INTRODUCTION

As a result of the extreme mathematical complexity involved in obtaining quantitative solutions to problems in finite elasticity, relatively few exact solutions to such problems appear in literature.\(^1,2,3\) Those solutions which are available deal with bodies which, in both the deformed and undeformed state, are of relatively simple geometric shapes. Moreover, even in the case of simple geometries, it is often necessary to introduce approximate numerical techniques in the final steps of the solution to obtain quantitative results.

Regardless of the initial assumptions and the methods used to formulate a nonlinear elasticity problem, if numerical methods are used to evaluate the results, the continuum is, in effect, approximated by a discrete model in the solution process. A logical alternative to this classical approach is to represent the continuum by a consistent discrete model at the onset; then further idealization in either the formulation or the solution may often be unnecessary. This approach is referred to as the finite element method,\(^4\) its generality gives promise that the method will find applications in a wide range of nonlinear structural problems.

Note.—Discussion open until November 1, 1967. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. This paper is part of the copyrighted Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, Vol. 93, No. ST3, June, 1967. Manuscript was submitted for review for possible publication on June 28, 1966.

\(^1\)Assoc. Prof. of Engrg. Mechanics, Univ. of Alabama in Huntsville, Huntsville, Ala.


M. J. Turner, et al., appear to have been the first to present applications of the finite element method to geometrically nonlinear structural problems. Subsequent contributions to the subject can be divided into two categories: (1) those which account for nonlinearity by successive corrections to the classical linearized problem and (2) those which formulate the nonlinear problem and then solve it numerically.

A large majority of the literature available on the subject, including the original work, falls into the former category. This procedure was used by J. H. Argyris to determine moderately large displacements in three-dimensional bodies and by H. C. Martin to analyze nonlinear plane stress problems. Martin's paper contains a survey of literature using the successive-correction approach.

The finite element formulation of nonlinear elasticity problems using the latter approach is that of Wissmann, who presented nonlinear stiffness relations based on the assumption of small strains. This approach has the advantage that nonlinear characteristics in the behavior of a system can be identified in the development and the relative importance of nonlinear terms can be easily evaluated. Oden presented a finite-element analysis of large displacements but small strains of certain plane-stress and membrane problems, and Oden and Sato derived nonlinear stiffness relations for investigations of finite strains in highly-elastic membranes. Using a somewhat different approach, E. B. Becker presented a numerical procedure for the analysis of finite in-plane deformations of rubber sheets subjected to prescribed boundary displacements. No finite element formulations that account for finite strains in three-dimensional bodies appear to have been published.

This paper extends the finite-element method to the analysis of three-dimensional elastic bodies that experience large displacements and finite strains. The writer is primarily concerned with the development of general nonlinear stiffness relations for finite deformations of elastic bodies, and only brief consideration is given to various numerical schemes for solving the resulting nonlinear algebraic equations. Nonlinear stiffness relations are de-


rived for tetrahedral, triangular, and one-dimensional elements for several types of highly elastic materials. The assembly of elements is accomplished by a series of elementary group transformations.

Notation.—Linear matrix algebra and symbolic notation provide a concise and meaningful notation in linear structural theory. In the formulation of nonlinear structural problems, however, ordinary matrix notation is cumbersome and inadequate. The development of discrete models capable of depicting finite deformations in solids involves multidimensional arrays and, hence, falls outside of the realm of the algebra of rectangular matrices.

For this reason, indicial notation and the summation convention are adopted herein. Upper-case Latin indices indicate points in space and lower-case Latin indices indicate elements in an array. Greek indices are used to indicate specific finite elements in a structural system.

GEOMETRY OF DISCRETE SYSTEMS

Consider a three-dimensional body of arbitrary shape subjected to a general system of applied forces. A discrete model of this system is formed by dividing the undeformed body into a finite number \( e \) of tetrahedral regions termed "finite elements." The geometry of each element is thus defined by four points, termed "nodes," connected together by straight lines. In such a model, curved parts of the boundary surfaces of the body are represented by networks of flat surfaces that are faces of tetrahedral elements (see Fig. 1).

To specify the location of nodes, \( e \) fixed rectangular Cartesian coordinate systems are established, one in the neighborhood of each finite element. These reference frames are referred to as "local coordinate systems" and are denoted \( x_i^\alpha \) \((i = 1, 2, 3; \alpha = 1, 2, \ldots, e)\). The local coordinates of the four nodes of element \( \alpha \) are denoted \( x_{Ni\alpha} \) \((N = 1, 2, 3, 4; i = 1, 2, 3; \alpha = 1, 2, \ldots, e)\). When expressed in terms of its local coordinates, the behavior of each element is described separately and is assumed to be independent of the geometry and the behavior of other elements as well as being independent of the location of the element in the assembled structure.

The geometry of the assembled structure is described in terms of an additional system of fixed rectangular coordinates \( Z_1, Z_2, Z_3 \) referred to as "global coordinate system." The location of node \( N \) in the assembled structure is given in global coordinates by \( Z_{Ni} \) \((N = 1, 2, \ldots, m; i = 1, 2, 3)\) in which \( m \) is the total number of nodes in the assembled structure.

The local coordinate axes \( x_i^\alpha \) in the neighborhoods of each element are rotated into local systems of axes \( x_i^\alpha \) which are parallel to the global coordinates by simple orthogonal transformations of the form

\[
\tilde{x}_i^\alpha = \lambda_{ij}^\alpha x_j^\alpha
\]

in which an underscore suspends the summation. Here \( \lambda_{ij}^\alpha \) is the cosine of the angle between the \( x_j \) axis of element \( \alpha \) and the \( Z_i \) axis of the global coordinate system.

The connectivity of the system is established by relating the local and global coordinates of each node. This is accomplished by the transformation

\[
\tilde{x}_{Ni\alpha} = \Omega_{MN\alpha} Z_{Ni}
\]

in which \( \Omega_{MN\alpha} \) equals unity if node \( M \) of element \( \alpha \) is identical to node \( N \) in the assembled structure and equals zero if otherwise. The ranges of the indices in this equation are \( M, 1, 2, 3; N = 1, 2, \ldots, m; \) and \( \alpha = 1, 2, \ldots, \).

**FIG. 1.—FINITE ELEMENT REPRESENTATION OF THREE-DIMENSIONAL BODY**

This simple transformation establishes dependencies between local and global coordinates and, in effect, assembles all of the finite elements into one discrete system.

