Finite Plane Strain of Incompressible Elastic Solids by the Finite Element Method

J. TINSLEY ODEN

(University of Alabama in Huntsville)
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Summary: The finite element method is extended to the problem of finite plane strain of elastic solids. A highly elastic body subjected to two-dimensional deformations is represented by an assembly of triangular elements of finite dimension. The displacement fields within each element are approximated by linear functions of the local coordinates. Non-linear stiffness relations involving generalised node forces and displacements are derived from energy considerations. For demonstration purposes, the non-linear stiffness equations are applied to the problems of finite simple shear and generalised shear. For finite simple shear, it is shown that these relations are in exact agreement with finite elasticity theory. Convergence rates of finite element representations of these problems are briefly examined.

1. Introduction

This paper investigates the application of the finite element method to the problem of finite plane strain of incompressible elastic bodies. Previous investigations have dealt with applications of the finite element concept to plane stress problems involving finite deformations and with general finite element formulations for the analysis of finite deformations of three-dimensional elastic and viscoelastic bodies of general shape.

The development of finite element formulations for the problem of finite plane strain of elastic solids involves several considerations not encountered in plane stress formulations. Available constitutive equations for highly elastic materials, such as natural rubber and certain solid propellant fuels, assume that such materials are incompressible. The deformation of incompressible bodies determines the stress tensor only to within a hydrostatic pressure. In the general theory of finite elastic deformations, this hydrostatic pressure is sometimes determined from equilibrium conditions and static (kinetic) boundary conditions. In plane stress problems, for example, the hydrostatic pressure is determined from the condition that the normal stress in the direction normal to the plane in which the body deforms is zero. With isochoric deformations, the hydrostatic pressure is arbitrary.

In the present investigation, the incompressibility condition is regarded as a condition of internal constraint, and the hydrostatic pressure appears as a Lagrange multiplier in the definition of an energy functional. This procedure follows the
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general variational principles for constrained continua presented by Truesdell and Toupin⁹, suggested for incompressible finite elements of three-dimensional bodies by Oden⁴ and used for infinitesimal deformations of incompressible solids by Hermann¹⁰. The approach adopted in the present investigation leads to stiffness relations which are non-linear in the generalised displacements. Solutions to representative finite plane strain problems are examined; it is shown that the present formulation leads to exact solutions in the case of finite simple shear. The rate of convergence of the procedure is also examined for the case of generalised shear.

Notation

- $A_0$: undeformed area of finite element
- $C_1, C_2$: Mooney constants
- $F_a$: components of body force
- $I_1, I_2, I_3$: strain invariants
- $P_{Na}$: global generalised forces
- $S_i$: components of surface force
- $U_{Na}$: global generalised displacements
- $V, V^*$: energy functionals
- $W$: strain energy density
- $c_{on}$: displacement coefficients
- $k_N$: rigid-body displacement coefficients
- $p$: hydrostatic pressure
- $p_{N,S}$: local generalised force
- $u_{N,S}$: local generalised displacements
- $\gamma_{ij}$: strain tensor
- $\delta_{ij}$: Kronecker delta
- $\epsilon_{ij}^{ab}$: permutation symbol
- $\lambda$: extension ratio
- $\psi$: strain invariant
- $\sigma^{ab}$: stress tensor
- $\Omega_{MN}$: mapping function

2. Finite Plane Deformations

Consider a continuous body which undergoes finite deformations parallel to a plane. To describe the deformation of this body, a fixed Cartesian coordinate system...
$x_i (i = 1, 2, 3)$ is established in the undeformed state so that the plane in which the body deforms is parallel to the $x_1x_2$ plane. In addition to the plane deformations, the body may be subjected to a uniform extension parallel to the $x_3$ axis.

