A Technical Note:

A FINITE-ELEMENT ANALOGUE OF THE NAVIER-STOKES EQUATIONS

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INTRODUCTION

Applications of numerical techniques to problems of unsteady fluid flow have long been looked upon as the only means for obtaining quantitative solutions to problems involving complex geometries and boundary conditions. A large literature exists on discrete models of fluid flow obtained by using finite difference approximations of the governing differential equations. Models of linear problems based on the finite-element concept have only recently appeared in the literature, but these exhibit many advantages over conventional methods of discretation due to the simplicity with which boundary conditions can be applied and the ease with which complex and multiply-connected domains can be approximated.

Applications of the finite element method to a restricted class of problems in potential flow and the flow of viscous incompressible fluids have recently been presented\(^1\),\(^2\). These have either required the availability of an associated variational principle or have considered incompressible flow under prescribed pressure fields or compressible flow in which the continuity equation is implicitly satisfied and the fluid density is known as a function of time. As such, they do not represent completely general models of general fluid flow or of the Navier-Stokes equations. It is the purpose of this

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note to present brief derivations of the finite element equations describing a discrete model of compressible and incompressible Stokesian fluids.

ENERGY BALANCES

Consider isothermal motion of an arbitrary fluid. If the continuity equation and the principle of balance of linear momentum is satisfied, then a global form of the law of conservation of energy can be written

$$\int \frac{Dv_i}{Dt} \rho dV + \int t_{ij} d_i_j dV = \Omega$$

(1)

where $\rho$ is the mass density, $v_i$ are the components of the velocity field, $t_{ij}$ is the Cauchy stress tensor, $d_{ij}$ is the rate-of-deformation tensor, and $\Omega$ is the mechanical power of the external forces:

$$\frac{Dv_i}{Dt} = \frac{\partial v_i}{\partial t} + v_k v_{i,k}$$

(2)

$$d_{ij} = \frac{1}{2} (v_i,j + v_j,i)$$

(3)

$$\Omega = \int F_i v_i dV + \int S_i v_i dA$$

(4)

Here $F_i$ and $S_i$ are the body and surface forces and the comma denotes partial differentiation with respect to a fixed system of spatial cartesian coordinates $x_i$.

In addition to (1), we have required that the continuity equation

$$\frac{\partial \rho}{\partial t} + (\rho v_k),_k = 0$$

(5)

be satisfied at every point in the continuum.

FINITE ELEMENT MODEL OF FLUID FLOW

We now construct a finite element model of the region $\mathcal{R}$ through which the
fluid flows. This consists of a collection of a finite number of connected subregions, called finite elements, which are generally assumed to be of some relatively simple geometric shape. We identify a number of nodal points in and on the boundaries of each element, so that the whole assembly is viewed as being connected together at various boundary nodes. We then isolate a typical finite element and consider the flow of fluid through it independent of the other elements.

Let \( \rho \) and \( v_i \) denote the density and velocity fields associated with a typical element \( e \). We proceed by constructing local approximations of these fields over the element which are uniquely determined by the values of \( \rho \) and \( v_i \) at the node points of the element. The local approximations are of the form

\[
\rho_{e}(\mathbf{x}) = \psi^N(x) \rho_{N}^{(e)} \\
v_i(x) = \psi^N(x) v_{N1}^{(e)}
\]

(6)

where \( \rho_{N}^{(e)} \) and \( v_{N1}^{(e)} \) are the values of the local fields \( \rho_{e}(\mathbf{x}) \) and \( v_i(x) \) at node \( N \) of element \( e \). The repeated nodal indices \( N \) in (6) are to be summed from 1 to \( N_e \), where \( N_e \) is the total number of nodes of element \( e \). The local interpolation functions \( \psi^N(x) \) are generally selected so that \( \rho \) and \( v_i \) are continuous across interelement boundaries once the elements have been connected to form the complete discrete model. Procedures for connecting elements together and applying boundary conditions are well-documented and will not be discussed here. The functions \( \psi^N(x) \) also have the properties

\[
\psi^N(x_M) = \delta^N_M = \begin{cases} 
1 & \text{for } N = M \\
0 & \text{for } N \neq M
\end{cases}
\]

where \( \delta^N_M \) is the Kronecker delta and \( x_M = x_{N1} \) denotes the coordinates of node \( M \) of the element.