**HOMOGENEOUS DEFORMATIONS**

Attention is now confined to a typical element of the system and the element index \( \alpha \) is temporarily dropped for clarity. Let \( u_1, u_2, \) and \( u_3 \) denote the components of displacement parallel to the local coordinates \( x_1, x_2, \) and \( x_3 \), respectively. Then, according to Green and Zerna, the Lagrangian strain tensor in local coordinates is given by

\[
\gamma_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right)
\]

wherein \( i, j, \) and \( k \) range from 1 to 3. This relationship holds for large strains and gross deformations of the body.
In general, straight lines connecting node points in the undeformed body become curved lines in the deformed body. However, if the node points are selected sufficiently close to one another, node lines in the deformed body are closely approximated by a system of straight line segments, as is indicated in Fig. 2. This is equivalent to assuming that the displacement fields corresponding to a given element are linear functions of the local coordinates of that element; that is,

\[ u_i = d_i + a_{ij} x_j \]  \hspace{1cm} (4)

in which \( d_i \) are the rigid-body translations of the element and the \( a_{ij} \) are undetermined constants. Whereas a finite element may undergo rigid-body motions relative to other elements, the entire assembly of elements (the body under consideration) is assumed to be constrained so that rigid motions are eliminated. Thus, the terms \( d_i \) may not influence the strains, but they may influence the directions of applied loads in the deformed body.

Evaluating Eq. 4 at each of the four node points of the element gives the set of relations

\[ u_{Ni} = d_i + a_{ij} x_{Nj} \]  \hspace{1cm} (5)

in which \( i, j = 1, 2, 3 \) and \( N = 1, 2, 3, 4 \). In Eq. 5, the \( u_{Ni} \) are the displacement components of node \( N \) and \( x_{Nj} \) are the local coordinates of node \( N \). Note also that

\[ \frac{\partial u_{Ni}}{\partial x_j} = a_{ij} \]  \hspace{1cm} (6)
Eq. 5 represents a system of twelve simultaneous equations in the twelve unknowns $d_i$, $a_{ij}$. If these equations are grouped into three sets of four simultaneous equations, each set corresponding to a different direction $i$, then the twelve equations reduce to three sets of four independent equations which are not difficult to invert. It is found that the solutions are

$$d_i = k_N u_{Ni}$$

and

$$a_{ij} = c_{jN} u_{Ni}$$

in which

$$k_1 = \frac{1}{6\nu_o} \varepsilon_{ijk} x_{2i} x_{3j} x_{4k}$$

$$k_2 = \frac{1}{6\nu_o} \varepsilon_{ijk} x_{3i} x_{1j} x_{4k}$$

$$k_3 = \frac{1}{6\nu_o} \varepsilon_{ijk} x_{1i} x_{2j} x_{4k}$$

$$k_4 = \frac{1}{6\nu_o} \varepsilon_{ijk} x_{1i} x_{3j} x_{2k}$$

and

$$c_{IN} = \frac{\delta_{IN}}{6\epsilon_o}$$

in which $\nu_o$ = the volume of the undeformed element and $\varepsilon_{ijk}$ = the permutation symbol which is defined as being 1, -1, or zero, respectively, if the indices $ijk$ form an even permutation, an odd permutation, or do not form a permutation of the integers from 1 to 3. In Eq. 10, the quantities $\delta_{IN}$ are the cofactors of elements in a matrix $b_{IN}$ which is defined as

$$b_{IN} = \begin{cases} x_{Ni} & \text{for } i = 1, 2, 3; N = 1, 2, 3, 4 \\ 1 & \text{for } i = 4; \quad N = 1, 2, 3, 4 \end{cases}$$

Note that only twelve of the sixteen co factors of $b_{IN}$ appear in Eq. 10 and that $6\epsilon_o$ is the determinant of $b_{IN}$. Note also that $i$ and $j$ range from 1 to 3 in these relations and the repeated index $N$, in Eqs. 7 and 8, is to be summed from 1 to 4.

With the quantities $d_i$ and $c_{IN}$ given by Eqs. 9 and 10, substitution of Eqs. 7 and 8 into Eqs. 4 and 6 defines the components of displacement and their derivatives in terms of the node displacements

$$u_i = k_N u_{Ni} + c_{jN} u_{Ni} x_j$$

and

$$\frac{\partial u_j}{\partial x_i} = c_{jN} u_{Ni}$$

Finally, substituting Eq. 13 into Eq. 3 gives

$$\gamma_{ij} = \frac{1}{2} \left( c_{jN} u_{Ni} + c_{IN} u_{Nj} + c_{jN} c_{jM} u_{Nk} u_{MK} \right)$$

in which $i, j, k = 1, 2, 3$ and $M, N = 1, 2, 3, 4$. Note that since the strain tensor is determined from a prescribed displacement field, the compatibility conditions are automatically satisfied throughout the element. Moreover, the
strains are constant throughout each element and the displacement components are continuous across element boundaries.

Three invariants may be formed from a symmetrical tensor of second order. In subsequent considerations, it is convenient to form invariants of the deformation tensor \( \delta_{ij} + 2\gamma_{ij} \), in which \( \delta_{ij} \) is the Kronecker delta. These invariants are given by the formulas

\[
I_1 = 3 + 2\gamma_{ij} \tag{15a}
\]

\[
I_2 = 3 + 4\gamma_{ij} + 2\left(\gamma_{ij} \gamma_{ij} - \gamma_{ij} \gamma_{ij}\right) \tag{15b}
\]

and

\[
I_3 = \text{det}(\delta_{ij} + 2\gamma_{ij}) \tag{15c}
\]

In Eq. 15c, the prefix det indicates the determinant of the argument within. These invariants can be written in terms of the generalized node displacements by simply introducing Eq. 14 into Eqs. 15.

**GENERALIZED FORCES**

Assuming that the deformation process is reversible and isothermal or reversible and adiabatic, an elastic potential function \( W \) exists which represents the strain energy per unit volume of the undeformed element. Since, in general, \( W \) is a function of the strains, it is also a function of the node displacements by virtue of Eq. 14. The total strain energy of a finite element is then

\[
U = \int \nu v d\nu_0 \tag{16}
\]

in which the integration is performed over the volume of the undeformed element.

