This paper presents a systematic numerical procedure for the analysis of nonlinear behaviour in general pneumatic structures. Recent advances in the application of the finite element method to the evaluation of finite strains and large displacements of elastic membranes are reviewed and extensions of the method to the analysis of large motions of reinforced fabrics, anisotropic materials, plastics, viscoelastic, and nonlinearly elastic materials are presented. Local yielding of metallic elasto-plastic membranes subjected to external pressure is also examined. By using linear displacement approximations and triangular finite elements, general nonlinear stiffness relations are derived. These lead to systems of nonlinear algebraic or ordinary differential equations in the generalized displacements and velocities which are solved numerically. Numerical results are included along with comparisons with available experimental data.

1. Introduction

1.1. Opening Remarks

Until recent years, the behaviour of the majority of practical structures could be adequately described by linear theory. The deformations of most structural systems under working loads are usually so small as to be scarcely detectable with the unaided eye, and the stress-strain relations for such common materials as steel, aluminum, and even concrete can, for practical purposes, be treated as linear. Solutions of linear problems involving two and three-dimensional structures of general shape with complex boundary conditions, however, are often untractable by classical means and, even with the gross simplifications afforded by linear theories, many important problems remain unsolved.

With the bulk of the available methods of analysis being applicable to only linear systems and with these methods being often inadequate in the face of complex geometries, the engineer must look upon the recent trend toward the use of highly flexible pneumatic structures with some bewilderment. The behaviour of inflatable pneumatic structures is inherently nonlinear: such structures often acquire their primary load-carrying capacity after undergoing deformations which, even under small pressures, may be so large that the original undeformed shape is unrecognizable. Strains appreciably greater than unity are not uncommon, and in such cases Hooke's law is not applicable. Moreover, the materials used to construct pneumatic structures are often anisotropic and nonlinearly elastic and, to further complicate matters, the directions and magnitudes of the applied loads change with the deformation. To emphasize this point, one need only refer to recent experimental studies on pneumatic structures wherein discrepancies on the order of 400% were encountered when measured stresses were compared with those predicted by conventional linear theories.

It is clear that in order to accurately analyse general pneumatic structures one must turn to the general nonlinear theories of structural mechanics, many of which have long been regarded as of only academic interest. Although only a small number of exact solutions to general nonlinear structural problems are available in the literature and although these, without exception, are concerned with only the most simple geometry, deformation, and loading, at least one can find here a rigorous and complete foundation on which to base nonlinear analyses.

J. Tinsley Oden, Dr., Associate Professor of Engineering Mechanics, University of Alabama Research Institute, Huntsville, Alabama, USA

W. K. Kubitza, Dr., Professor of Engineering Mechanics, same address
Fortunately, the trend toward the use of nonlinear structural systems has been accompanied by significant developments in both large-scale digital computers and general methods of numerical analysis. With the aid of these tools, many of the nonlinear structural theories can be employed to obtain useful information concerning the nonlinear behaviour of pneumatic structures. Chief among the numerical schemes is the so-called finite element method wherein continuous structural systems are replaced by discrete models whose properties are consistent with the general field equations defining the behaviour of the continuum. Notable progress has recently been made in applying this method to complex linear and nonlinear structural problems. A comprehensive review of applications of the finite element method to the analysis of nonlinear behaviour in elastic membranes as well as several extensions of the method to the analysis of large deformations of elastic, elasto-plastic, and viscoelastic pneumatic structures is the subject of this paper.

1.2. Previous Related Work

Since 1950, the technical literature has contained numerous applications of the finite element approach to a wide variety of linear structural problems. Solutions to complex plane-stress, plate, and shell problems are available, along with several solutions to three-dimensional elastic bodies. Applications to nonlinear problems, however, have been significantly less extensive. It appears that the first successful application of the finite element concept to the analysis of geometrically nonlinear problems was presented by Turner et al [1]. These authors solved certain nonlinear problems by dividing a large deformation into a number of steps. Within each step the structure is assumed to behave linear and an instantaneous (linear) stiffness matrix is computed in the deformed geometry. Argyris [2, 3], Gallagher and Padlog [4], and Martin [5] were among several investigators who later applied this successive correction technique to large deflection and stability analyses. In these papers, the correction to the linear stiffness matrix is often referred to as the "geometric" stiffness of the structure, and it is ordinarily a function of the stresses associated with some reference equilibrium state. A survey of the literature using this approach is contained in the paper by Martin [5] and a general formula for geometric stiffness matrices was presented by Oden [6]. Following a different approach, Wissmann [7, 8] obtained nonlinear finite element formulations for certain problems involving large displacements, but small strains of elastic structures.

Extensions of the finite element method to the analysis of finite deformations of elastic membranes and three-dimensional bodies were presented by Oden [9, 10] and Oden and Sato [11]. Using a somewhat different approach, Becker [12] obtained a numerical solution to the problem of finite in-plane deformations of rubber sheets subjected to prescribed boundary displacements. These nonlinear formulations contain the linear stiffness matrices as special cases and lead to systems of nonlinear algebraic equations which must be solved numerically. At the same time, they are considerably more general than the successive correction methods mentioned earlier in that they contain characteristics which are encountered only in highly nonlinear structural behaviour.

1.3. Scope

In the discussion to follow, the basic philosophy of the finite element representation of flexible pneumatic structures is presented. General kinematic properties of thin membranes are cast in the form of algebraic equations defining the motion of an assembly of finite elements. The first law of thermodynamics is called upon to provide general relationships between kinematic and kinetic variables associated with the behaviour of finite elements of arbitrary pneumatic structures. This leads to the general equation of motion of finite elements of thin membranes, and includes such properties as anisotropy, nonlinear viscoelasticity, thermoviscoelasticity, uniaxial homogeneity, and plasticity with no restrictions on the magnitudes of the deformations [Eq. (20)]. In order to obtain quantitative results, the general formulation is modified so that it applies to a number of important special cases. These include a review of the analysis of finite deformations of elastic membranes given in Refs. [9, 10, and 11] and applications to elasto-plastic and viscoelastic structures. This is followed by discussions of procedures for assembling the finite elements, computing changes in loading due to deformation, and solution of the nonlinear equations generated in the analysis. Finally, the solutions of several representative problems are presented and numerical values are compared with available experimental data.
2. Geometric and Kinematic Considerations

2.1. The Discrete Model

Classically, the analysis of continuous systems begins with investigations of the properties of small differential elements of the continuum under investigation. Relationships are established between mean values of various quantities associated with the infinitesimal elements and partial differential equations governing the behavior of the entire domain are obtained by allowing the dimensions of the elements to approach zero as the number of elements becomes infinitely large. In contrast to this classical approach, in the present study we begin with investigations of the properties of elements of finite dimensions. We may employ equations of the continuous system in order to arrive at the properties of these elements; but the dimensions of the elements remain finite in the analysis, integrations are replaced by finite summations, and the differential equations of the continuous structure are replaced by systems of algebraic or ordinary differential equations. The continuous system with infinitely many degrees of freedom is thus represented by a discrete model which has finite degrees of freedom. Moreover, if certain kinematic conditions are satisfied, then, as the number of finite elements is increased and their dimensions are decreased, the behavior of the discrete system converges monotonically to that of the continuous system.

Consider, for example, the general pneumatic structure shown in Fig. 1a subjected to a general system of applied loads. To define the initial geometry of this system, a fixed rectangular cartesian coordinate system $Z_1, Z_2, Z_3$ is established and referred to as the global reference frame. In general, an infinite number of coordinates $Z_i$ are required to completely specify the initial configuration of the membrane; but in the present analysis, we reduce this continuously distributed system to a discrete one by representing the structure as an assembly of a finite number $E_e$ of flat triangular elements, as indicated in Fig. 1b. The vertices of these triangles are referred to as the node points of the discrete model. Thus, if $n$ denotes the total number of nodes in the system and if $Z_{Ni} (N=1,2,...; i=1,2,3)$ denote the global coordinates of a typical node $N$, then the set of number $Z_{Ni}$ define the geometry of the discrete system.

A rigid rotation of a typical local system $\mathbf{x}_{ie}$ into a local system $\mathbf{x}_{ie}'$ whose coordinate lines are parallel to the corresponding global coordinate axes, is accomplished by the orthogonal transformation

$$\mathbf{x}_{ie}' = \mathbf{R}_{ie} \mathbf{x}_{ie} \quad \text{(no sum on e)} \quad (1)$$

---

*Fig. 1: Finite element representation of a pneumatic structure*
2.1. Strains and Displacements

In order that the present discussion be self-contained, we reproduce in this section the derivation of the general nonlinear stiffness relations for finite elements of membrane structures \([9,10,11]\). For the present, we confine our attention to a typical finite element and we temporarily drop the element identification index \(e\) for clarity.

Under a general deformation, line elements originally straight in the undeformed structure become curved lines in the deformed structure. Hence, an initially flat element is generally deformed into a curved surface. However, if the node points are selected sufficiently close of one another, node lines in the deformed configuration are closely approximated by straight-line segments. Then plane elements remain plane after the deformation. This is equivalent to assuming that the displacement fields within each element are linear functions of the local coordinates of the element.

