the bar under consideration buckles at the classical critical load \( P_{cr} = \pi^2 E I / \phi_0 L^2 \), where \( E I \) is the instantaneous flexural rigidity at \( \phi = 0 \). According to the classical elastica theory, an increase in axial load is required to increase the deformation (end slope) of the bar so that the stiffness of the bar increases with an increase in end slope. A quite different behavior is encountered in the present case, as is indicated by the plot of \( P/P_{cl} \) versus \( \alpha \) given in Fig. 5. For \( \phi_0 L = 0.40 \) it is seen that a sudden reduction in load occurs immediately after the column buckles. In cases in which the material is extremely stiff after “yielding” (that is, in those cases in which \( \epsilon = D_1/D_0 \geq 0.5 \) (say)), the column stiffness apparently becomes zero at \( \alpha \) ~ 40 deg and the curve acquires a positive slope as \( \alpha \) becomes larger. At this point the behavior approaches asymptotically that predicted by the classical elastica for a bar with flexural

### Table 2 \( D_1/D_0 = 0.50 \)

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>0°</th>
<th>1°</th>
<th>2°</th>
<th>4°</th>
<th>6°</th>
<th>8°</th>
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<tbody>
<tr>
<td>( P/P_{cl} )</td>
<td>1.000</td>
<td>0.999</td>
<td>0.998</td>
<td>0.996</td>
<td>0.994</td>
<td>0.992</td>
<td>0.989</td>
<td>0.986</td>
<td>0.983</td>
<td>0.980</td>
<td>0.977</td>
<td>0.974</td>
</tr>
<tr>
<td>( \phi_0 L )</td>
<td>0.000</td>
<td>0.011</td>
<td>0.022</td>
<td>0.044</td>
<td>0.067</td>
<td>0.090</td>
<td>0.112</td>
<td>0.235</td>
<td>0.458</td>
<td>0.680</td>
<td>0.865</td>
<td>0.800</td>
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</table>

### Table 3 \( D_1/D_0 = 0.25 \)

<table>
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<tr>
<th>( \alpha )</th>
<th>0°</th>
<th>1°</th>
<th>2°</th>
<th>4°</th>
<th>6°</th>
<th>8°</th>
<th>10°</th>
<th>20°</th>
<th>40°</th>
<th>60°</th>
<th>80°</th>
<th>100°</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P/P_{cl} )</td>
<td>1.000</td>
<td>0.999</td>
<td>0.998</td>
<td>0.997</td>
<td>0.995</td>
<td>0.993</td>
<td>0.991</td>
<td>0.989</td>
<td>0.987</td>
<td>0.984</td>
<td>0.982</td>
<td>0.979</td>
</tr>
<tr>
<td>( \phi_0 L )</td>
<td>0.000</td>
<td>0.011</td>
<td>0.022</td>
<td>0.044</td>
<td>0.067</td>
<td>0.090</td>
<td>0.112</td>
<td>0.235</td>
<td>0.458</td>
<td>0.680</td>
<td>0.800</td>
<td>0.800</td>
</tr>
</tbody>
</table>

For the classical critical load \( P_{cr} = \pi^2 E I / \phi_0 L^2 \), where \( E I \) is the instantaneous flexural rigidity at \( \phi = 0 \). According to the classical elastica theory, an increase in axial load is required to increase the deformation (end slope) of the bar so that the stiffness of the bar increases with an increase in end slope. A quite different behavior is encountered in the present case, as is indicated by the plot of \( P/P_{cl} \) versus \( \alpha \) given in Fig. 5. For \( \phi_0 L = 0.40 \) it is seen that a sudden reduction in load occurs immediately after the column buckles. In cases in which the material is extremely stiff after “yielding” (that is, in those cases in which \( \epsilon = D_1/D_0 \geq 0.5 \) (say)), the column stiffness apparently becomes zero at \( \alpha \) ~ 40 deg and the curve acquires a positive slope as \( \alpha \) becomes larger. At this point the behavior approaches asymptotically that predicted by the classical elastica for a bar with flexural
rigidity $D_1$. For $\varphi \lambda = 0.05$ and $\epsilon = 0.25$, a case not too different from many mild steel columns, there is a sharp decrease in load after buckling and the slope remains negative for a large portion of the range of values of $\alpha$ considered, Fig. 6. This, of course, is in agreement with experimental data available on the postbuckling behavior of elastoplastic columns, e.g., [7]. It is noted, however, that as long as $\epsilon > 0$, there is always a value of $\alpha$ for which the slope of the $P/P_{\infty}$ versus $\alpha$ curve will begin to increase and to approach asymptotically the curve corresponding to the postbuckling behavior predicted by classical elatic theory for an elastic column with flexural rigidity $D_1$.

References