The precise form of \( W \) depends, of course, on the type of material of which the element is composed. In the case of heterogeneous materials, \( W \) is also a function of the local coordinates \( x_i \). However, in the present study nonhomogeneity is accounted for by assigning different material properties to each finite element and regarding each element as being homogeneous within itself. Thus, \( W \) is solely a function of the strains and, in view of Eq. 14, it is a constant with respect to the coordinates \( x_i \) for each finite element. It follows that \( W \) can be factored outside of the integral in Eq. 16 and that, therefore,

\[
U = \nu_0 W \tag{17}
\]

It is further assumed that the external forces acting on the element are also derivable from a potential function. Let \( f_i \) denote the components of body force per unit volume and \( S_i \) denote the components of surface tractions per unit surface area of the deformed element. Then the potential energy of the extrinsic forces acting on the element is

\[
\Omega = \int f_i u_i \sqrt{I_3} d\nu_0 - \int S_i u_i ds \tag{18}
\]
in which \( ds \) is the an element of surface area of the deformed element. Introducing Eq. 12 into this expression gives
\[
\Omega = -\rho_{NI} u_{NI} \quad \cdots \quad (19)
\]
in which \( \rho_{NI} = \int f_i \left( k_N + c_{jN} x_j \right) \sqrt{J} \, d\nu_o + \int s_i \left( k_N + c_{jN} x_j \right) d\nu \quad \cdots \quad (20) \]
The quantities \( \rho_{NI} \) are the generalized forces corresponding to the generalized node displacements \( u_{NI} \). Thus \( \rho_{NI} \) is the force at node \( N \) of the deformed element acting in direction \( i \).

The cartesian coordinates \( y_i \) of material points in the deformed body are
\[
y_i = x_i + u_i \quad \cdots \quad (21)
\]
so that the coordinates of node points of the deformed element are given by
\[
y_{NI} = x_{NI} + u_{NI} \quad \cdots \quad (22)
\]
 Thus, it is a simple matter to define the geometry of the deformed element in terms of the node displacements for the purpose of evaluating the integrals in Eq. 20.

Consider, for example, the case in which a boundary element is subjected to a uniform pressure \( q \) over the exterior face of the tetrahedron which contains nodes 1, 2, and 3. Then,
\[
S_i = qA_i \quad \cdots \quad (23)
\]
in which \( A_i \) is the area vector normal to the surface,
\[
A_i = \epsilon_{ijk} (y_{2j} y_{3k} - y_{1j} y_{3k} - y_{2j} y_{1k}) \quad \cdots \quad (24)
\]
If no body forces are present, it follows from Eq. 20 that, in this case,
\[
\rho_{NI} = qA_i \left( k_N A + c_{jN} A^3 \int x_j \, dy_1 \, dy_2 \right) \quad \cdots \quad (25)
\]
in which \( A = \sqrt{A_1 A_4} \quad \cdots \quad (26) \)
Thus, it is possible to account for changes in the external loading due to deformations of the body.

Returning now to Eqs. 17 to 19, we have for the total potential energy of the finite element
\[
V = v_o W - \rho_{NI} u_{NI} \quad \cdots \quad (27)
\]
According to the principle of minimum potential energy, \( V \) is a relative minimum when the generalized displacements correspond to an equilibrium configuration. Thus, if the finite element is in equilibrium,
\[
\frac{\delta V}{\delta u_{NI}} = 0 = v_o \frac{\delta W}{\delta u_{NI}} - \rho_{NI} \quad \cdots \quad (28)
\]
from which it follows that
\[
\rho_{NI} = v_o \frac{\delta W}{\delta u_{NI}} \quad \cdots \quad (29)
\]
In the case of isotropic materials, \( W \) can be expressed as a function of the strain invariants defined in Eqs. 15, or

\[
W = W(I_1, I_2, I_3) \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots (30)
\]

For such materials, Eq. 29 can be written in the form

\[
\begin{aligned}
\hat{P}_{NI} &= \frac{v_o}{2} \left[ \frac{\partial W}{\partial I_1} \left( \frac{\partial I_1}{\partial \gamma_{ij}} + \frac{\partial I_1}{\partial \gamma_{ij}} \right) + \frac{\partial W}{\partial I_2} \left( \frac{\partial I_2}{\partial \gamma_{ij}} + \frac{\partial I_2}{\partial \gamma_{ij}} \right) \\
&\quad + \frac{\partial W}{\partial I_3} \left( \frac{\partial I_3}{\partial \gamma_{ij}} + \frac{\partial I_3}{\partial \gamma_{ij}} \right) \right] \frac{\partial \gamma_{ij}}{\partial u_{NI}} \\
&\quad + \frac{2 \delta_{jk}}{\partial I_1} \left( \frac{\partial W}{\partial I_1} + 4(\delta_{jk} + \delta_{jk} \gamma_{rr} - \gamma_{jk}) \frac{\partial W}{\partial I_2} \right) \\
&\quad \quad + (2 \delta_{jk} + 4 \delta_{jk} \gamma_{rr} - 4 \gamma_{jk} + 4 \epsilon_{mnj} \epsilon_{rsk} \gamma_{mr} \gamma_{ns}) \frac{\partial W}{\partial I_3} \right] \cdots \cdots \cdots \cdots (31)
\end{aligned}
\]

where, in differentiating with respect to \( \gamma_{ij} \), it is understood that all other strain components are held constant, including \( \gamma_{ji} \). Introducing Eqs. 15 into this result, it is found that

\[
\begin{aligned}
\hat{P}_{NI} &= v_o c_{jN} \left( \delta_{kl} + c_{kM} u_{Ml} \right) \left[ \frac{2 \delta_{jk}}{\partial I_1} \left( \frac{\partial W}{\partial I_1} + 4(\delta_{jk} + \delta_{jk} \gamma_{rr} - \gamma_{jk}) \frac{\partial W}{\partial I_2} \right) \\
&\quad \quad + (2 \delta_{jk} + 4 \delta_{jk} \gamma_{rr} - 4 \gamma_{jk} + 4 \epsilon_{mnj} \epsilon_{rsk} \gamma_{mr} \gamma_{ns}) \frac{\partial W}{\partial I_3} \right] \cdots \cdots \cdots \cdots (32)
\end{aligned}
\]

This equation represents the general nonlinear stiffness relation for finite deformations of an isotropic, elastic finite element. The relation is expressed in terms of node displacements by introducing Eq. 14. Again note that in the above equations upper-case indices range from 1 to 4 while lower-case indices range from 1 to 3. Unless noted otherwise, this will be the case in subsequent equations.