Assuming, for simplicity, that the element is initially in the \(x_1, x_2\) plane and denoting by \(u_i\) the components of displacement referred to the local coordinates of the element under consideration, it follows that

\[ u_i = d_i + a_{i\alpha} x_{N\alpha} \quad i = 1, 2, 3 \quad \alpha = 1, 2 \]  

\[ (2') \]

where \(d_i\) are the rigid-body translations of the element and the \(a_{i\alpha}\) are undetermined constants. By evaluating Eq. (2) at each of the three nodes of the element, we arrive at nine simultaneous equations in the nine unknowns \(d_i, a_{i\alpha}\). Solving these we find that

\[ d_i = k_i N u_{Ni} \]  

\[ (3) \]

and

\[ a_{i\alpha} = c_{a\alpha} N u_{Ni} \]  

\[ (4) \]

where \(u_{Ni}\) is the displacement of node \(N\) in the \(x_i\) direction,

\[ k_i = \frac{1}{2A_0} \left( x_{12} x_{32} - x_{13} x_{22} \right) \]  

\[ (5) \]

\[ k_2 = \frac{1}{2A_0} \left( x_{12} x_{31} - x_{11} x_{32} \right) \]

\[ k_3 = \frac{1}{2A_0} \left( x_{11} x_{22} - x_{12} x_{21} \right) \]

and

\[ c_{a\alpha} N = \frac{1}{2A_0} \left[ x_{32} - x_{32} x_{32} - x_{12} x_{12} - x_{22} \right] \]

\[ (6) \]

In these equations, \(x_{N\alpha} (N = 1, 2, 3; \alpha = 1, 2)\) are the local coordinates of node \(N\) and \(A_0\) is the area of the undeformed triangle.

Substituting Eqs. (3) and (4) into Eq. (2) gives

\[ u_i = k_i N u_{Ni} + c_{a\alpha} N u_{Ni} x_{\alpha} \]

\[ \alpha = 1, 2; \quad N, i = 1, 2, 3 \]  

\[ (7) \]

According to Green and Adkins [13], the Lagrangian strain tensor for a thin membrane is given by

\[ \gamma_{\alpha\beta} = \frac{1}{2} \left( \frac{\partial u_{\alpha}}{\partial x_\beta} + \frac{\partial u_{\beta}}{\partial x_\alpha} + \frac{\partial u_{\beta}}{\partial x_\alpha} \right) \]

\[ \gamma_{\alpha\beta} = 0 \]

\[ \gamma_{33} = \frac{1}{2} (\lambda^2 - 1) \]  

\[ (8) \]

where \(\lambda\) is a scalar function representing the extension ratio at the middle surface in a direction normal to the surface. For very thin membranes, the strains are essentially uniform over the thickness and

\[ \lambda = \frac{h}{h_0} \]  

\[ (9) \]

where \(h_0\) and \(h\) are the thicknesses of the membrane before and after deformation respectively.
Introducing Eq. (7) into Eqs. (8), we find for the strains in the discrete system
\[ \gamma_{ad} = \frac{1}{2} (c_{a\alpha} N^\alpha N^\beta + c_{\beta\beta} N^\alpha N^\alpha + c_{a\alpha} N^\beta N^\gamma N^\delta N^\delta) \]
\[ \gamma_{a\beta} = 0 \]
\[ 33 = \frac{1}{2} (\lambda^2 - 1) \quad (10) \]

The strain components are thus constant throughout the finite element and, since they are determined from prescribed displacement fields, they automatically satisfy the equations of compatibility throughout the element. Note also that the components of displacement along the boundaries of an element are linear functions of the local coordinates. Since each edge contains two nodes and since the displacements are linear along each edge, it follows that the displacements are continuous across element boundaries. These properties of the approximation insure monotonic convergence of the solutions as the finite-element representation is refined.

Three invariants can be formed from every symmetric second-order tensor. In the analysis of deformable membranes, it is convenient to form the invariants of the deformation tensor \( \delta_{ij} + 2 \gamma_{ij} \), where \( \delta_{ij} \) is the Kronecker delta \( (\delta_{ij} = 1 \text{ for } i = j \text{ and } \delta_{ij} = 0 \text{ for } i \neq j) \). These invariants are given by the formulas
\[ l_1 = \lambda^2 + 2(1 + \gamma_{ad}) \]
\[ l_2 = -\lambda^2 + \lambda^2 l_1 + 1 \]
\[ l_3 = \lambda^2 \phi \quad (11) \]
where
\[ \phi = 1 + 2 \gamma_{ad} + 2 \epsilon_{a\beta} \epsilon_{\mu\nu} \gamma_{\alpha\lambda} \gamma_{\beta\mu} \quad (12) \]
and \( \epsilon_{a\beta} \epsilon_{\mu\nu} \) are the two-dimensional permutation symbols \( (\epsilon_{12} = 1, \epsilon_{21} = -1, \epsilon_{11} = \epsilon_{22} = 0) \).

3. Thermodynamics of Finite Elements

3.1. The General Equation of Motion of a Finite Element

Having defined the discrete system in the previous section, we now turn to certain fundamental principles of mechanics in order to obtain general equations of motion for a finite element. We begin with the first law of thermodynamics:
\[ \dot{x} + \dot{U} = \Omega + \dot{Q} \quad (13) \]
where \( \dot{x} \) is the time rate of change of kinetic energy, \( \dot{U} \) is the time rate of change of internal energy, \( \Omega \) is the power developed by the external forces, and \( \dot{Q} \) is the heat input.

By definition,
\[ \dot{x} = \frac{1}{2} \iint \dot{u}_i \dot{u}_i \, dv \]
\[ U = \int \dot{\xi} \, dv \]
\[ \Omega = \int F_i \dot{u}_i \, dv + \int S_i \dot{u}_i \, ds \]
\[ Q = \int \dot{\varphi} \, dv + \int \dot{q}_i n_i \, ds \quad (14) \]

In these equations, \( \varphi \) is the mass density, \( dv \) is the differential volume, \( \dot{u}_i \) are the velocity components, \( \dot{\xi} \) is the internal energy per unit mass, \( F_i \) are the body forces per unit mass, \( S_i \) are the surface tractions per unit of surface area \( s \), \( H \) is the heat input per unit mass, \( q_i \) are the components of heat flux, and \( n_i \) are the components of a unit vector normal to the boundary surfaces of the element.

Through arguments similar to those used in obtaining Eq. (7), it is found that
\[ \dot{u}_i = \frac{\partial \tilde{u}_i}{\partial t} = k_N \dot{u}_i + c_a N^\alpha \dot{u}_i \quad (15) \]
and
\[ \dot{q}_i = k_N q_i + c_a N^\alpha \dot{u}_i \quad (16) \]
where \( n_i \) is the velocity of node \( N \) in direction \( i \) and \( q_i \) is the heat flux at node \( N \) in direction \( i \). Moreover, if \( \rho_0 \), \( \varphi_0 \), and \( \rho_\varphi \) denote the mass densities and the volumes of the finite element in the undeformed and the deformed states respectively, then according to the principle of conservation of mass,
\[ \rho_0 \varphi_0 = \rho \varphi \quad (17) \]

Introducing Eqs. (15), (16), and (17) into Eqs. (14) and simplifying, we find
\[ \dot{x} = \frac{1}{2} m_{NM} \dot{u}_N \dot{u}_M \]
\[ U = \int \rho_0 \dot{\xi} \, dv \]
\[ \Omega = \int \rho_0 \dot{\varphi} \, dv \]
\[ Q = b_{NM} \dot{q}_N + \rho_0 \dot{\varphi} \, H \quad (18) \]
where
\[ m_{NM} = \int \rho_0 (k_{NM} + c_a N^\alpha) (k_M + c_b M^\beta) \, dv \]
\[ P_{Ni} = \int_V (k_N + c_a N_a N_a) \delta o^o \text{dv}_o + \int_0^1 (k_N + c_a N_a N_a) \text{ds} \]  

(19)

\[ b_{Ni} = \int_0^1 (k_N + c_a N_a N_a \text{ds} \]  

The array \( m_{NM} \) is the so-called consistent mass matrix for the finite element and the quantities \( P_{Ni} \) are the components of generalized force corresponding to the generalized displacements \( u_{Ni} \). Physically, \( P_{Ni} \) is the generalized force at node \( N \) in direction \( i \). The quantities \( b_{Ni} \) are the generalized thermal gradients usually referred to as thermal loads. Note that the heat input \( H \) is regarded as a constant for each finite element.