Introducing (6) into (1) and simplifying, we obtain
where \( \mathcal{V}_e \) is the volume of the element, \( \mathbf{p}_i^N \) are the components of generalized force at node \( N \), and \( a^{MQN}, b^{QRN}_a \) are multidimensional arrays:

\[
a^{MQN} = \int_{\mathcal{V}_e} \psi^M(\mathbf{x}) \dot{\psi}_0^0(\mathbf{x}) \dot{\psi}^N(\mathbf{x}) d\mathcal{V}
\]

\[
b^{QRN}_a = \int_{\mathcal{V}_e} \psi^M(\mathbf{x}) \dot{\psi}_0^0(\mathbf{x}) \dot{\psi}_a^0(\mathbf{x}) \dot{\psi}^N(\mathbf{x}) d\mathcal{V}
\]

\[
\mathbf{p}_i^N = \int_{\mathcal{V}_e} F_i \dot{\psi}^N(\mathbf{x}) d\mathcal{V} + \int_{A_e} S_i \dot{\psi}^N(\mathbf{x}) dA
\]

In these equations we have dropped the element identification label \( e \) for simplicity; \( M, N, Q, R, i, j, m = 1, 2, \ldots, Ne \), and \( i, j, m = 1, 2, 3 \) and all repeated indices are summed.

Since (1), and consequently (8), must hold for arbitrary continuous velocity fields, the term in brackets in (8) must vanish. Thus, we obtain for the equations of motion for a typical fluid element the system of nonlinear equations

\[
a^{MQN} \rho M \dot{\mathbf{v}}_Q + b^{QRN}_a \rho N \mathbf{v}_Q \mathbf{v}_R = \mathbf{p}_i^N
\]

This result applies to arbitrary fluids since the form of the constitutive equation for stress is, as yet, unspecified.

By following a similar procedure, we also obtain a finite element model of (5), the continuity equation:

\[
c^{NK} \rho M \dot{\mathbf{v}}_N + d^{NMR}_K \rho N \mathbf{v}_N = 0
\]
where
\[ c^{NN} = \int_{V_e} \psi^M(x) \psi^N(x) \, dV \]  
(14)

\[ d^N_{kR} = \int_{V_e} \psi^N(x) \left( \psi^M(x) \frac{\partial^2 \psi^N(x)}{\partial x_k^2} \right) \, dV \]  
(15)

Here \( n_k \) are the components of a unit vector normal to the bounding surface area of the element \( A_e \).

**COMPRESSIBLE STOKESIAN FLUIDS**

For adiabatic flows of compressible Stokesian fluids, the stress tensor \( t_{ij} \) is of the form
\[ t_{ij} = (-\pi(\rho^{-1}, \theta) + \lambda \psi_{kk}) \delta_{ij} + 2\mu \psi_{ij} \]  
(16)

where \( \pi \) is the thermodynamic pressure which must be given by an equation of state for the fluid, \( \theta \) is the absolute temperature, and \( \lambda \psi \) and \( \mu \psi \) are the dilatational and shear viscosities, respectively.

The tensor \( d_{ij} \) for the finite element is obtained in terms of the nodal velocities by introducing (6) into (3). If we then incorporate (16) into (12), we arrive at the finite equations for compressible Stokesian fluids:
\[ a^{\alpha N} \rho M \psi_{Q1} + b^{\alpha N} \rho M \psi_{Qh} \psi_{R1} \]
\[ + \int_{V_e} \left[ (\lambda \psi^N \psi_k^N - \pi) \delta_{ij} + 2\mu \psi_{ij} \psi_{Nj} \right] \psi_{Nj} \, dV = p^N \]  
(17)

To these equations we must add the finite element analogue of the continuity equation, Eq. (15).

**INCOMPRESSIBLE STOKESIAN FLUIDS**

In the case of incompressible fluids, \( \pi \) becomes the hydrostatic pressure
\[ p, \text{ the density } p \text{ is a constant, and the incompressibility condition} \]

\[ d_{kk} = 0 \]  \hspace{1cm} (18)

must be satisfied. Then (16) reduces to

\[ t_{ij} = -p\delta_{ij} + 2\mu \psi_{ij} \]

(19)

and the equation of motion for an element becomes

\[ m^{NM} \dot{v}_N + e^{NMR}_{\psi} v_{M} v_{\psi} + \int_{\Omega} \psi^{N}_{\psi} \left( 2\mu \psi^{\psi}_{\psi} v_{M} - p\delta_{ij} \right) d\Omega = p^{\psi} \]  \hspace{1cm} (20)

wherein \( m^{NM} \) and \( e^{NMR}_{\psi} \) are the mass and convected mass matrices for the element:

\[ m^{NM} = \int_{\Omega} \rho \psi^{N}(x) \psi^{\psi}(x) d\Omega \]  \hspace{1cm} (21)

\[ e^{NMR}_{\psi} = \int_{\Omega} \rho \psi^{N}(x) \psi^{\psi}(x) \psi^{\psi}_{\psi}(x) d\Omega \]  \hspace{1cm} (22)

Although (13) is now implicitly satisfied, (20) represents \( 3N_e \) nonlinear differential equations in the \( 3N_e + 1 \) unknown nodal velocities \( v_{N1} \) and the uniform element hydrostatic pressure \( p \). To complete the system, an additional equation is needed. This is furnished by the incompressibility condition (18), which, for the finite element, is satisfied in an average sense by

\[ \int_{\Omega} d_{kk} d\Omega = v_{Nk} \int_{\Omega} \psi^{N}_{\psi}(x) d\Omega = 0 \]  \hspace{1cm} (23)

Equations (20) and (23) complete the description of motion of a finite element of an incompressible Stokesian fluid.