Let $y_i$ denote the Cartesian coordinates in the deformed body of a particle which has coordinates $x_i$ in the undeformed state. Then

$$y_i = x_i + u_i; \quad x_3 = y_3/\lambda,$$

where $u_i (i = 1, 2)$ are the components of displacement and $\lambda$ is the constant extension ratio normal to the plane of deformation. The Lagrangian strain tensor is defined by Ref. 8 as

$$\gamma_{ij} = \frac{1}{2} \left( \frac{\partial y_m}{\partial x_i} \frac{\partial y_m}{\partial x_j} - \delta_{ij} \right).$$

where $\delta_{ij}$ is the Kronecker delta. Introducing equations (1) into (2) we find, for the case of finite plane strain,

$$\gamma_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),$$

$$\gamma_{as} = 0,$$

$$\gamma_{s3} = \frac{1}{2} (\lambda^2 - 1).$$

Three invariants can be formed from the deformation tensor $\delta_{ij} + 2\gamma_{ij}$. For finite plane strain, the principal invariants are

$$I_1 = 2 (1 + \gamma_{as}) + \lambda^2,$$

$$I_2 = 1 + 2\lambda^2 + 2 (1 + \lambda^2) \gamma_{as} + \psi,$$

$$I_3 = \lambda^2 (1 + 2\gamma_{as} + \psi),$$

in which

$$\psi = 4 (\gamma_{as} \gamma_{s3} - \gamma_{as} \gamma_{s3}) = 2 \epsilon_{as} \epsilon_{s3} \gamma_{as} \gamma_{s3},$$

where $\epsilon_{as}$ and $\epsilon_{s3}$ are the two-dimensional permutation symbols.

3. Finite Element Representation

The continuous body is now represented by an assembly of a finite number of flat triangular elements as shown in Fig. 1. The dimensions of these elements are assumed to be sufficiently small in comparison with characteristic dimensions of the body that the displacement field within the element is adequately approximated by linear functions of the coordinates $x_i$ of the form

$$u_i = a_i + b_{i\rho} x_{\rho},$$

in which $a_i$ and $b_{i\rho}$ are undetermined constants. By evaluating equation (6) at each of the three node points of an element, six equations are obtained which can be
solved to obtain $u_a$ and $b_{ab}$ in terms of the node displacements. If these solutions are introduced into equation (6), it is found that the displacement field corresponding to element $e$ is

$$u_{at} = k_N u_{N,t} + c_{ab} u_{ab} x_a.$$  

(7)

where

$$k_N = \frac{1}{4 A_0} \varepsilon_{SMK} \alpha_{MN} x_{MN}.$$  

(8a)

and

$$c_{ab} = \frac{1}{2 A_0} \left[ \begin{array}{ccc} x_2 - x_3 & x_3 - x_1 & x_1 - x_2 \\ x_1 - x_3 & x_3 - x_2 & x_2 - x_1 \end{array} \right].$$  

(8b)

In these equations, $u_{N,t}$ are the displacement components of node $N$ of element $e$. $\varepsilon_{SMK}$ is the three-dimensional permutation symbol, $A_0$ is the area of undeformed triangle, $x_{MK}$ are the Cartesian coordinates of node $M$. Note that the sum is to be taken on all repeated indices in equations (7) and (8a), including $N$, $M$, and $K$. Ranges of these indices are: $\alpha, \beta, \lambda, \mu = 1, 2$; $M, N, K = 1, 2, 3$; $e = 1, 2, \ldots, E$, where $E$ is the total number of elements. Unless noted otherwise, subsequent developments pertain to a typical finite element which, for the present, is considered to be independent of other elements in the discrete model. Thus, the element identification index $e$ is temporarily dropped for simplicity.

Substituting equation (7) into equation (3a) gives, for the strain components in a typical finite element,

$$\gamma_{ab} = \frac{1}{4} (c_{ab} u_{N,b} + c_{ab} u_{aN} + c_{ab} c_{MN} u_{MN} u_{MN})$$  

(9)

with, of course, $\gamma_{ab} = 0$

and $\gamma_{ab} = \frac{1}{4} (\lambda^2 - 1) = \text{constant}$. 

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4. Non-Linear Stiffness Equations

It is now assumed that the body under consideration is perfectly elastic and that there exists a potential function $W$ which represents the strain energy per unit of undeformed volume. If the body is isotropic, $W$ can be expressed as a function of the three invariants defined in equations (4). The total potential energy of a typical finite element is given by

$$V = \int_{V_0} W \, dv - \int_{V_0} F_a u_a \, dv - \int S_a \, ds. \tag{10}$$

where $v_0$ is the volume of the undeformed element, $F_a$ are components of body force per unit volume, and $S_a$ are components of the surface tractions per unit surface area (arc) $ds$.