Introducing Eqs. (18) into Eq. (13), we obtain the general result

\[ m_{NM} \dot{u}_{Mi} + \int_0^1 P_{Ni} \delta o^o \text{dv}_o = P_{Ni} \dot{u}_{Ni} + b_{Ni} \dot{q}_{Ni} \text{dv}_o + P_o \dot{v} H \]  

(20)

where \( \dot{u}_{Ni} \) are the components of acceleration of the nodes. This relation represents the most general discrete representation of the equation of motion for a finite element of a continuous media. It is applicable to the analysis of large deformations of nonlinearly elastic, plastic, viscoelastic, and thermoviscoelastic media since no restrictions have as yet been placed on the constitutive equations for the material of which the element is composed or the magnitude of the deformation. The second and third terms on the right-hand side of Eq. (20) represent thermal effects on the motion of the element. The rate of change of internal energy \( \dot{\varepsilon} \) is, in general, also a function of the heat flux and the temperature gradients. These effects will not be considered further in this discussion; for our purpose, it suffices to merely point out that thermal effects can be included in the analysis by retaining the terms \( b_{Ni} \dot{q}_{Ni} \) and \( P_o \dot{v} H \) in Eq. (20) and by introducing, in addition to a constitutive equation involving the stress, a second constitutive equations which relates heat flux to thermal gradients, deformation rates, strains, etc. Thus, with thermal effects omitted, Eq. (20) reduces to

\[ P_{Ni} \dot{u}_{Ni} = m_{NM} \dot{u}_{Mi} \dot{u}_{Ni} + \int_0^1 P_{Ni} \delta o^o \text{dv}_o \]  

(21)

3.2. Internal Energy

To apply this equation to specific materials, it is necessary to obtain \( \dot{\varepsilon} \) as a function of the generalized displacements and their time derivatives. According to Eringen [14], if couple stresses and thermal effects are omitted,

\[ \dot{\varepsilon} = \sigma_i \dot{d}_{ij} \]  

(22)

where \( \sigma_i \) is the stress tensor and \( \dot{d}_{ij} \) is the deformation rate tensor. For the finite element representation, it is easily shown that

\[ \dot{d}_{ij} = \sigma_i c_j \]  

(23)

We now assume that the material is homogenous so that \( \dot{\varepsilon} \) is not a function of the local coordinates \( x_i \). In view of Eq. (22) it is a constant with respect to \( x_i \) for the finite element and can therefore be factored outside of the integral in Eq. (21). Noting also that

\[ \dot{\varepsilon} = \sqrt{3} \]  

(24)

where \( \sqrt{3} \) is the third invariant in Eq. (12), and introducing Eqs. (22), (23), and (24) into Eq. (21), we arrive at the equation

\[ \left[ m_{NM} \dot{u}_{Mi} + \sqrt{3} \int_0^1 \delta s c_i \dot{r}_{Ni} (\delta_i s + c_i s M^N M_k) \dot{u}_{Mi} \right] \dot{u}_{Ni} = 0 \]  

(25)

Since this equation must hold for all velocities \( \dot{u}_{Ni} \), we have

\[ m_{NM} \dot{u}_{Mi} + \sqrt{3} \int_0^1 \delta s c_i \dot{r}_{Ni} (\delta_i s + c_i s M^N M_k) = P_{Ni} (t) \]  

(26)

It is now necessary to introduce into this equation the appropriate relation expressing the stress in terms of the strain, strain rates, higher-order strain rates, etc. in order to obtain the finite element representation for specific materials.

4. Nonlinear Stiffness Relations

We now examine applications of Eqs. (21) and (26) to various types of membranes.

4.1. Elastic Materials

In the case of elastic materials, the stress \( \sigma_{ij} \) is derivable from a potential function \( W \) which represents the strain energy per unit of undeformed volume. If each element is homogeneous, then \( \dot{\varepsilon} \) and \( W \) are not functions of the spatial coordinates \( x_i \). It follows that

\[ \int_0^1 P_{Ni} \delta o^o \text{dv}_o = \int_0^1 W \text{dv}_o = v_o \dot{W} \]  

(27)
This result must be valid for arbitrary node velocities of the element. Therefore

\begin{equation}
(3,6)
\end{equation}

\[ w_2 ' - \omega_2 + (\epsilon_2 + \epsilon_3) \alpha_3^2 \alpha_3^2 = \left( \omega_2 - \epsilon_2 \right)^2 + \omega_2 \] \( \alpha_3^2 \) \( \alpha_3^2 \)

\begin{equation}
(37)
\end{equation}

Equation (30) is the general equation of motion for finite forced oscillations of an elastic membrane. For specific materials, the appropriate form of \( \text{W} \) must be introduced into this equation.

The important problem of small oscillations about a state of large deformation is obtained as a special case of Eq. (30) by denoting

\begin{equation}
(31)
\end{equation}

where \( \bar{u}_{\text{Ni}} \) is the prescribed large displacement and \( w_{\text{Ni}} \) is a small perturbation. Then

\begin{equation}
(32)
\end{equation}

where

\begin{equation}
(33)
\end{equation}

Here \( f_{\text{NM}} \) is a known function of \( \bar{u}_{\text{Mi}} \) and is independent of time. The forces \( \bar{p}_{\text{Ni}} \) are known functions of time and Eq. (32) is linear in the dependent variables \( w_{\text{Ni}} \) and their derivatives.

If the membrane is in equilibrium, Eq. (30) reduces to [10, 11]

\begin{equation}
(34)
\end{equation}

For elastic membranes, the stresses are calculated by means of the formulas

\begin{equation}
(35)
\end{equation}

\begin{equation}
(36)
\end{equation}

\[ \sigma_{\alpha \beta} = 2 \delta_{\alpha \beta} \left( \frac{\partial w}{\partial \alpha} + \lambda \frac{\partial w}{\partial \beta} \right) + \left( \frac{2}{\lambda^2} \frac{\partial w}{\partial \alpha} + \rho \right) \sigma_{\alpha \beta} \]

\[ \sigma_{33} = \lambda^2 \frac{\partial w}{\partial \alpha} + \lambda^2 \left( \alpha_1^2 - \lambda^2 \right) \frac{\partial w}{\partial \alpha^2} + \rho \]

where \( \lambda^2 \) is defined in Eq. (9), \( \rho \) is the hydrostatic pressure, and

\begin{equation}
(38)
\end{equation}

It is assumed that the membrane is very thin so that the strains are essentially uniform over the thickness and \( \sigma_{33} \), the stress normal to the deformed surface, is negligible in comparison with \( \sigma_{\alpha \beta} \). Then \( \rho \) can be determined from the condition

\begin{equation}
(39)
\end{equation}

\[ p = -\lambda^2 \frac{\partial w}{\partial \alpha} + \lambda^2 \left( \alpha_1^2 - \lambda^2 \right) \frac{\partial w}{\partial \alpha^2} \]

Thus

\begin{equation}
(40)
\end{equation}

\[ \sigma_{\alpha \beta} = 2 \delta_{\alpha \beta} \left( \frac{\partial w}{\partial \alpha} + \lambda \frac{\partial w}{\partial \beta} \right) + \left( \frac{4}{\lambda^2} \frac{\partial w}{\partial \alpha} - \lambda^2 \right) \frac{\partial w}{\partial \alpha^2} \]

Rivlin and Saunders [15, 16] verified experimentally that the strain energy function for most isotropic, incompressible rubbers is of the form

\[ W = C_1 (\alpha_1^2 - 3) + \psi (\alpha_2^2 - 3) \]

where \( C_1 \) is a material constant and the function \( \psi \) depends upon the type of rubber. In analytical work, the most common form of \( W \) is the well-known Mooney form [17]:

\[ W = C_1 (\alpha_1^2 - 3) + C_2 (\alpha_2^2 - 3) \]

where \( C_1 \) and \( C_2 \) are experimentally determined constants.
where \( C \) is a constant. Rivlin [19] refers to such materials as neo-Hookean.

To obtain nonlinear stiffness relations for finite elements of rubber membranes, first note that for incompressible membranes

\[
\begin{align*}
I_1 &= \lambda^2 + 2(1 + \gamma_{aa}) \\
I_2 &= \frac{1}{\lambda^2} + 2\lambda^2 (1 + \gamma_{aa}) \\
\lambda^2 &= (1 + 2\gamma_{aa} + \varepsilon_{aa} \varepsilon_{\lambda\lambda} \varepsilon_{\alpha\alpha} \varepsilon_{\beta\beta})^{-1}
\end{align*}
\]

and

\[
\frac{\partial^2 \lambda^2}{\partial \gamma_{aa}} = -2\lambda^4 \gamma_{aa} \quad \frac{\partial (\lambda^2)}{\partial \gamma_{aa}} = 2 \gamma_{aa}
\]

\[
\frac{\partial I_1}{\partial \gamma_{aa}} = 2 (\delta_{aa} - \lambda^4 \gamma_{aa})
\]

\[
\frac{\partial I_2}{\partial \gamma_{aa}} = 2 \gamma_{aa} (1 - 2\lambda^4 - 2 \lambda^4 \gamma_{aa}) + 2 \lambda^2 \delta_{aa}
\]

where \( \gamma_{aa} \) is defined in Eq. (38). For the finite element,

\[
\begin{align*}
\lambda^2 &= \left(1 + c_{an}(u_{an} + \frac{1}{2} c_{an} u_{an} u_{an} u_{an}) + \varepsilon_{aa} \varepsilon_{\lambda\lambda} (c_{an} u_{an})
\right. \\
&\quad + \left. \frac{1}{2} c_{an} \varepsilon_{\lambda\lambda} (c_{an} u_{an} u_{an} u_{an}) + \varepsilon_{aa} \varepsilon_{\lambda\lambda} \varepsilon_{\alpha\alpha} \varepsilon_{\beta\beta} \right)^{-1}
\end{align*}
\]

and

\[
\gamma_{aa} = \delta_{aa} + \varepsilon_{aa} \varepsilon_{\lambda\lambda} (c_{an} u_{an} u_{an} u_{an}) + \varepsilon_{aa} \varepsilon_{\lambda\lambda} \varepsilon_{\alpha\alpha} \varepsilon_{\beta\beta} u_{an} u_{an} u_{an} u_{an}
\]

wherein \( i, j, k, l, m, n, k = 1, 2, 3; a, \beta, \gamma \), \( \mu = 1, 2 \).