Stresses in the element are calculated by means of the formula

\[
\tau_{ij} = \frac{1}{2\sqrt{I_3}} \left( \frac{\partial W}{\partial \gamma_{ij}} + \frac{\partial W}{\partial \gamma_{ji}} \right) \cdots \cdots \cdots \cdots (33)
\]

in which \( \tau_{ij} \) is the stress tensor per unit area of the deformed body. Since

\[
\frac{\partial W}{\partial u_{Nk}} = \frac{1}{2} \left( \frac{\partial W}{\partial \gamma_{ij}} + \frac{\partial W}{\partial \gamma_{ji}} \right) \frac{\partial \gamma_{ij}}{\partial u_{Nk}} = \sqrt{I_3} \tau_{ij} \frac{\partial \gamma_{ij}}{\partial u_{Nk}} \cdots \cdots \cdots \cdots (34)
\]

Eq. 28 can be written in the alternate form

\[
\hat{P}_{Nk} = v_o \sqrt{I_3} \tau_{ij} \frac{\partial \gamma_{ij}}{\partial u_{Nk}} \cdots \cdots \cdots \cdots (35)
\]

Introducing Eq. 14 and noting that

\[
\frac{\partial u_{Nk}}{\partial u_{Mk}} = \delta_{MN} \delta_{ik} \cdots \cdots \cdots \cdots (36)
\]

Eq. 33 becomes

\[
\hat{P}_{Nk} = 2v_o \sqrt{I_3} \ c_{IN} \left( \delta_{jk} + c_{jM} u_{MK} \right) \tau_{ij} \cdots \cdots \cdots \cdots (37)
\]

The strain energy function

In order to apply Eqs. 29 or 32 to specific problems, it is necessary to know the strain energy \( W \) as a function of the strains or the strain invariants.
R. S. Rivlin\textsuperscript{14} noted that for compressible materials \( W \) can be expressed as a polynomial function of the strain invariants of the form

\[
W = \sum_{\alpha\beta\gamma} A_{\alpha\beta\gamma} (I_1 - 3)^\alpha (I_2 - 3)^\beta (I_3 - 1)^\gamma \quad \cdots \cdots \quad (38)
\]

in which \( A_{\alpha\beta\gamma} = \) constants. When written in this manner, \( W \) is zero in the unstrained body, provided \( A_{000} \) is zero.

R. A. Toupin and B. Bernstein\textsuperscript{15} obtained an alternate but equivalent formula for compressible materials by expressing \( W \) as a polynomial in the strains of the form

\[
W = \frac{1}{2} \left[ E_{ijkl} \gamma_{ij} \gamma_{kl} + \frac{1}{3} E_{ijklm} \gamma_{ij} \gamma_{kl} \gamma_{mn} + \cdots \right] \quad \cdots \cdots \quad (39)
\]

in which \( E_{ijkl}, E_{ijklm}, \ldots \), etc. are second order, third order, \ldots, etc. elastic constants. In the case of small strains, \( W \) is approximated by taking only the first term in the polynomial, and Eq. 39 reduces to the energy function for the classical Hookean material

\[
W = \frac{1}{2} E_{ijkl} \gamma_{ij} \gamma_{kl} \quad \cdots \cdots \quad (40)
\]

In the case of incompressible materials, the third strain invariant \( I_3 \) equals unity, the volumes of the undeformed and the deformed elements are equal, and \( W \) is known only in terms of \( I_1 \) and \( I_2 \). For certain incompressible rubber-like materials, a strain-energy function of the form

\[
W = C_1 (I_1 - 3) + C_2 (I_2 - 3) \quad \cdots \cdots \quad (41)
\]

in which \( C_1 \) and \( C_2 \) = experimentally determined constants, has been proposed by M. Mooney\textsuperscript{16}. Rivlin and D. W. Saunders\textsuperscript{17,18} verified experimentally that Eq. 41 gives a suitable form of \( W \) for certain highly-elastic materials such as natural rubber. Elastic materials with strain-energy functions of the form given in Eq. 41 are often referred to as Mooney-Rivlin materials.

L. R. G. Treloar\textsuperscript{19}, using a statistical approach based on the molecular theory of highly-elastic materials, found for incompressible materials

\[
W = C (I_1 - 3) \quad \cdots \cdots \quad (42)
\]


in which \( C = \text{a constant} \). \( \textit{Rivlin}^{20} \) refers to material having this form of strain-energy function as neo-Hookean.

NONLINEAR STIFFNESS RELATIONS

Substituting the appropriate form of \( W \) into Eq. 29 and introducing Eq. 14 into the results gives the nonlinear stiffness relations for various types of elastic materials in terms of the node displacements. Stiffness relations for compressible and incompressible materials are given below.

**Compressible Material.**—Substituting Eq. 39 into Eq. 29 gives

\[
P_{Mk} = v_0 c_{IM} (\delta_{jk} + c_{jN} u_{Nk}) \left( E_{ijmn} \gamma_{mn} + \frac{1}{2} E_{ijmrns} \gamma_{mn} \gamma_{rs} + \ldots \right) \tag{43}
\]

In Eq. 43, it is assumed the the arrays of elastic constants \( E_{ijmn}, E_{ijmrns}, \ldots \), etc. are symmetric with respect certain indices.\(^2\),\(^15\) Again, strains are expressed in terms of node displacements by introducing Eq. 14.

