

CSE386M/EM386M
FUNCTIONAL ANALYSIS IN THEORETICAL MECHANICS
Fall 2022, Final Exam, 9:00 AM - noon, Mon, Dec 12, POB 6.304

1. A linear algebra “sanity check”.

Consider \mathbb{R}^2 . Let A be the counterclockwise rotation by angle α , see Fig. 1.

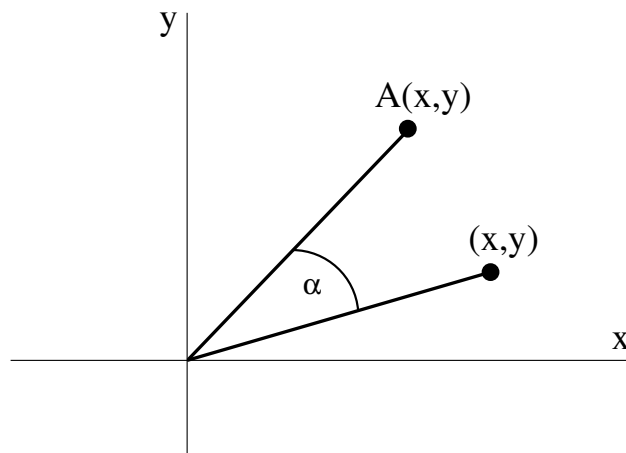


Figure 1: Rotation by angle α in a plane.

- (a) Is A a linear map ? Explain (2 points).
- (b) Write down the matrix representation for map A in the canonical basis (3 points).
- (c) Explain why all linear maps from \mathbb{R}^2 into itself - $L(\mathbb{R}^2, \mathbb{R}^2)$, form a vector space. What is the dimension of the space (2 points) ?
- (d) Do the rotations by an arbitrary angle α form a vector subspace of $L(\mathbb{R}^2, \mathbb{R}^2)$? Explain your answer. If yes, what is the dimension of this subspace (3 points) ?
- (e) Define adjoint for a linear operator in a general Hilbert setting (2 points).
- (f) Compute the adjoint of map A with respect to the canonical inner product in \mathbb{R}^2 . Is A self-adjoint ? (3 points).
- (g) Define an orthonormal matrix (2 points).
- (h) Is matrix representation of map A an orthonormal matrix ? Explain, why ? (3 points)

Answers:

- (a) Yes, it is. Rotation and multiplication by a number, commute. Similarly, vector addition and rotation commute as well.
- (b) Representing vector x in the canonical basis, $x = x_1e_1 + x_2e_2$, we use the linearity to obtain:

$$Ax = A(x_1e_1 + x_2e_2) = x_1Ae_1 + x_2Ae_2 = x_1 \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} + x_2 \begin{pmatrix} -\sin \alpha \\ \cos \alpha \end{pmatrix},$$

and so,

$$Ax = (\cos \alpha x_1 - \sin \alpha x_2, \sin \alpha x_1 + \cos \alpha x_2).$$

The matrix representation in the canonical basis is:

$$\begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}.$$

- (c) Functions defined on any set (in our case \mathbb{R}^2) with values in a vector space (in our case \mathbb{R}^2), equipped with pointwise addition and scalar multiplication, form a vector space. One has only to argue that the linear maps form a subset closed with respect to the vector space operations and, therefore, form a vector subspace of all functions defined on \mathbb{R}^2 . This follows from the fact that a linear combination of linear maps is a linear map itself. Dimension of $L(X, Y)$ is always equal to the product of $\dim X = n$ and $\dim Y = m$ (in our case = 4). This follows from the isomorphism between $L(X, Y)$ and $m \times n$ matrices.
- (d) Well, they do not. One possible way to see this is to notice that, for a product of a number γ and a rotation to be a rotation, the terms in the corresponding matrix representation would have been bounded in absolute value by one. But, e.g., $\gamma \cos \alpha$ can be as large as we wish.
- (e) The notion of the adjoint involves two Hilbert spaces X and Y with inner products $(\cdot, \cdot)_X$ and $(\cdot, \cdot)_Y$. Given a linear map $A : X \rightarrow Y$, we define the adjoint map $A^* : Y \rightarrow X$ by:

$$A^* = R_X^{-1}A^T R_Y$$

where $A : Y^* \rightarrow X^*$ is the transpose of A , and R_X, R_Y are Riesz maps for X and Y , resp. Equivalently,

$$(Ax, y)_Y = (x, A^*y)_X \quad x \in X, y \in Y.$$

- (f) The matrix for the adjoint A^* is the transpose of the matrix representation of A , i.e.,

$$\begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}.$$

In other words, the adjoint is given by:

$$A^*y = (\cos \alpha y_1 + \sin \alpha y_2, -\sin \alpha y_1 + \cos \alpha y_2).$$

(g) Matrix A is *orthonormal* if $A^{-1} = A^T$.

(h) Yes, it is. Just check that $A^*A = AA^* = I$.

2. An integration exercise.

(a) State the Lebesgue Dominated Convergence Theorem (5 points).

(b) Let $\beta > 0$ be a positive constant. Prove that the integral

$$\int_{\pi/2}^{3\pi/2} \frac{e^{\beta+n \cos \theta}}{\sqrt{(\beta + n \cos \theta)^2 + (n \sin \theta)^2}} n d\theta$$

converges to zero as $n \rightarrow \infty$ (15 points).

Answers:

(a) See the book.