Introducing Eq. (42) and Eqs. (44)-(47) into Eqs. (34) and (40), we obtain the following equations for a finite element of a membrane of Mooney material [11],

\[
\begin{align*}
P_{Nk} &= 2u \varepsilon_{an}(\delta_{aj} + c_{an} u_{an} u_{an} u_{an}) \left\{ C_1 (\delta_{aj} - \lambda^4 \gamma_{aa}) \\
&\quad + C_2 [\gamma_{aa} (1 - 2\lambda^4 - 2 \lambda^4 \gamma_{aa}) + \lambda^2 \gamma_{aa}] \right\}
\end{align*}
\]

\[
\sigma_{aa} = 2k (C_1 (\delta_{aj} - \lambda^4 \gamma_{aa}) \\
&\quad + C_2 \left( \lambda^2 \gamma_{aa} + (1 - 2\lambda^4 - 2 \lambda^4 \gamma_{aa}) \right) \gamma_{aa})
\]

where

\[
\gamma_{\alpha\beta} = c_{\alpha\beta}(u_{\alpha\beta} + \frac{1}{2} c_{\alpha\beta} u_{\alpha\beta} u_{\alpha\beta})
\]

Similarly, for neo-Hookean membranes, we find

\[
P_{Nk} = 2u \varepsilon_{an}(\delta_{aj} + c_{an} u_{an} u_{an} u_{an}) (\delta_{aj} - \lambda^4 \gamma_{aa})
\]

Eqs. (48a) and (50a) are general nonlinear stiffness relations governing the static behaviour of finite elements of Mooney and neo-Hookean membranes. Upon connecting the elements appropriately and applying boundary conditions, these lead to systems of nonlinear algebraic equations in the node displacements. Once solved, the results are introduced into Eqs. (10), (48b), and (50b) to obtain final strains and stresses in the structure.

4.1.2. Plastics and Nonlinearly Elastic Materials

Recent experiments on plastics and synthetic materials have attempted to arrive at approximate energy functions for such materials even though their behaviour is not always perfectly elastic. These have led to highly nonlinear forms for the potential function \( W \). For example, experiments on a dimethyl siloxane rubber by Hutchinson, Becker, and Landel [20] have indicated an energy function of the form

\[
W = B_1 (1 - 3) + B_2 (1 - 3)^2 + B_3 \left[1 - \exp[b_1 (1 - 3)]\right]
\]

\[
+ B_4 \left[1 - \exp[b_2 (1 - 3)]\right]
\]

where \( B_1, B_2, B_3, B_4, b_1, \) and \( b_2 \) are material constants.

In this case, the nonlinear stiffness relation is given by

\[
P_{Nk} = 2u \varepsilon_{an}(\delta_{aj} + c_{an} u_{an} u_{an} u_{an}) \left\{ B_1 (1 - 3) + B_2 (1 - 3)^2\right\}
\]

\[
- (1 - 3) \left[ b_3 \exp[b_1 (1 - 3)] + b_4 \exp[b_2 (1 - 3)]\right]
\]

\[
\left[ \gamma_{aa} (1 - 2 \lambda^4 - 2 \lambda^4 \gamma_{aa}) + \lambda^2 \gamma_{aa} \right]
\]

as before, stresses are computed by means of Eq. (40).

4.1.3. Metals and Reinforced Fabrics

Deformations of metallic and reinforced fabric pneumatic structures are usually characterized by large displacements accompanied by strains which are small in comparison with unity. In such cases the material is assumed to be homogeneous and elastic within each finite element and the well-known Hookean form of the strain energy function is applicable. Moreover,
the strain normal to the deformed surface can then be expressed as a linear function of the strains in the middle surface and the state-of-strain is described by the two-dimensional tensor \( \gamma_{\alpha\beta} \). The strain energy function is therefore of the form

\[
W = \frac{1}{2} E_{\alpha\beta\mu\nu} \gamma_{\alpha\beta} \gamma_{\mu\nu}
\]

(53)

where \( E_{\alpha\beta\mu\nu} \) is a multi-dimensional array of elastic constants. The stress tensor is given by

\[
\sigma_{\alpha\beta} = E_{\alpha\beta\mu\nu} \gamma_{\mu\nu} = E_{\alpha\beta\mu\nu} c_{\lambda N} (u_{N\mu} + \frac{1}{2} \epsilon_{\mu\nu} u_{M\lambda} u_{N\lambda})
\]

(54a)

and the nonlinear stiffness relation for the finite element is obtained by introducing Eq. (53) into Eq. (34):

\[
P_{Nk} = \sum_{\alpha=1}^{2}(\delta_{\beta M} + c_{\beta M} u_{Mk}) E_{\alpha\beta\mu\nu} c_{\lambda N} (\delta_{\mu \lambda} + \frac{1}{2} \epsilon_{\mu \lambda} u_{Ji} + \frac{1}{2} \epsilon_{\mu \lambda} u_{Ji} u_{Ji})
\]

(54b)

The array of material constants, \( E_{\alpha\beta\mu\nu} \), possess the symmetric properties

\[
E_{\alpha\beta\mu\nu} = E_{\beta\alpha\mu\nu} = E_{\alpha\beta\mu\nu} = E_{\lambda\mu\alpha\beta}
\]

(55)

and, for isotropic materials, it is given by

\[
E_{\alpha\beta\mu\nu} = \frac{E}{2(1+\nu)} (\delta_{\alpha\beta} \delta_{\mu\nu} + \delta_{\alpha\nu} \delta_{\mu\beta} + \frac{2\nu}{1-\nu} \delta_{\alpha\mu} \delta_{\beta\nu})
\]

(56)

where \( E \) is Young's modulus and \( \nu \) is Poisson's ratio.

The form of \( E_{\alpha\beta\mu\nu} \) in Eq. (56) is applicable to most engineering metals (steel, aluminum, and tungsten, etc.) and to isotropic fabrics. However, most of the reinforced fabric and composite materials are either completely anisotropic or transversely isotropic only with respect to a normal to the middle surface.

A typical example of such a reinforced fabric is indicated in Fig. 3. Here a fabric core material, which is assumed to be isotropic and to have a modulus \( E \) and Poisson's ratio \( \nu \), is reinforced by a network of fibers of modulus \( E \) and Poisson's ratio \( \nu \). The fibers form an angle \( \alpha \) with respect to the \( y_1 \)-axis, as shown. When the fiber spacing is relatively small, it is convenient to introduce mean values for the appropriate material constants of the composite structure. Thus, we introduce the following constants [21]:

\[
E_1 = kE + (1-k)E, \quad E_2 = \frac{E E}{kE + (1-k)E}
\]

(57)

\[
G_c = \frac{G G}{kG + (1-k)G}, \quad \nu_c = \frac{k \nu + (1-k)\nu}{kG + (1-k)G}
\]

where \( k \) is the ratio of the cross-sectional area of the fibers to the total cross-sectional area of the element, \( G \) and \( \bar{G} \) are the respective shear moduli of the core fabric and the fibers, \( E_1 \) and \( E_2 \) are respectively the effective mean moduli of elasticity in the \( y_1 \) and \( y_2 \) directions, \( G_c \) is the mean shear modulus, and \( \nu_c \) is the mean Poisson's ratio of the composite material. We then have

\[
E_{1111} = a_{11} \cos^4 \alpha + a_{22} \sin^4 \alpha + 2(a_{12} + 2G_c) \sin^2 \alpha \cos^2 \alpha
\]

\[
E_{1122} = E_{2211} = (a_{11} + a_{22} - 4G_c) \sin^2 \alpha \cos^2 \alpha
\]

\[
E_{2222} = a_{11} \sin^4 \alpha + a_{22} \cos^4 \alpha + 2(a_{12} + 2G_c) \sin^2 \alpha \cos^2 \alpha
\]

\[
E_{1212} = E_{1221} = E_{2112} = E_{2212} = E_{1112} = E_{1122} = E_{1212} = E_{1221} = E_{2112} = E_{2212} = 0
\]

where

\[
a_{11} = \frac{E_2}{1 - \nu_c^2 E_1/E_2}
\]

(59)

\[
a_{22} = \frac{E_1}{1 - \nu_c^2 E_1/E_2}
\]

\[
a_{12} = \frac{\nu_c E_1}{1 - \nu_c^2 E_1/E_2}
\]
for such materials can be written in the form

$$
\tau = \mathcal{F}(\mathbf{\sigma}, \mathbf{u})
$$

(60)

where \( \mathcal{F} \) is a tensor functional of the indicated arguments. Then the equations of motion for a finite element are

$$
\frac{\partial}{\partial t} \mathbf{u}^{N} = \mathcal{L}^{N} \mathbf{u}^{N} + \mathcal{F}^{N} \mathbf{u}^{N} + \mathbf{P}^{N}(t)
$$

(61)

Specific forms of these relations for given materials are obtained by introducing the appropriate expanded form of \( \mathcal{F} \) in Eq. (61).