It is interesting to note that if terms of higher than the third degree in the displacement gradients are neglected in Eq. 39, the higher (third) order elastic constants \( E_{ijmrns} \) still appear in the nonlinear stiffness relation. The methods of successive corrections to linearized problems proposed by \textit{Turner, et al.},\(^6\) \textit{Argyris,\(^7\) Martin,\(^7\) and others, ignore characteristics peculiar to the nonlinear problem. Hence, these methods are incapable of accounting for such higher-order elasticities.}

**Hookean Material.**—In the case of Hookean materials, \( W \) is given by Eq. 40, strains are assumed to be small in comparison with unity. Introducing Eq. 40 into Eq. 29 and expressing the strains in terms of the node displacements gives

\[
P_{Mk} = v_0 c_{IM} (\delta_{jk} + c_{jN} u_{Nk}) E_{ijmn} c_{mI} \left( \delta_{ir} + \frac{1}{2} c_{nJ} u_{Jr} \right) u_{Ir} \tag{44}
\]

This result agrees with that obtained by \textit{Wissmann}.\(^8\)

**Hookean Material: Small Displacements.**—If products of the node displacements are neglected in comparison with the displacements themselves, Eq. 44 reduces to the stiffness relation for a tetrahedron based on the linear theory.\(^21\)

\[
P_{Mk} = v_0 c_{IM} E_{ijmn} c_{mN} u_{Nn} \tag{45}
\]

It is interesting to note that, even in the case of small strains, stiffness matrices do not transform congruently as in the linear theory. The array \( F_{mijl} \) in Eqs. 44 and 45 represents the stiffness matrix of a unit cube of material, and the quantities in parentheses transform this matrix into one associated with the finite element. In the case of large deformations, however, it is seen that the transformation terms become functions of the displacements and can no longer be calculated from the geometry of the undeformed element. Furthermore, the two sets of transformation terms are of different forms; one is not the transpose of the other as in the linear case.


Incompressible Material.—In the case of an isotropic incompressible material, the strain energy function $W$ is given in terms of only $I_1$ and $I_2$. The stress developed in an incompressible body is not completely derivable from a strain energy function; rather, $W$ determines the stress only to within an additive scalar function $h$ which represents the hydrostatic pressure. The hydrostatic pressure performs no work. Thus, if the incompressibility condition

$$I_3 = 1 \quad \text{................................................. (46)}$$

is satisfied, $W(I_1, I_2)$ determines the deformations of the body but not the stress. In general, the hydrostatic pressure must be determined from equilibrium equations or static boundary conditions.

To be consistent with the linear displacement approximation, $h$ is assumed to be constant for each finite element. In stead of the potential energy $V$ in Eq. 27, a new functional $\widetilde{V}$ is introduced, in which

$$\widetilde{V} = V + v_b h (I_3 - 1) = v_b W - p_{N_i} u_{N_i} + v_b h (I_3 - 1) \quad \text{........... (47)}$$

Thus, Eq. 46 is regarded as a condition of internal constraint and by introducing $\widetilde{V}$, use has been made of the method of Lagrange multipliers. In the present case, the hydrostatic pressure behaves as the Lagrange Multiplier.\(^{22}\)

The equilibrium conditions are

$$\frac{\partial \widetilde{V}}{\partial u_{MK}} = 0 = v_o \frac{\partial W(I_1, I_2)}{\partial u_{MK}} - p_{MK} + v_b h \frac{\partial I_3}{\partial u_{MK}} \quad \text{............... (48)}$$

so that, instead of Eq. 29,

$$p_{MK} = v_o \left( \frac{\partial W(I_1, I_2)}{\partial u_{MK}} + h \frac{\partial I_3}{\partial u_{MK}} \right) \quad \text{............... (49)}$$

For the three-dimensional finite element shown in Fig. 1, Eq. 49 represents twelve simultaneous equations in the thirteen unknowns $u_{MK}$ and $h$. The thirteenth equation is the incompressibility condition in Eq. 46, which is equivalent to the condition

$$\det \left( \delta_{ij} + c_{ijN} u_{Nj} \right) = 1 \quad \text{................................................. (50)}$$

Thus, Eqs. 49 and 50 represent the nonlinear stiffness relations for a finite element of an isotropic, incompressible material. Once the hydrostatic pressure is determined, the stress tensor is calculated from the formula

$$\tau_{ij} = h \delta_{ij} + \frac{1}{2} \left[ \frac{\partial W}{\partial I_1} \left( \frac{\partial I_1}{\partial \gamma_{ij}} + \frac{\partial I_1}{\partial \gamma_{ji}} \right) + \frac{\partial W}{\partial I_2} \left( \frac{\partial I_2}{\partial \gamma_{ij}} + \frac{\partial I_2}{\partial \gamma_{ji}} \right) \right] \quad \text{............... (51)}$$

If only displacements and strains are required in an analysis, it is not necessary to include the effects of hydrostatic pressure in the nonlinear stiffness relation. Eq. 29 is still applicable in such cases, provided the invariants $I_1$ and $I_2$ are written so that the incompressibility condition (Eq. 46) is automatically satisfied. Moreover, in many cases, a previous knowledge of the state of stress in a body makes it possible to determine $h$ directly (e.g., plane stress). It is then possible to eliminate $h$ from Eq. 49 and to write the nonlinear stiffness relations in terms of only the generalized displacements. Some of these cases are considered in subsequent paragraphs.

---

Mooney-Rivlin Material.—In the case of Mooney-Rivlin materials, $W$ is given by Eq. 41 and Eq. 49 becomes

$$P_{Mk} = 2v_0 C_{1M} \left( \delta_{kj} + c_{jN} u_{Nk} \right) \left\{ \delta_{ij} C_1 + C_2 \left[ 2 \delta_{ij} + 2 \delta_{ij} c_{ml} \left( \delta_{mn} + \frac{1}{2} \delta_{mj} u_{Jn} \right) u_{Im} - (c_{jl} u_{ji} + c_{ji} u_{ji} + c_{ij} c_{jN} u_{Nj} u_{Nk}) \right] + h w_{ij} \right\} \tag{52}$$

where $w_{ij} = \delta_{ij} \left[ 1 + 2 c_{nj} \left( \delta_{mn} + \frac{1}{2} c_{nK} u_{Km} \right) u_{jm} \right] - (c_{ij} u_{ij} + c_{ji} u_{ji}) + c_{ij} c_{jN} u_{Nj} + 2 c_{ml} c_{rl} \epsilon_{imr} \epsilon_{jst} \left( \delta_{sn} + \frac{1}{2} c_{sJ} u_{Jn} \right) \delta_{tp} + \frac{1}{2} c_{tL} u_{Lp} u_{Im} u_{Kp} \tag{53}$

In these relations $i, j, k, m, n, p, r, s, t = 1, 2, 3$ and $I, J, K, L, M, N = 1, 2, 3, 4$.

Neo-Hookean Material.—The nonlinear stiffness relation for neo-Hookean materials is obtained directly from Eq. 52 by replacing $C_1$ by $C$ and equating $C_2$ to zero; thus,

$$P_{Mk} = 2v_0 C_{1M} \left( \delta_{kj} + c_{jN} u_{Nk} \right) \left( \delta_{ij} C + h w_{ij} \right) \tag{54}$$

Again note that the incompressible condition in Eq. 50 must also be applied when writing the nonlinear stiffness relations (Eqs. 49, 52, or 54) for a given finite element.