**EXAMPLE OF ONE-DIMENSIONAL FORMS**

Although a detailed exploration of results obtained using these equations is not within the scope of this note, it is informative to examine the forms
of the nonlinear equations for a simple one-dimensional case. Consider the case of one-dimensional compressible flow through a typical finite element of unit length. In this case, if \( x \) is a local coordinate, we can take as a first approximation

\[
\psi^1(x) = 1 - x \quad \psi^2(x) = x \quad (24)
\]

Then Eqs. (7) are satisfied and we find from (9), (10), (14) and (15) that in this case

\[
a^{xQ1} = \frac{1}{12} \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \quad a^{xQ2} = \frac{1}{12} \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \quad (25)
\]

\[
b^{xQ1n} = -a^{xQn} \quad b^{xQ2n} = a^{xQn} \quad (26)
\]

\[
c^{NM} = \frac{1}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad (27)
\]

\[
d^{NM1}_{1} = \frac{1}{6} \begin{bmatrix} -4 & 1 \\ -2 & -1 \end{bmatrix} \quad d^{NM2}_{1} = \frac{1}{6} \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \quad (28)
\]

Denoting

\[
\pi_i = \int_{i-1}^{i} \pi(\rho^{-1})dx - \int_{i}^{i+1} \pi(\rho^{-1})dx \quad (29)
\]

Introducing Eqs. 25-28 into 12 and 13, and connecting a series of finite elements together to form the total discrete mode, we find that at a typical node \( i \) the equation of motion is

\[
(\rho_i + \rho_{i+1})\ddot{v}_{i-1} + (\rho_{i-1} + 6\rho_i + \rho_{i+1})\ddot{v}_i + (\rho_i + \rho_{i+1})\ddot{v}_{i+1} \\
+ (\rho_i + \rho_{i-1})(\dot{v}_i - \dot{v}_{i-1})\dot{v}_{i-1} + (\rho_{i-1} + 3\rho_i)(\dot{v}_i - \dot{v}_{i-1})\dot{v}_i \\
+ (3\rho_i + \rho_{i+1})(\dot{v}_{i+1} - \dot{v}_i)\dot{v}_i + (\rho_i + \rho_{i+1})(\dot{v}_{i+1} - \dot{v}_i)\dot{v}_{i+1} \\
+ 12 (\lambda_n + 2\mu_n)(\dot{v}_{i-1} - 2\dot{v}_i + \dot{v}_{i+1}) - 12\pi_i = 0 \quad (30)
\]
and the continuity equation is

\[ \dot{\rho}_{i+1} + 4\dot{\rho}_i + \dot{\rho}_{i-1} + \rho_{i+1}(v_{i+1} - v_i) + \rho_i(v_{i+1} - v_{i-1}) \]

\[ - \rho_{i-1}(v_{i-1} - 2v_i) = 0 \quad (31) \]

Due to the averaging-character of finite element approximations, the forms of these equations are noticeably different than those of the corresponding finite difference approximations of the Navier-Stokes equations. For the case of steady flow, all of the interiord terms in Eq. 30 disappear and only the underlined terms remain. For incompressible flow of homogeneous fluids, \( \rho_{i-1} = \rho_i = \rho_{i+1} = \rho = \text{constant} \) and \( \pi_i \) becomes the difference in hydrostatic pressures between elements spanning nodes \( i-1, i \) and \( i, i+1 \). Equation 30 then reduces to

\[ \rho(\dot{v}_{i-1} + 4\dot{v}_i + \dot{v}_{i+1}) + \rho(v_{i+1}^2 - v_i^2) + \rho(v_{i+1} - v_i)v_i + 6(\lambda_n + 2\mu_n)(v_{i-1} - 2v_i + v_{i+1}) - 6\pi_i = 0 \quad (32) \]

while the continuity equation becomes

\[ v_{i-1} - 2v_i + v_{i+1} = 0 \quad (33) \]

which, in this case, is identical to the central difference approximation of the incompressibility condition

\[ \text{div } \mathbf{v} = \mathbf{v} \cdot \nabla = \frac{d^2\mathbf{v}}{dx^2} = 0 \quad (34) \]

**CONCLUSION**

Finite element analogues describing the motion of compressible and incompressible Stokesian fluids may be developed without resorting to variational principles by considering energy balances over an element. These involve systems of nonlinear ordinary differential equations in the nodal velocities.
and, for compressible fluids, the nodal densities $\rho_N$. In the case of compressible fluids, a finite element model of the continuity equation must be derived to supplement the equations of motion. For incompressible flows, an incompressibility condition involving the nodal velocities must be added, both to ensure incompressibility in an average sense over the element and to compute element hydrostatic pressures if they are not specified a priori.

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**REFERENCES**