Although a detailed discussion of finite element formulations for nonlinearly viscoelastic materials is outside the scope of the present investigation, the general procedure is amply demonstrated by a simple example. Consider the case of plane stress in a thin membrane undergoing large displacements but strains which are small in comparison with unity, and, for simplicity, assume that the membrane is constructed of a simple linearly viscoelastic material of Voigt type:

$$
\mathbf{\sigma}^{0} = A \mathbf{a} \mathbf{\lambda} \mathbf{u}^{N} + B \mathbf{b} \mathbf{\lambda} \mathbf{u}^{N}
$$

(62)

Here \( A \mathbf{a} \mathbf{\lambda} \mathbf{u} \) and \( B \mathbf{b} \mathbf{\lambda} \mathbf{u} \) are arrays of matrix parameters and may be functions of time. Then Eq. (26) becomes

$$
m_{NM} \mathbf{u}^{N} + c_{NM} \mathbf{c} \mathbf{u}^{N} = \mathbf{a} \mathbf{\lambda} \mathbf{u}^{N} \left[ \mathbf{a} \mathbf{\lambda} \mathbf{u} \mathbf{u}^{N} + B \mathbf{b} \mathbf{\lambda} \mathbf{u} \mathbf{u}^{N} \right] = \mathbf{p}(t)
$$

(63)

We obtain a system of linear differential equations describing the in-plane motion of flat viscoelastic elements by deleting products and squares of the displacements and velocities in Eq. (63) and limiting the ranges of lower-case indices to 2:

$$
m_{NM} \mathbf{u}^{N} + B \mathbf{b} \mathbf{\lambda} \mathbf{u}^{N} \mathbf{c} \mathbf{u}^{N} \mathbf{M}^{N} + A \mathbf{a} \mathbf{\lambda} \mathbf{u}^{N} \mathbf{c} \mathbf{u}^{N} \mathbf{M}^{N} = \mathbf{p}(t)
$$

(64)

It is seen that the above procedure provides a systematic and rational means for identifying the material damping coefficients for any material for which the stress is given explicitly as a function of strains, strain rates, and other kinematic variables.

4.3. Elasto-plastic Materials

It is not difficult to modify finite element formulations so as to account for yielding and plastic deformation of metallic membranes. Since, in view of Eqs. (10), the strain components are uniform throughout each triangular element, so also in the stress Eq. (54). Thus, if local yielding is imminent, it is characterized by uniform yielding and strain hardening of the associated local finite elements. This means that elastic-plastic boundaries cannot move continuously during an incremental loading process; but this characteristic of the discrete model need not lead to divergent or inconsistent results.

In this investigation, the approach suggested by Pope [22, 23] is extended so as to apply to large displacements of elasto-plastic membranes. The elasto-plastic behaviour of a typical finite element is analyzed through an incremental loading process. During each increment, the material responds linearly, but the overall response obtained by summing the incremental values may be highly nonlinear.

Let \( \mathbf{\sigma}^{0} \), \( \mathbf{\gamma}^{0} \), \( \mathbf{p}^{0} \), and \( \mathbf{u}^{0} \) denote the known values of the stress, strain, node forces, and node displacements at some reference state \( o \) in a typical finite element and let \( \delta \mathbf{\sigma}^{0} \), \( \delta \mathbf{\gamma}^{0} \), \( \delta \mathbf{p}^{0} \), and \( \delta \mathbf{u}^{0} \) denote small increments in these quantities. If these increments are sufficiently small, it can be easily shown that

$$
\delta \mathbf{\gamma}^{0} + \delta \mathbf{p}^{0} = \mathbf{c}^{0} \mathbf{\lambda} \mathbf{\delta} \mathbf{\gamma}^{0} + \mathbf{c}^{0} \mathbf{\delta} \mathbf{p}^{0}
$$

(65a)

$$
\delta \mathbf{\gamma}^{0} + \delta \mathbf{p}^{0} = \mathbf{c}^{0} \mathbf{\lambda} \mathbf{\delta} \mathbf{\gamma}^{0} + \mathbf{c}^{0} \mathbf{\delta} \mathbf{p}^{0}
$$

(65b)

$$
\delta \mathbf{p}^{N} = \mathbf{c}^{0} \mathbf{\lambda} \mathbf{\delta} \mathbf{\gamma}^{0} + \mathbf{c}^{0} \mathbf{\delta} \mathbf{p}^{0}
$$

(65c)

where \( \delta \mathbf{\gamma}^{0} \) is the elastic strain increment and \( \delta \mathbf{\gamma}^{0} \) is the plastic strain increment. The term \( \mathbf{c}^{0} \mathbf{\lambda} \mathbf{\delta} \mathbf{\gamma}^{0} \) in Eq. (65c) represents the influence of large displacements on the relationship between stresses and the node forces. The term \( \mathbf{c}^{0} \mathbf{\delta} \mathbf{p}^{0} \) can be neglected in the case of small displacements.
The elastic strain increment is related to the stress increment according to
\[
\delta \varepsilon_{ab} = \delta \sigma_{ab} - \delta \sigma^p_{ab} = E^0 \varepsilon_{ab} \lambda_\mu \delta \gamma_{\lambda \mu}
\] (66)

In which the array \( E^0 \varepsilon_{ab} \lambda_\mu \) may, in general, be a function of \( \varepsilon_{ab} \).

The yield condition may be represented by a convex yield surface in stress space which is given by [24]
\[
f(\varepsilon_{ab}) = 0
\] (67)
The yield function \( f \) is symmetrical with respect to \( \varepsilon_{ab} \) and \( \varepsilon_{ab} \) and depends upon the strain history of the membrane.

Since it is assumed that plastic deformation is independent of the hydrostatic stress, we can rewrite (67) in the form
\[
f(\varepsilon_{ab}) = 0
\] (68)
where \( \varepsilon_{ab} \) is the stress deviator:
\[
\varepsilon_{ab} = \varepsilon_{ab} - \frac{1}{3} \delta_{ab} \varepsilon_{kk}
\] (69)

When the stress increment \( \delta \sigma_{ab} (\sigma_{33} = 0) \) does not have a positive component in the direction of an inward normal to the yield surface and when the stress point \( \sigma_{ab} \) lies on the yield surface, the material yields and the plastic strain increment is given by
\[
\delta \gamma_{ab}^P = \mu \frac{\partial f}{\partial \sigma_{ab}}
\] (70)
The factor \( \mu \) depends upon the strain history and is independent of all components of \( \delta \sigma_{ab} \) except that normal to the yield surface.

At initial yielding, the yield condition (68) is satisfied. After an additional increment of plastic strain, the yield condition is given by
\[
f + t_{ab} \delta \gamma_{ab}^P + \frac{\partial f}{\partial \sigma_{ab}} \delta \sigma_{ab} = 0
\] (71)
where \( t_{ab} \) describes the strain-hardening properties of the material. Introducing Eqs. (66), (68), and (70) into (71) and simplifying the results, we find that [33, 34, 35]
\[
\mu = \frac{\partial f}{\partial \sigma_{ab}} - \frac{\partial f}{\partial \sigma_{ab}} \frac{\partial \sigma_{ab}}{\partial \sigma_{ab}} - \frac{\partial f}{\partial \sigma_{ab}}
\] (72)

Thus
\[
\delta \gamma_{ab}^P = G_{ab} \lambda_\mu \delta \sigma_{ab}
\] (73)
\[
G_{ab} \lambda_\mu = \frac{\partial f}{\partial \sigma_{ab}} \frac{\partial f}{\partial \sigma_{ab}}
\] (74)

Introducing Eqs. (73) and (66) into (65b) and solving for \( \delta \sigma_{ab} \) we obtain
\[
\delta \sigma_{ab} = H_{ab} \lambda_\mu \varepsilon_{ab} \mu_i (\delta_{ij} + \varepsilon_{ij} M_i) \delta \sigma_{Ni}
\] (75)
where \( H_{ab} \lambda_\mu \) is the inverse of \( G_{ab} \lambda_\mu + \varepsilon_{ij} M_i \):
\[
H_{ab} \lambda_\mu (\varepsilon_{ij} + \varepsilon_{ij} M_i) = \delta_{ij} \lambda_\mu
\] (76)

Finally, Eq. (65c) becomes
\[
\delta \sigma_{Ni} = \left[ c_{an} \left( \varepsilon_{ab} + \varepsilon_{ab} M_i \right) H_{ab} \lambda_\mu \varepsilon_{ab} \mu_i \right] \lambda_\mu (\delta_{ij} + \varepsilon_{ij} M_i) \delta \sigma_{Ni}
\] (77)