### BARS, PLATES, AND MEMBRANES

All of the above stiffness relations apply to a tetrahedral element of a three-dimensional body. By adjusting the ranges of the indices, these equations can be degenerated to yield nonlinear stiffness relations for triangular plate elements, initially flat membranes, and one-dimensional straight bars. These elements are shown in Fig. 3.

**Membrane Element.**—The tetrahedron is reduced to a triangle in the $x_1, x_2$-plane with nodes $N = 1, 2, 3$. In this case,

$$\gamma_{ij} = \frac{1}{2} \left( c_{iN} u_{Nj} + c_{jN} u_{Ni} + c_{iN} c_{jM} u_{Nj} u_{Nk} \right) \tag{55a}$$

and

$$\gamma_{33} = \frac{1}{2} \left[ \lambda^2 - 1 \right] \tag{55b}$$

in which $i$ and $j$ now range from 1 to 2 but $k$ and $N$ range from 1 to 3. The quantity $\lambda$ in Eq. 55b is the ratio of the thickness of the deformed membrane to that of the undeformed membrane. The displacement coefficients $c_{jN}$ are given by

$$c_{jN} = \frac{1}{2A_o} \left[ (x_{22} - x_{22}) (x_{32} - x_{32}) (x_{12} - x_{12}) \right] \tag{56}$$

in which $A_o$ is the area of the triangle.

**Plane Element, Finite Plane Strain.**—The general formulas can be applied to problems of finite plane strain by assuming that the finite element is a tri-
FIG. 3.—DEGENERATE FINITE ELEMENTS

FIG. 4.—FLAT PLATE SUBJECTED TO UNIFORM EXTENSION ALONG ONE AXIS
angle in the \(x_1, x_2\) - plane. Eqs. 55 and 56 also apply to this case, except that all lower-case indices now range from 1 to 2. The problem of finite plane strain superimposed on uniform finite extension is also encompassed by the above equations if, instead of setting \(\gamma_{33}\) equal to zero, the strain normal to the plane of the plate is computed using Eq. 55b, \(\lambda\) now being a known constant.

**Straight Bar.**—Nonlinear stiffness relations for a one-dimensional bar element are obtained by degenerating the triangular element into a straight line coincident with the \(x_1\) - axis. In this case,

\[
\gamma_{11} = (c_{11} u_{11} + c_{12} u_{21}) + \frac{1}{2} (c_{11} u_{11} + c_{12} u_{21})^2 \quad \ldots \ldots \ldots \ldots \ldots (57)
\]

and all other strain components are zero. Thus, for the bar element, all lower-case indices in Eqs. 43, 44, 45, 49, 52, and 54 are equal to unity and the upper-case indices range from 1 to 2. The displacement coefficients are given by

\[
c_{jkN} = \frac{1}{L} \begin{bmatrix} -1 & 1 \end{bmatrix} \quad \ldots \ldots \ldots \ldots \ldots (58)
\]
in which \(L\) is the length of the bar.

**PLANE PROBLEMS IN FINITE ELASTICITY**

In order to demonstrate the application of the theory developed previously, consideration is now given to problems of plane stress and finite plane strain of incompressible materials.

**Finite Plane Strain.**—In the case of finite plane strain of an incompressible Mooney-Rivlin material,

\[
I_2 = \frac{1}{\lambda^2} = \lambda^4 + \lambda^2 I_1 \quad \ldots \ldots \ldots \ldots \ldots (59)
\]

and \(\lambda\) is a known constant throughout the body. Thus \(W\) becomes a function of only \(I_1\). The strain components are given by Eq. 53.

As a result of the dependency established by Eq. 59, the nonlinear stiffness relation for finite plane strain acquires the simplified form

\[
\beta_{jkN} = 4 \nu \epsilon_{1M} (\delta_{jk} + c_{MN} u_{Nk}) (C_1 + \lambda^2 C_2) \epsilon_{im} \epsilon_{nj} \gamma_{mn} \ldots \ldots \ldots (60)
\]
in which \(\epsilon_{im}\) and \(\epsilon_{nj}\) are the two-dimensional permutation symbols. In this case, the incompressibility condition is automatically satisfied, \(i, j,\) and \(k\) range only from 1 to 2, and \(M\) and \(N\) range from 1 to 3.

If Eq. 52 is modified so that it applies to the case of plane strain, a nonlinear stiffness relation involving the hydrostatic pressure is obtained of the form

\[
\beta_{jkN} = \nu \epsilon_{1M} (\delta_{jk} + c_{MN} u_{Nk}) \left[ 2 \delta_{ij} (C_1 + \lambda^2 C_2) + \left( \frac{3}{\lambda^2} C_2 \right) \right] \quad (61)
\]

This equation, along with the incompressibility condition

\[
\lambda^2 (1 + 2 \gamma_{ii} + 2 \epsilon_{im} \epsilon_{jn} \gamma_{ij} \gamma_{mn}) = 1 \quad \ldots \ldots \ldots \ldots \ldots (62)
\]
must be applied to each finite element.

**Plane Stress.**—Eq. 59 and 60 are also applicable to problems of plane stress...
of an incompressible material. In the case of plane stress, however, \( \lambda^2 \) is no longer a constant but is related to the strains by the formula

\[
\lambda^2 = [(1 + 2\gamma_{11})(1 + 2\gamma_{22}) - 4\gamma_{12}^2]^{-1} \quad (63)
\]

The hydrostatic pressure is determined directly from the condition

\[
\tau_{33} = 0 \quad (64)
\]
from which it is found that

\[
h = -2\lambda \left[ C_1 + 2C_2(1 + \gamma_{11} + \gamma_{22}) \right] \quad (65)
\]

In this case,

\[
w_{ij} = \lambda^2 (\delta_{ij} + 2\epsilon_{jm}\epsilon_{in}\gamma_{mn}) \quad (66)
\]
in which \( \epsilon_{jm} \) and \( \epsilon_{in} \) are the two-dimensional permutation symbols.