Although different finite elements in the system may have different material properties, individual elements are assumed to be homogeneous. Thus, $W$ is not a function of the initial coordinates $x_i$ and can be factored outside the integral in equation (10). Taking this into account, and substituting equation (7) into equation (10), the potential energy becomes

$$V = v_0 W - p_{N_a} u_{N_a} \tag{11}$$

in which

$$p_{N_a} = \int F_a (k_N + c_{PN} x_P) \, dv + \int S_a (k_N + c_{SA} x_S) \, ds. \tag{12}$$

The quantities $p_{N_a}$ ($N = 1, 2, 3; \alpha = 1, 2$) are the components of "consistent" generalised force. Physically, $p_{N_a}$ is the generalised force at node $N$ in direction $\alpha$. It is important to note that the forces $S_a$ are, in general, functions of the displacements, since their direction and the area on which they act may change with the deformation.

According to the principle of minimum potential energy

$$\frac{\partial V}{\partial u_{N_a}} = 0. \tag{13}$$

Thus

$$p_{N_a} = v_0 \frac{\partial W}{\partial u_{N_a}}. \tag{14}$$

Subsequent considerations are restricted to incompressible isotropic materials. For such materials, $W$ is a function of only invariants $I_1$ and $I_2$, and the incompressibility condition

$$I_3 = 1 \tag{15}$$

must be satisfied. In particular, we consider highly elastic materials with strain energy functions of the Mooney form

$$W = C_1 (I_1 - 3) + C_2 (I_2 - 3), \tag{16}$$

where $C_1$ and $C_2$ are material constants.
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Introducing equation (15) into equations (4), it is found that

\[ I_1 = c - \psi \]  
(17a)

\[ I_2 = c - \lambda^2 \psi \]  
(17b)

\[ c = 1 + \lambda^3 + \frac{1}{\lambda^2}. \]  
(17c)

Substituting equations (5), (16), and (17) into equation (14) leads to a non-linear stiffness relation for finite plane strain of a finite element of an isotropic, incompressible, elastic body; but the incompressibility condition is satisfied only in an average sense over the element. Moreover, an arbitrary hydrostatic pressure \( p \) does not affect \( W \) and hence is not accounted for in the stiffness relation. To obtain a non-linear stiffness relation which includes hydrostatic pressure, we introduce a new energy functional \( V^* \) defined by

\[ V^* = V + \nabla p (I_2 - 1). \]  
(18)

Here equation (15) is regarded as a condition of internal constraint and the hydrostatic pressure plays the role of the Lagrange multiplier. This procedure follows that suggested by Truesdell and Toupin\(^4\) and Hermann\(^5\) and used by Oden\(^6\) in the formulation of three-dimensional problems. To be consistent with the linear displacement approximation, the hydrostatic pressure is regarded as a constant for each finite element.

Minimising \( V^* \) with respect to \( u_{na} \), without introducing equation (15) leads to the alternative form of the non-linear stiffness relation which involves the hydrostatic pressure:

\[ p_{na} = 2\nu_0 (\delta_{na} + c_{n\alpha} u_{\alpha n}) \{ c_{n\alpha} [C_1 + (1 + \lambda^2) C_2 + \lambda^2 p] + 2 (C_3 + \lambda^2 p) e_{,n} e_{,a} C_{,n} \gamma_{,aa} \}. \]  
(19)

Equation (19) represents six simultaneous non-linear equations for each finite element in the seven unknowns \( u_{na} \) and \( p \). The seventh equation needed is obtained by introducing the incompressibility condition in equation (15).

It is interesting to note that, for certain types of deformation, equation (15) is automatically satisfied for any type of material (e.g., isochoric deformations such as simple shear). The incompressibility condition then provides no information and the hydrostatic pressure is indeterminate. In such cases, additional information, usually in the form of conditions on the stress components, must be introduced if \( p \) is to be evaluated.