Equation (77) represents the stiffness relation between the incremental loads \( \delta \sigma_{Ni} \) and their corresponding incremental displacements. These equations are inverted for each load increment; the solutions \( \delta \sigma_{Ni} \) are introduced into Eq. (75) to obtain the associated stress increment. Incremental strains \( \delta \gamma_{ab}^P \) and \( \delta \gamma_{ab}^P \) are then calculated by means of Eqs. (66) and (73). Once the incremental values \( \delta \sigma_{Ni} \), \( \delta \sigma_{ab} \), and \( \delta \sigma_{ab} \) are determined, they are added algebraically to those of the reference state (i.e., \( \sigma_{ab}^{(0)} \)) to obtain a new reference state (i.e., \( \sigma_{ab}^{(0)} = \sigma_{ab}^{(0)} + \delta \sigma_{ab}^{(0)} + \delta \sigma_{ab}^{(0)} + \delta \sigma_{ab}^{(0)} \)). At the end of each cycle the factor \( \mu \) of each element is examined. If \( \mu < 0 \), the element is added to the elastic region of the membrane and the analysis is repeated; if the mean stress over the load increment is on or outside the yield surface, the analysis is repeated with the element permitted to deform plastically. Accuracy is improved by choosing the load increments such that one element, at the most, yields during each load increment. Other details of the procedure are identical to those of the small displacement case and can be found in references [22] and [23].
5. Formulation of the Structural Problem

5.1. Global Equations of Motion

The nonlinear equations derived in the previous section describe the behavior of a single finite membrane within its local reference frame; these relations are independent of the loading on the membrane, the boundary conditions, and the location of the element in the assembled system. It is now necessary to connect the elements at appropriate node points and to sum their properties so as to represent a pneumatic structure of specified shape with specified boundary conditions. To accomplish this, it is convenient to first rotate the node forces, displacements, velocities, accelerations, and local coordinates associated with each element so that they are parallel to the global reference frame \( Z_i \). This is accomplished through the transformations

\[
\begin{align*}
\tilde{P}_{Mie} &= U_{Mie} P_{Mke} \\
\tilde{u}_{Mke} &= \frac{\theta_{Mie}}{\theta_{Mke}} u_{Mke}
\end{align*}
\]  
(78)

where the underscore \( (e) \) indicates that no sum is to be taken on the repeated index \( e \). In these equations, \( \tilde{P}_{Mie} \) and \( \tilde{u}_{Mke} \) are respectively, the node forces, displacements, velocities, and accelerations of node \( M \) of element \( e \) in the direction of \( \tilde{x}_{Mie} \) and \( \tilde{x}_{Mke} \). The ranges of the indices in these equations are \( M = 1, 2, 3 \) and \( e = 1, 2, \ldots, E_e \) where \( E_e \) is the total number of finite elements.

It was pointed out earlier that the set of numbers \( Z_{Ni} (i = 1, 2, 3; N = 1, 2, \ldots, n) \) describes the geometry of the assembled (connected) system whereas \( \tilde{x}_{Mie} (N = 1, 2, 3; e = 1, 2, \ldots, E_e) \) describes that of the individual elements. The connectivity of the system is established by relating the members of the set \( Z_{Ni} \) to those of \( \tilde{x}_{Mie} \) by the transformation

\[
\tilde{x}_{Mie} = \Omega_{Mie} Z_{Ni} \quad (M = 1, 2, 3; N = 1, 2, \ldots, n)
\]  
(79)

where

\[
\Omega_{Mie} = \frac{1}{ \theta_{Mie} } \tilde{x}_{Mie} Z_{Ni} \quad (M = 1, 2, 3; N = 1, 2, \ldots, n)
\]

The transformation indicated in Eq. (79) defines a mapping of points in the set \( Z_{Ni} \) into local sets \( \tilde{x}_{Mie} \) and, in effect, assembles the elements into a single unit. The process is illustrated symbolically for the case \( E_e = 4, n = 5 \) in Fig. 4.

Similarly, if \( P_{Nke}, U_{Nke}, \dot{U}_{Nke}, \text{ and } \ddot{U}_{Nke} \) denote the values of node forces, displacements, velocities, and accelerations in the assembled system, it can be shown that

\[
\begin{align*}
\tilde{P}_{Mke} &= \Omega_{Mke} M_{Nke} \\
\tilde{u}_{Mke} &= \frac{\Omega_{Mke}}{\Omega_{Mke}} U_{Nke} \\
\ddot{u}_{Mke} &= \frac{\Omega_{Mke}}{\Omega_{Mke}} \dot{U}_{Nke} \quad (81)
\end{align*}
\]

In this case, the repeated indices \( N \) and \( e \) are summed throughout their entire ranges: \( N = 1, 2, \ldots, n_e \), \( e = 1, 2, \ldots, E_e \).

Application of Eqs. (78) through (81) assembles the finite elements into a single discrete system. When the local variables appearing in local equations of motion [such as Eqs. (21) or (26)] are transformed in accordance with Eqs. (78) and (81),
where the resulting relation is referred to as a global equation of motion. In particular, Eq. (21) becomes

$$P_{Ni} \ddot{U}_i = M_{NM} \ddot{U}_m + \int \rho \ddot{e} dV$$

(82)

where

$$M_{NM} = \sum_{e=1}^{N} \rho_{Me} m_{Se}$$

In this case, $N, M = 1, 2, \ldots, n; R, S = 1, 2, 3$ and the integration is carried out over the entire volume $V_0$ of the undeformed structure. Equation (82) is the general global equation of motion of a finite element.

Global equations for the case of static behaviour are of special interest. In this case, the local equations of motion reduce to nonlinear stiffness relations of the form

$$P_{Mm} = k(u_{Nk})$$

(84)

where $k(u_{Nk})$ is the appropriate nonlinear function of the node displacements. For example, the function $k(u_{Nk})$ for synthetic rubbers, nonlinearly elastic materials, and Hookonian metals are defined by Eqs. (40a), (52), and (54) respectively.

When the components of node forces and displacements are rotated so that they are parallel to the coordinates $x_i e_i$, Eq. (84) becomes

$$P_{Mm}^{\prime} = \delta_{im} k(u_{Nk})$$

(85)

Finally, the global stiffness relations are obtained through the transformations indicated in Eq. (81):

$$P_{Nk} = K_{Nk}(u_{Rs})$$

(86)

where

$$K_{Nk}(u_{Rs}) = \sum_{m=1}^{N} \rho_{Me} \delta_{im} h_{kem}$$

(87)

Boundary conditions are applied by prescribing generalized (global) displacements at appropriate boundary nodes. Then Eqs. (86) reduce to a system of independent nonlinear algebraic equations in the unknown node displacements.

5.2. External Pressure

Up to this point, the relations derived are applicable only to situations in which the loads do not change in direction as the structures deform. Since, for pneumatic structures, this is obviously a severe restriction, a procedure [7, 11] for accounting for changes in the external loading due to deformation is examined.

It is first assumed that the dimensions of each finite element are sufficiently small that the pressure $q$ can be regarded as uniform over the surfaces of each finite element. Then, if $A$ denotes the area of the deformed element, the total force exerted normal to the plane of the element is

$$\bar{F} = Aq$$

(88)

Let $n_i$ denote the components of an outward unit vector normal to $A$. Then the components of the pressure force $\bar{F}$ are given by

$$\bar{F}_i = n_i Aq$$

(89)

To determine the components $n_i$, the coordinates of nodes of the deformed element are denoted

$$\gamma_{Ni} = x_{Ni} + u_{Ni}$$

(90)

For convenience in writing, the origin of the reference frame $y_i$ is transferred to node 3 of the deformed element. If the resulting coordinate system is denoted $z_i$, it follows that

$$z_{Ni} = \gamma_{Ni} - \gamma_{3i}$$

(91)

Now consider two unit vectors emanating from the origin of the coordinates $z_i$ (node 3). The components $n_i$ of the unit normal are obtained by forming the vector product of these two vectors;

$$n_i = \frac{1}{2} A e_{ijk} z_i z_j z_k$$

(92)

where $e_{ijk}$ is the permutation symbol. Thus, equation (89) can be written

$$q_i = \frac{1}{2} q e_{ijk} z_i z_j$$

(93)

The next external force at each node is obtained by simply representing $q$ by three forces, one at each node, whose components are

$$Q_i = \frac{1}{3} q e_{ijk} (\gamma_{1i} \gamma_{2k} + \gamma_{2i} \gamma_{3k} + \gamma_{3i} \gamma_{1k})$$

(94)

Introducing Eq. (91) gives

$$Q_i = \frac{1}{3} q e_{ijk} (\gamma_{1i} \gamma_{2k} + \gamma_{2i} \gamma_{3k} + \gamma_{3i} \gamma_{1k})$$

(95)
This result defines the generalized external force in the deformed element produced by external pressure. Note that no node identification index is needed since \( q_{ie} \) is the same for each node of the element.

To complete the analysis, these forces are now transformed into components \( \vec{Q}_{ie} \) parallel to the local reference frame \( \vec{x}_i \). The result for element \( e \) is

\[
\vec{Q}_{ie} = \frac{1}{12} q_e \sum_{p=1}^{3} E_{pmN} \delta_{ik} (\vec{x}_{N|e} + \vec{u}_{N|e})(\vec{x}_{M|e} + \vec{u}_{M|e})
\]

(96)

wherein the underscore again indicates that no sum is to be taken on the repeated index \( e \).