Finally, substituting Eqs. 41, 59, 63, and 65 into Eq. 49 and simplifying gives the nonlinear stiffness relation

\[
P_{jk} = 2v_0 c_{IM}(\delta_{jk} + c_{jN} u_{Nk}) \left\{ C_1(\delta_{ij} - \lambda^4 f_{ij}) + C_2[\lambda^2 \delta_{ij} + f_{ij}(1 - 2\lambda^4 - 2\lambda^4 \gamma_{rr})]\right\} \quad (67)
\]
in which \( f_{ij} = \delta_{ij} + 2\epsilon_{jm}\epsilon_{in}\gamma_{mn} \quad (68) \)

Again note that in these equations \( i, j, k, m, n, \) and \( r \) range from 1 to 2, and \( M \) and \( N \) range from 1 to 3.

The stiffness relation for plane stress in a neo-Hookean material is obtained as a special case of Eq. 67 by setting \( C_2 \) equal to zero and replacing \( C_1 \) by \( C_0 \).

Example.—Consider, as a simple example, the square incompressible flat plate shown in Fig. 4. For simplicity, the plate is divided into only eight identical triangular elements. It is assumed that the plate undergoes a uniform extension of \( 2\Delta \) in the \( x_1 \) - direction, as is indicated in Fig. 4, and that the plate is in a state of plane stress.

In this case, it is convenient to establish only one set of coordinates, \( x_1, x_3 \), that originate at the center of the plate. Due to symmetry, the deformation of only two plates, I and II, need be considered. The node points of each element are numbered from 1 to 3 and the nodes in the assembled structure are numbered from 1 to 9.

First considering plate, I, it is seen that

\[
u_{11} = u_{12} = u_{21} = u_{22} = u_{32} = 0 \quad (69a)
\]
and

\[
u_{31} = \Delta \quad (69b)
\]
and, from Eq. 54,

\[
c_{IN} = \frac{1}{a} \begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \quad (70)
\]

Substituting Eqs. 69 and 70 into Eq. 14 and introducing the results into Eq. 63 gives

\[
\lambda^2 = \frac{1}{1 + 2\frac{\Delta}{a} + \left(\frac{\Delta}{a}\right)^2} \quad (71)
\]
The node forces are now calculated by substituting Eqs. 69 and 70 into Eq. 67. It is found, for example, that

\[ p_{31} = -p_{21} = at_0 \left( 1 + \frac{\Delta}{a} \right) (1 - \lambda^4) (C_1 + C_2) \]  
\[ p_{32} = -p_{22} = -2at_0 \left[ \frac{\Delta}{a} + \frac{1}{2} \left( \frac{\Delta}{a} \right)^2 \right] [ \lambda^4 C_1 - (1 - \lambda^2 - \lambda^4) C_2 ] \]

and \[ p_{11} = p_{12} = 0 \]

(72a) \hspace{1cm} (72b) \hspace{1cm} (72c)

A similar procedure is now applied to plate II. For this plate,

\[ u_{11} = u_{12} = u_{22} = u_{32} = 0 \]
\[ u_{31} = u_{21} = \Delta \]

and \[ c_{1N} = \frac{1}{a} \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \]

(73a) \hspace{1cm} (73b) \hspace{1cm} (74)

Substituting these results into Eq. 63 gives an expression for \( \lambda^2 \) identical to that in Eq. 71 and node forces are calculated using Eq. 66, as indicated previously.

The connectivity of the system is established by relating node displacements and forces of the individual elements to node displacements and force of the assembled structure. For example, if \( U_{Ni} \) are the displacements of nodes of the assembled structure, relations of the form

\[ u_{11}(I) = u_{11}(II) = U_{11} \]
\[ u_{12}(I) = u_{12}(II) = U_{12} \]
\[ u_{31}(I) = u_{21}(II) = U_{31} \]

and so forth, establish dependencies among node displacements of individual elements and, in effect, assemble the elements into one system. The corresponding node forces \( p_{Ni} \) on the assembled structure, are obtained in a similar manner. General formulas for such transformations are given in the following section.

In order to demonstrate the variations in the node forces with the prescribed displacement \( \Delta \), the forces on element I were evaluated for a Mooney-Rivlin material for which \( C_1 = 24.0 \) psi and \( C_2 = 1.5 \) psi. These values correspond to those obtained experimentally by Rivlin and Saunders\(^\text{17}\) for certain vulcanized rubbers. A plot of the results of these calculations is given in Fig. 5.

**THE ASSEMBLED STRUCTURE**

Let \( p_{Mi\alpha} \) and \( u_{Mi\alpha} \) denote the node forces and displacements corresponding to a finite element \( \alpha \). Then a nonlinear stiffness relation can be found of the form

\[ p_{Mi\alpha} = K(u_{Mi\alpha}) \]

(76)

in which \( K(u_{Mi\alpha}) \) is the appropriate nonlinear function of the node displacements. For example, the function \( K \) for Mooney-Rivlin materials is defined in Eq. 52; for neo-Hookean materials \( K \) is defined by Eq. 54, etc.
When the forces and displacements are rotated so that they are parallel to the global coordinate system \( Z \), Eq. 76 becomes

\[
\bar{P}_{MK\alpha} = \bar{K}(\bar{u}_{MK\alpha}) \tag{77}
\]

in which

\[
\bar{P}_{MK\alpha} = \lambda_{kl\alpha} \bar{P}_{MI\alpha} \tag{78a}
\]

\[
u_{MJ\alpha} = \lambda_{lk\alpha} u_{MK\alpha} \tag{78b}
\]

and

\[
\bar{K}(\bar{u}_{MK\alpha}) = \lambda_{kl\alpha} K(\lambda_{lk\alpha} u_{MI\alpha}) \tag{78c}
\]

and \( \lambda_{lk\alpha} \) is the array of direction cosines for element \( \alpha \) defined in Eq. 1. In these equations an underscore indicates that no sum is to be taken on the repeated index.

Now let \( P_{NI} \) and \( U_{NI} \) denote the node forces and displacements corresponding to node \( N \) of the entire assembly of elements after they have been appropriately connected together. These forces and displacements are referred
to the global coordinate system. The node forces and displacements of individual elements are related to those of the assembled structure by the transformations

\[ P_{Nk} = \Omega_{NM} \alpha \tilde{p}_{Mk} \alpha \]  
\[ \tilde{u}_{Mk} = \Omega_{MN} \alpha U_{Nk} \]

in which \[ \Omega_{MN} \alpha \] is the array defined in Eq. 2.