The stress tensor, measured per unit area in the deformed body and referred to convected coordinates \( x^a \), is given by

\[ \sigma^{ab} = \frac{1}{2\sqrt{(I_2)}} \left( \frac{\partial W}{\partial y_{,a}} + \frac{\partial W}{\partial y_{,b}} \right), \quad \sigma^{aa} = 0, \quad \sigma^{ab} = \frac{\lambda^3}{\sqrt{(I_2)}} \frac{\partial W}{\partial y_{,a}}. \]  
(20)
If $W$ is of the Mooney form,

$$
s^{0} = 2C_1\delta^{0} + 2C_2 \left[ \delta^{0} \left( \lambda^2 + \frac{1}{\lambda} \right) - 2\epsilon^{1}e^{\delta x}y_{x} \right] + \rho \left( \delta^{0} + 2\epsilon^{1}e^{\delta x}y_{x} \right) + p
$$

$$
\sigma^{0} = 0
$$

$$
\sigma^{0} = 2\gamma^{2}C_1 + 4\lambda^2 \left( 1 + \frac{c_{12}u_{S_{1}} + \frac{1}{2} c_{12}c_{22}u_{S_{2}}u_{S_{3}} }{C_1} \right) C_1 + p.
$$

The stiffness relations in equations (19) apply to a single finite element. To appropriately connect all of the elements together so as to obtain the complete discrete model of a given elastic body, group transformations of the form proposed by Wissmann¹, Oden², and Oden and Sato³ are used. Briefly, let $P_{NS}$ and $U_{NS}$ denote the generalised node forces and displacements at node $N$ of the assembled system of finite elements. In this case, $N$ ranges from 1 to $k$, where $k$ is the total number of nodes in the assembled system. Then

$$
P_{NS} = \Omega_{NS} P_{NS}, \quad u_{NS} = \Omega_{NS} U_{NS},
$$

in which $\Omega_{NS} = 1$ if node $M$ of element $e$ is identical to node $N$ of the assembled system and $\Omega_{NS} = 0$ if otherwise. In equations (22), $N = 1, 2, \ldots, k; M = 1, 2, 3; \alpha = 1, 2; \text{and } e = 1, 2, \ldots, E$, where $E$ is the total number of finite elements, and all repeated indices are to be summed throughout their admissible ranges.

Substituting equation (19) into equation (22) leads to global stiffness relations for the entire assembly of the elements of the form

$$
P_{NS} = K_{NS} (U_{NS}).
$$

The functions $K_{NS}(U_{NS})$ are non-linear in $U_{NS}$. After appropriate boundary conditions are applied, equations (23) reduce to a system of independent non-linear algebraic equations in the unknown node displacements. These are solved numerically. Stresses and strains in individual elements are then obtained from equations (9), (20), and (21).

5. Sample Problems

Consideration is now given to the application of the non-linear stiffness relations derived previously to some relatively simple problems of finite plane strain.

![Figure 2.](image-url)
for which the exact solutions are known. For convenience, the material is assumed to have a strain energy function of the Mooney form.

5.1. SIMPLE SHEAR

The classical problem of simple shear involves finite plane deformations of a rectangular section of a material in which straight line elements parallel to, say, the \( x \), \( y \) axis are displaced relative to one another in the \( y \) direction, but remain straight and parallel in the deformed body. A cubic specimen in simple shear is shown in Fig. 2(a).

Let \( A, B, C, \) and \( D \) denote corner points in the specimen, and let \( \Delta \) denote the total horizontal displacement of \( A \) and \( B \) relative to \( C \) and \( D \). Assuming that \( \lambda = 1 \) and employing the finite element network shown in Fig. 2(b), it is found that, for the \( k \)th element on the line \( AC \) (shaded in the figure),

\[
\begin{bmatrix}
    0 & -1 & 1 \\
    1 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
    u_{1k} \\
    u_{2k} \\
    u_{3k}
\end{bmatrix} = \begin{bmatrix}
    k\Delta \\
    (k-1)\Delta
\end{bmatrix} \quad (24)
\]

Introducing these values into equations (9), we find, for the strains,

\[
\gamma_{11} = \gamma_{22} = \gamma_{33} = 0, \quad \gamma_{12} = \gamma_{21} = \frac{\Delta}{2}, \quad \gamma_{23} = \frac{1}{2} \Delta^2. \quad (25)
\]

Thus, the strain components are independent of \( n \) (and \( k \)) and the same result is obtained for all elements in the network, regardless of the number of elements used. This follows immediately, of course, from the fact that in simple shear all elements experience pure homogeneous strain, which is precisely the mode of deformation assumed in deriving stiffness relations for finite elements. Thus, the finite element solution is exact for the case of simple shear.