In the case of pressure loadings, the components \( \vec{Q}_{ie} \) take the place of the node forces \( \vec{P}_{N|e} \) of Eqs. (78) and (85). It is seldom necessary to transform these components into the global system, however, since it is more convenient to first transform displacements into the \( \vec{x}_i \) system with the aid of Eq. (1) and then to transform the resulting forces into the global system.

5.3. Solution of Nonlinear Equations

In the case of time-dependent phenomena, finite element formulations lead to systems of simultaneous nonlinear differential equations of the form indicated in Eqs. (26), (63), and (82). The solution of such systems of equations is a formidable task, even with the aid of modern digital computers. Generalizations of the well-known Runge-Kutta techniques may lead to acceptable results in some cases; but general procedures for solving such large systems of coupled nonlinear differential equations are, at best, still in the early stages of development.

It is important to note, however, that numerical procedures are available for the solution of nonlinear algebraic equations; and by incrementing the time variable \( t \), the original set of differential equations reduce to a system of nonlinear algebraic equations for each time increment. Moreover, finite element representations of static behaviour in pneumatic structures also lead to systems of nonlinear algebraic equations. In view of this, the solution of large systems of nonlinear differential equations is not considered further in this study. Rather, consideration is given to procedures for solving systems of nonlinear algebraic equations, it being understood that, at the cost of greatly increasing the computing time, these procedures can also be applied to certain systems of nonlinear differential equations.

Several numerical schemes for solving simultaneous nonlinear algebraic equations are available in the literature; but not all of these are suitable for systems of equations as large as those encountered in the present investigation. A comprehensive review and comparison of numerical procedures for equations of this type was recently contributed by Remmler, Cowood, Stanton, and Hill [25], wherein numerical experimentation showed that the classical Newton-Raphson method and the Fletcher-Powell method are among the most efficient and reliable techniques available. To these may be added the method of incremental loading, which is somewhat related to the Newton-Raphson method, except that the loading is assumed to be applied in small increments during each of which the structure responds linearly. This latter technique is particularly well-suited for the analysis of stability and plastic behaviour. The numerical results to be presented subsequently were obtained using variations of the Newton-Raphson method. Thus, for the present discussion, it suffices to merely outline this procedure. Details of this and other numerical procedures can be found in the report by Remmler et al [25] and in the papers by Sprang [26] and Brooks [27].

Consider a system of nonlinear stiffness relations of the form in Eq. (86). Assuming that after appropriate boundary conditions have been applied there remain \( r \) unspecified components of node displacements, this system represents a set of \( r \) independent nonlinear equations in the unknown displacements \( \vec{U}_{N|k} \). For simplicity, suppose that these equations are represented in matrix form as

\[
H(\vec{U}) = \vec{0}
\]

where \( H \) is a \( r \times 1 \) column matrix, each row of which represents an independent nonlinear stiffness equation, and \( \vec{U} \) is the solution vector. To solve these equations, we expand \( H \) in a Taylor series about an arbitrary point \( \vec{U}^0 \) which represents an initial estimate of the solution \( \vec{U} \). The vector \( \vec{U}^0 \) may, for example, correspond to the linearized solution. Taking only two terms, we find

\[
H(\vec{U}) = H(\vec{U}^0) + J(\vec{U}^0)(\vec{U} - \vec{U}^0)
\]

(98)

where \( J \) is the Jacobian matrix

\[
J = \left[ \frac{\partial H}{\partial U} \right]
\]

(99)
6.2. Elasto-plastic Behaviour of a Metallic Membrane

The finite element formulation described in Section 4.3 was used in the analysis of plastic behaviour of a square aluminum membrane subjected to external pressure. A bilinear stress-strain law of the form indicated in Fig. 6 was assumed with $\sigma = 2,514 \text{ kg/cm}^2$, $\gamma = 0.0034$, $E_e = 20E_p = 740,000 \text{ kg/cm}^2$. These properties correspond to the aluminum alloy 2014-73 and agree closely with those used in experiments on rectangular shell plating by Neubert and Sommer [29]. A thin metallic sheet, 60 cm square and 0.14 cm thick, is subjected to a uniformly distributed hydrostatic pressure. As the pressure is slowly increased, a region near the center of the plate yields and plastic flow is initiated.

This behaviour was analyzed by using the finite element representation shown in Fig. 7. Fig. 8 shows the computed variation in the center displacement of the sheet with external pressure compared with the experimental results of...
Neubert and Sommer [29] and with results obtained using approximate theories proposed by Foppl [30] and Hencky [31]. Again note that the rather coarse network was adequate to obtain displacements in excellent agreement with experimental data.

6.3. Finite Stretching of a Rubber Sheet

In order to indicate the rate of convergence of the results as the finite-element network is refined, we reproduce here results similar to those obtained earlier by Oden and Soto [11]. In this example, an initially square rubber sheet, 0.127 cm thick, is stretched in its plane to twice its original length. The material is assumed to be of the Mooney type, with material constants $C_1$ and $C_2$ of 1.75 and 0.15 kg/cm$^2$ respectively. Thus, Eqs. (48) are applicable.

Various finite element representations of the sheet are shown in Fig. 9 along with the variations in the total edge force.

Fig. 6: Bilinear stress-strain law

Fig. 7: Finite element representation of a square metallic membrane

Fig. 8: Variation in center displacement of a square plate with external pressure

Fig. 9: Finite stretching of a square rubber sheet
with the total number of finite elements. In this example, the edge force converged monotonically to approximately 16.32 kg.

6.4. Inflation of an Initially Flat Rubber Membrane

In a recent paper, Hart-Smith and Crisp [32] presented experimental data on the inflation of thin rubber membranes. Although these investigators used an exponential form of the strain energy function, sufficient information was given to deduce equivalent Mooney constants for the material used. Specifically, we consider the inflation of an initially flat, circular, synthetic rubber membrane subjected to uniform external pressure. The membrane is initially 50.8 mm in diameter and 0.2 mm thick and is held fixed around its edges in a metal clamp.

In the finite element analysis of this membrane, it was assumed that the rubber possessed a strain energy function of the Mooney form [Eq. (41)] so that the nonlinear stiffness relations in Eqs. (48) were applicable. Values of the Mooney constants of $C_1 = 9.5$ and $C_2 = 1.75$ kg/cm² were derived from the data given in [32]. The case considered is that in which the membrane is subjected to a uniform pressure of 0.097 kg/cm². According to the experimental data, this corresponds to an extension ratio at the crown of 5.5.

It is important to note that the inflating pressure is a highly nonlinear function of the extension ratio at the crown and, consequently, of the displacements. Thus, more than one equilibrium configuration can exist for a given pressure. No provisions for determining all possible equilibrium states for a specified pressure were incorporated in the present analysis, and the particular configuration obtained depends upon the choice of initial values employed in the iterative solution of the nonlinear stiffness relations.

Several finite element networks were used in the analysis, beginning with a single 30° degree element and eventually using the 10-element representation shown in Fig. 10. For a given finite element network, the rate of convergence of the Newton-Raphson method depends on the choice of initial values of the displacements. Convergence rates are considerably higher for plane problems (such as that in Fig. 9) than in the case of large out-of-plane deformations. In the present example, rates of convergence were increased by first analyzing a coarse finite-element representation of the membrane using a small number of iterations. The results were then used as starting values for a more refined representation, the displacements of the added node points being obtained through linear interpolation.

Fig. 10 shows the computed profile of the inflated sheet compared with the profiles obtained experimentally and theoretically by Hart-Smith and Crisp. We observe that the agreement is quite good, the maximum difference between the displacements computed by the finite-element analysis and the experimental values being approximately six per cent.

6.5. Experiments on Rubber Membranes

As a final example, we consider briefly the results of experiments performed at the Structures and Materials Laboratory of the University of Alabama Research Institute on thin natural rubber membranes. In these experiments, circular disks, 0.0068 in (0.0173 cm) thick and 15.0 in (38.1 cm) in diameter, of pure gum natural rubber sheet were clamped around their edges in a metal clamp. The disks were marked at the center and on ten equally spaced concentric circles. The disks were
then inflated in stages of pressures of approximately 100 mm of water, which corresponded to a polar extension ratio of around $\lambda = 5$. After resisting maximum pressures for 45 minutes, the specimens were then deflated in stages until all applied pressure was removed.

A finite element representation with 96 elements, four nodes along $30^\circ$ radial lines was used to determine the inflated profile of a typical specimen subjected to a pressure of 61 mm of water. Again, the material was assumed to be of Mooney type, with constants $C_1 = 1.14 \text{ kg/cm}^2$ and $C_2 = 0.14 \text{ kg/cm}^2$ determined by the method of Hart-Smith and Crisp [32] and data in Fig. 13. No attempt was made to predict the obvious viscoelastic character of the behaviour. Results of these calculations are given in Fig. 14.
It is a pleasure to acknowledge the assistance of Mr. T. Sato, who developed the computer programs used in all of the calculations. This work was supported by the National Aeronautics and Space Administration through General Research Grant NAG-381, and the synthetic rubber materials used in the experiment were donated by the Materials Laboratory of NASA's Marshall Space Flight Center.