Substituting Eqs. 79 into Eq. 77 gives the complete nonlinear stiffness relation for the entire assembled system,

\[ P_{Nk} = \Omega_{MN} \alpha \bar{K}(\Omega_{MN} \alpha U_{Nk}) \]  

The ranges of the indices in this equation are \( N = 1, 2, 3, \ldots, m; M = 1, 2, 3, 4; \alpha = 1, 2, \ldots, e; \) and \( k = 1, 2, 3 \) for three-dimensional elements. Here \( m \) is the number of nodes of the assembled structure and \( e \) is the total number of finite elements. For plane problems involving flat triangular elements, \( M = 1, 2, 3 \) and \( k = 1, 3 \).

After the appropriate boundary conditions are applied, Eq. 80 leads to a system of nonlinear algebraic equations in the node displacements \( U_{Nk} \). Node forces and displacements of individual elements are obtained by substituting the solutions of these equations into Eqs. 78 and 79. Final stresses and strains developed in each element can then be evaluated using Eq. 14 and Eqs. 33 or 51.

The systems of equations obtained in finite element formulations of finite elasticity problems are, in general, highly nonlinear in the unknown node displacements. A comprehensive review of numerical schemes for solving such systems of equations has been given by H. A. Sprang, and a comparison of various methods based on numerical experiments has been presented by S. H. Brooks.

Returning to Eq. 80, it is now assumed that the appropriate boundary conditions have been applied and that a system of independent simultaneous nonlinear equations in the node displacements has been obtained. The entire set of equations is written in the form

\[ F_{Nk} = 0 \]  

in which \( F_{Nk} = P_{Nk} - \Omega_{MN} \alpha \bar{K}(\Omega_{MN} \alpha U_{Nk}) \)

The array \( F_{Nk} \) is expanded in a Taylor series about an arbitrary set of initial displacements \( U_{Nk}^0 \) which, for example, may be taken as the solutions of the classical linearized problem. Taking only two terms of the series, it is found that

\[ F_{Nk} = F_{Nk}^0 + \frac{\partial F_{Nk}^0}{\partial U_{Mk}} (U_{Mk} - U_{Mk}^0) = 0 \]
in which $U_{n_k}$ is the corrected solution an

$$F^0_{n_k} = F_{n_k}(U^0_{n_k}) \quad \cdots \quad \cdots \quad \cdots \quad (84)$$

Eqs. 83 are now linear in $U_{n_k}$. These are solved for the corrected displacements $U_{n_k}$ and the process is repeated until Eq. 81 is satisfied to a desired degree of accuracy.

A recurrence formula is obtained by solving Eq. 83 for corrected set of displacements of the $j + 1^{th}$ cycle;

$$U_{n_k}^{j+1} = U_{n_k}^j - \left[ \frac{\partial F^j_{n_k}}{\partial U_{n_k}} \right]^{-1} F^j_{n_k} \quad \cdots \quad \cdots \quad \cdots \quad (85)$$

**CONCLUSIONS**

General nonlinear stiffness relations for three-dimensional finite element of an elastic continuum can be derived on the basis of a linear displacement assumption. The resulting equations are applicable to the analysis of gross deformations and finite strain of both compressible and incompressible three-dimensional bodies of arbitrary shape. By simply changing the ranges of indices of terms in the general equations, nonlinear stiffness relations for triangular plates, membranes, and straight bars are obtained. These equations show that the customary procedure of accounting for nonlinearities by successive corrections to linearized problems is inadequate in the case of finite strains and cannot include effects of higher-order elastic constants. Further, it is concluded that stiffness matrices do not transform congruently, as in the linear case. The final assembly of elements into one system can be accomplished through a series of simple transformations.

**APPENDIX.—NOTATION**

The following symbols are used in this paper:

- $A_o$ = area of triangular element;
- $a_{ij}$ = coefficients in a linear form;
- $b_{1N}$ = a rectangular array;
- $b_{1N}$ = cofactors of $b_{1N}$;
- $C, C_1, C_2$ = material constants;
- $c_{1N}$ = displacement coefficients;
- $E_{iijkl}, E_{ijklmn}$ = multidimensional arrays of elastic constants;
- $e$ = number of finite elements in system;
- $f_i$ = body force per unit volume;
- $f_{ij}$ = a rectangular array;
- $h$ = hydrostatic pressure;
- $I_1, I_2, I_3$ = strain invariants;
- $K(u_{Nj})$ = nonlinear stiffness function;
- $m$ = number of nodes in assembled system;
\( p_{Nk} \) = force at node N in direction k of element \( \alpha \); 
\( p_{Nk} \) = force at node N in direction k of assembled system; 
\( q \) = applied pressure; 
\( S_i \) = surface traction per unit of deformed surface area; 
\( t_o \) = undeformed plate thickness; 
\( U_{Nk} \) = displacement of node N of assembled system in direction k; 
\( u_{Nk} \) = displacement of node N of element \( \alpha \) in direction k; 
\( V, V_o \) = potential energy functionals; 
\( v, v_o \) = volumes of deformed and undeformed elements; 
\( W \) = strain energy function; 
\( \lambda_{ij} \) = a rectangular array; 
\( x_{Ni} \) = local coordinates of node N of element \( \alpha \); 
\( z_{Ni} \) = global coordinates of node N in assembled system; 
\( \gamma_{ij} \) = Lagrangian strain tensor; 
\( \delta_{ij} \) = the Kronecker delta; 
\( \epsilon_{ij}, \epsilon_{ijk} \) = permutation symbols; 
\( \lambda_{ij} \) = orthogonal transformation matrix for element \( \alpha \); 
\( \lambda \) = extension ratio; 
\( \tau_{ij} \) = stress tensor per unit area of deformed body; and 
\( \Omega_{MN} \) = a multidimensional transformation array.
ABSTRACT: The development of nonlinear stiffness relations for three-dimensional finite elements of an elastic continuum is considered. On the basis of linear displacement approximations, consistent finite element representations are formulated for geometrically nonlinear problems involving large displacements and large strains. General nonlinear stiffness relations are derived for compressible materials and for incompressible materials of the Mooney-Rivlin and neo-Hookean type. Stiffness relations for triangular plates, membranes, and straight bars are also derived. It is shown that the connection of elements into the assembled system can be accomplished by a series of group transformations. A simple example is provided to demonstrate parts of the theory.