For a Mooney material, we find from equation (21) that

\[
\begin{align*}
\sigma_{11} &= 2C_1 + 2C_2 (2 + \Delta^2) + p (1 + \Delta^2) \\
\sigma_{22} &= -(2C_1 + p) \Delta \\
\sigma_{33} &= 2C_1 + 4C_2 + p.
\end{align*}
\quad (26)
\]

The incompressibility condition is automatically satisfied (isochoric deformation) and equations (26) are in exact agreement with the solutions given by Green and Adkins.

5.2. GENERALISED SHEAR

In the case of generalised shear, sides \( AC \) and \( BD \) of the specimen are allowed to deform in a general manner (Fig. 3(a)). For an incompressible, isotropic solid...
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Figure 3.
(a) Generalised shear deformations and (b) deformed finite element network.

with strain energy of the Mooney form. Green and Adkins (Ref. 8, p. 122) have shown that the curve $AC$ (and $BD$) in the deformed state takes the shape of a quadric parabola. Interesting results are obtained in a finite element analysis of this problem in which the displacements of node points along lines parallel to $AC$ are prescribed to fit a simple parabola. The deformed finite element network for $2n^2$ elements (4 divisions along each side) is shown in Fig. 3(b).

For the $k^{th}$ element,

$$\begin{align*}
\mu_{1k} &= \Delta \left( \frac{k}{n} \right)^2, \quad \mu_{1k} = \mu_{4k} = \Delta \left( \frac{k-1}{n} \right)^2, \\
\gamma_a &= \Delta \left( \frac{2k}{n} - \frac{1}{n} \right), \\
\gamma_a &= \Delta \left( \frac{2k}{n} \right)^2 - \frac{2k}{n^2} + \frac{1}{n^2}.
\end{align*}$$

(27)

the remaining components being zero. Stresses are calculated as before. For example, we find, for the component $\sigma^{11}$ in element $k$,

$$\sigma^{11} = 2C_1 + 4C_2 + p + (2C_1 + p) \frac{\Delta^2}{n^2} (2k-1)^2. \quad (28)$$

Comparing this result with the exact solution, we find that the difference $\epsilon$ is given by

$$\epsilon = (p + 2C_1) \Delta^2 \left( 2y - 2 + \frac{1}{n} \right). \quad (29)$$

5.3. INFINITE INCOMPRESSIBLE LOG

As a final example, we consider the problem of finite deformation of an infinite, incompressible rubber log under the symmetric line loading indicated in Fig. 4(a). The finite element representation indicated in Fig. 4(b) is used, and it is assumed that the material is of the Mooney type. In this case, displacements are not prescribed.

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Figure 4. Incompressible log, under line loading.

Figure 5. Deformed shape.
and, taking advantage of symmetry, application of equations (19) and (22) leads to a system of 22 simultaneous, non-linear algebraic equations in the 22 unknown nodal displacements and hydrostatic pressures. For illustration purposes, the line loading \( P \) was taken to be 200 lb/in\(^2\) (1379 kN/m\(^2\)) and for the Mooney constants \( C_1 = 43.75 \) lb/in\(^2\) (302 kN/m\(^2\)) and \( C_2 = 62.5 \) lb/in\(^2\) (91 kN/m\(^2\)) were used.

The non-linear equations were solved with the aid of a digital computer in six iterations, using the Newton-Raphson method. Results in the form of the deformed profile are indicated in Fig. 5. In addition to the model in Fig. 4(b), several coarser finite element representations were used, and results indicate that the nodal displacements are not significantly in error. Hydrostatic pressures, of course, converge more slowly since they are assumed to be constant over each element. For the present example, the values of \( p \) obtained can, at best, represent only rough averages of the true hydrostatic pressures. Thus, whereas a rather crude network is often adequate to predict nodal displacements, a significantly finer network is needed to depict variations in stress and hydrostatic pressure.

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**References**


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