Notation

Indical notation and the summation convention are used throughout this paper. Upper-case Latin indices indicate points in space and lower-case indices indicate elements of an array. In general, Greek indices are associated with local coordinate systems and range from 1 to 2. The following symbols are used:

- $a, b$: Constants in displacement approximation
- $A, A_o$: Area of deformed and undeformed element
- $b_N$: Thermal load vector at node $N$
- $c_N$: Node displacement coefficients
- $C, C_1, C_2$: Material constants
- $d_i$: Components of rigid-body translation
- $d_{ij}$: Deformation rate tensor
- $e$: Element identification index
- $E$: Total number of finite elements
- $E_1, E_2, E_3$: Elastic moduli
- $E_{ab}$: Multi-dimensional array of material constants
- $f(x_i)$: Yield surface
- $F_i$: Body force per unit mass
- $g_{ab}$: A surface tensor
- $G, G_c, G_c$: Shear moduli
- $H$: Heat input per unit mass
- $I_1, I_2, I_3$: Strain invariants
- $M_N$: Consistent mass matrices
- $n$: Total number of nodes
- $p$: Hydrostatic pressure
- $P_{N_k}$: Generalized node forces of element $e$ in local coordinates
- $P_{N_k}$: Generalized node forces in global coordinates
- $q$: External pressure
- $q_{1, q_N}$: Components of heat flux
- $Q$: Heat input
- $Q_{1, q_{1e}}$: Element node forces due to $q$
- $u_{1e}$, $u_{N_{1e}}$: Displacement components in local coordinates
- $u$: Total internal energy
- $u_{N_k}$: Displacement of node $N$ in $z_i$-direction in local coordinates
- $v_i, v_o$: Volumes of deformed and undeformed elements
- $W$: Strain energy per unit of undeformed volume
- $x_{1e}$: Local coordinates of element $e$
- $y_{1e}$, $y_{N_k}$: Local coordinates of node $N$ of element $e$
- $y_{N_{1e}}$: Local coordinates of deformed element $e$
- $y_{N_{1e}}$: Local coordinates of node $N$ of element $e$ after deformation
- $Z_i$: Global coordinates
- $Z_{N_k}$: Global coordinates of node $N$
- $\beta_{ij}$: Orthogonal transformation matrix of element $e$
- $\gamma_{ij}$: Lagrange strain tensor
- $\delta_{ij}$: Kronecker delta
- $e_{ab}$: Two-dimensional permutation symbols
- $\alpha$: Kinetic energy
- $\lambda$: Extension ratio
- $\nu, \overline{\nu}, \nu_c$: Poisson's ratios
- $\xi$: Internal energy per unit mass
- $\rho$: Mass densities
- $\sigma_{ab}$: Stress tensor
- $\sigma_{ab}$: Deviatoric stress tensor
- $\Omega$: Power of external forces
- $\omega_{NM_e}$: Multi-dimensional array

References


[5] Martin, H.C.,
On the Derivation of Stiffness Matrices for the
Analysis of Large Deflection and Stability Problems,
Conference on Matrix Methods in Structural Mechanics, Wright-Patterson AFB, Dayton, Ohio, October 1965

[6] Oden, J.T.,
Calculation of Geometric Stiffness Matrices for Complex Structures,
AIAA Journal, Vol. 4, No. 6, August 1966, pp. 1480-1482

[7] Wissmann, J.W.,
Numerische Berechnung nichtlinear elastischer Körperformationen,
Dissertation, Hannover, 1963

[8] Wissmann, J.W.,
Nonlinear Structural Analysis; Tensor Approach,
Conference on Matrix Methods in Structural Mechanics, Wright-Patterson AFB, Dayton, Ohio, October 1965

[9] Oden, J.T.,
Analysis of Large Deformations of Elastic Membranes by the Finite Element Method,
Proceedings, IASS Congress on Large Span Shells, Leningrad, September, 1966

[10] Oden, J.T.,
Numerical Formulation of Nonlinear Elasticity Problems,
Journal of the Engineering Mechanics Division, ASCE, June 1967

[11] Oden, J.T. and Soto, T.,
Finite Strains and Displacements of Elastic Membranes by the Finite Element Method,
International Journal of Solids and Structures, June 1967

[12] Becker, E.B.,
A Numerical Solution of a Class of Problems of Finite Elastic Deformation,
Dissertation, University of California, Berkeley, California, 1966

[13] Green, A.E. and Adkins, J.E.,
Large Elastic Deformations and Non-linear Continuum Mechanics,
Oxford University Press, London, 1960

[14] Eringen, A.C.,
Nonlinear Theory of Continuous Media,

[15] Rivlin, R.S. and Saunders, D.W.,
Large Elastic Deformations of Isotropic Materials,

[16] Rivlin, R.S. and Saunders, D.W.,
The Free Energy of Deformation for Vulcanized Rubbers,
Transactions of the Faraday Society, No. 48, 1952, pp. 200-206

[17] Mooney, M.,
A Theory of Large Elastic Deformations,
Journal of Applied Physics, No. 11, 1940, pp. 582-592

[18] Treloar, L.R.G.,
The Physics of Rubber Elasticity,

[19] Rivlin, R.S.,
Large Elastic Deformations of Isotropic Materials;
1 Fundamental Concepts,

[20] Hutchinson, W.D., Becker, G.W., and Landel, R.F.,
Determination of the Stored Energy Function of Rubber-like Materials,

[21] Boresi, A.P., Longhaor, H.L. and Miller, R.E.,
Buckling of Axially-Compressed Bilayered Fiber-Reinforced Elastic Cylindrical Shells,

[22] Pope, G.G.,
The Application of the Matrix Displacement Method in Plane Elasto-Plastic Problems,
Conference on Matrix Methods in Structural Mechanics, Wright-Patterson, AFB, Dayton, Ohio, October 1965

[23] Pope, G.G.,
A Discrete Element Method for the Analysis of Plane Elastoplastic Stress Problems,
The Aeronautical Quarterly, February 1966

[24] Hill, R.,
The Mathematical Theory of Plasticity,
Oxford University Press, London, 1950

Solutions of Systems of Nonlinear Equations,
Lockheed MSC/HREC, NASA 8-20178, October 1966

[26] Sprang, H.A.,
A Review of Minimization Techniques for Nonlinear Functions,

[27] Brooks, S.H.,
A Comparison of Maximum Seeking Methods,

[28] Kuhn, P.,
Stresses in Aircraft Structures,
Neubert, M. and Sommer, A.,
Rechteckige Blechhaut unter gleichmässig verteiltem Flüssigkeitsdruck,
Luftfahrtechnik, Vol. 17, No. 7, July 20, 1940,
pp. 207-210 (also published as NACA Technical Memorandum, 965, December 1940)

Föppl, A. and Föppl, L.,
Drang und Zwang,
Vol. 1, Oldenburg, 1924

Hencky, H.,
Die Berechnung dünner rechteckiger Platten mit verschwindender Biegungsteifigkeit,
Zeitschrift für Angewandte Mathematik und Mechanik, (ZAMM), Bd. 1, Heft 2, April 2, 1921,
pp. 81-89

Hart-Smith, L.J. and Crisp, J.D.C.,
Large Elastic Deformations of Thin Rubber Membranes,

Prager, W.,
The Stress-Strain Laws of the Mathematical Theory of Plasticity - A Survey of Recent Progress,
Journal of Applied Mechanics, Vol. 15, 1948,
pp. 226-233

Drucker, D.C.,
A Definition of Stable Inelastic Material,
Journal of Applied Mechanics, Vol. 26, 1959,
pp. 101-106

Hovner, K.S.,
On the Formulation of Iterative Solution of Small Strain Plasticity Problems,
Quarterly of Applied Mathematics, January 1966,
pp. 323-335

Discussion

Question:
Could you explain somewhat more the Newton-Raphson method? How many steps of iteration did you use?

Answer:
The Newton-Raphson method is perhaps one of the oldest methods to solve nonlinear equations. It has several disadvantages. One is that it requires an initial guess, you must guess the solution. The initial guess could be the solution to the linear problem, you could linearize the equations and solve them. Basically, suppose ... (follows the description of the method given in section 5.3, Eqs. (97) to (100)). Of course finding the corrected solution to the system of nonlinear equations is only one step in the process. Then you have to start all over and go through the process again. And usually, depending on how good you can guess, five or six iterations are sufficient. In many problems you can start out with a very coarse network, use one or two elements, and solve this with one or two iterations, you do not care too much about accuracy. Then you use linear interpolation and take a more refined network. In this case you get a better set of answers. The solution of simultaneous nonlinear equations involves a great deal of computer time, and consequently I think that a lot of work needs to be done in the development of reliable methods for solving nonlinear equations. When you speak of application of finite elements to linear systems you often hear talk of a thousand elements and so forth. In application to nonlinear problems I guess the state of the art is that we can handle 40 to 50 elements, and not much more. It is not a question of the size of the computer, it is usually a matter of computing time. Conceivably we have computers that would be big enough, but it might take ten hours to solve a problem.