(b) Rewrite the integral in the form,

$$\int_{\pi/2}^{3\pi/2} \frac{e^{\beta+n \cos \theta}}{\sqrt{(\beta/n + \cos \theta)^2 + (\sin \theta)^2}} d\theta$$

For $\theta \in (\pi/2, 3\pi/2)$, the denominator converges to one, whereas the numerator converges to zero (exponential with a negative exponent), as $n \rightarrow \infty$. Consequently the integrand converges a.e. to zero. In order to apply the Lebesgue Dominated Convergence Theorem, we need to show only that the integrand is dominated by an integrable function, for all n . The numerator is bounded by e^β . For the denominator, we have

$$\begin{aligned} \left(\frac{\beta}{n} + \cos \theta\right)^2 + \sin^2 \theta &= \frac{\beta^2}{n^2} + \frac{2\beta}{n} \cos \theta + 1 \\ &\geq \frac{\beta^2}{n^2} - \frac{2\beta}{n} + 1 \\ &= \left(\frac{\beta}{n} - 1\right)^2 \end{aligned}$$

Thus, for sufficiently large n , the denominator is bounded below by a positive number (independent of angle θ).

3. A topology problem.

- State the definition of a *continuous function* f from a topological space X into real numbers \mathbb{R} (2 points).
- Let X be a metric space. Define a metric in product space $X \times X$. Is it unique ? (3 points).
- Prove that metric

$$d : X \times X \rightarrow \mathbb{R}$$

is a continuous function in the very topology it generates (15 points).

Answers:

- Let \mathcal{B}_x denote bases of neighborhoods in space X . Function $f : X \rightarrow \mathbb{R}$ is continuous at x_0 iff

$$\forall \epsilon > 0 \quad \exists B \in \mathcal{B}_{x_0} \text{ such that } x \in B \Rightarrow |f(x) - f(x_0)| < \epsilon.$$

- Let ρ be a metric on X . A metric d in $X \times X$ is defined by:

$$d((x_1, x_2), (y_1, y_2)) := \|\rho(x_1, y_1), \rho(x_2, y_2)\|_p$$

where $\|\cdot\|_p$, $p \in [1, \infty]$ denotes any p -norm on \mathbb{R}^2 . As all p -norms on \mathbb{R}^2 are equivalent, so are the corresponding metrics on $X \times X$.

- As all metrics above are equivalent, we can choose $p = 1$. We have:

$$\begin{aligned} d(x, y) &= \rho(x_1, y_1) + \rho(x_2, y_2) && \text{(definition of metric } d) \\ &\leq \rho(x_1, x_1^0) + \rho(x_1^0, y_1^0) + \rho(y_1^0, y_1) \\ &\quad + \rho(x_2, x_2^0) + \rho(x_2^0, y_2^0) + \rho(y_2^0, y_2) && \text{(triangle inequality for } \rho \text{ applied multiple times)} \\ &= d(x^0, y^0) + d(x, x^0) + d(y^0, y) && \text{(definition of } d). \end{aligned}$$

Consequently,

$$d(x, y) - d(x^0, y^0) \leq d(x, x^0) + d(y^0, y).$$

Switching x, y with x^0, y^0 ,

$$d(x^0, y^0) - d(x, y) \leq d(x, x^0) + d(y^0, y)$$

and, so,

$$|d(x, y) - d(x^0, y^0)| \leq d(x, x^0) + d(y^0, y).$$

Given now $\epsilon > 0$, choose balls $B(x_0, \frac{\epsilon}{2})$ and $B(y_0, \frac{\epsilon}{2})$, to obtain,

$$|d(x, y) - d(x^0, y^0)| < \epsilon \quad \text{for all } x \in B(x_0, \frac{\epsilon}{2}) \text{ and } y \in B(y_0, \frac{\epsilon}{2}).$$

4. A metric space problem. Let X be a set and $\rho_1(x, y), \rho_2(x, y)$ two metrics on X . Define:

$$d(x, y) := \max\{\rho_1(x, y), \rho_2(x, y)\}. \quad (0.1)$$

- Is d also a metric on X ? Prove or disprove. (5 points).
- Define the notion of *equivalent metrics* (2 points).
- Define the notion of *topologically equivalent metrics* (2 points).
- If the answer to the first question is positive, you have three topologies in X corresponding to the three metrics. Discuss the relative strength of the corresponding topologies (which one is stronger or weaker than others ?)(11 points).

Answers:

- Yes, it is.

Positive definiteness: If $d(x, y) = 0$ then both $\rho_1(x, y) = \rho_2(x, y) = 0$ which implies that $x = y$.

Symmetry: We have:

$$\rho_i(x, y) = \rho_i(y, x), \quad i = 1, 2.$$

Apply $\max_{i=1,2}$ to both sides.

Triangle inequality: Start with:

$$\rho_i(x, y) \leq \rho_i(x, z) + \rho_i(z, y) \leq \max_{j=1,2} \rho_j(x, z) + \max_{j=1,2} \rho_j(z, y), \quad j = 1, 2,$$

and take maximum with respect to i on both sides.

- Let $B^d(x, \epsilon)$ and $B^{\rho_i}(x, \epsilon)$ denote balls corresponding to metrics d and ρ_i , resp. Inequality

$$\rho_i(x, y) \leq d(x, y), \quad i = 1, 2$$

implies that

$$B^d(x, \epsilon) \subset B^{\rho_i}(x, \epsilon), \quad i = 1, 2.$$

Consequently, if $\mathcal{B}^d, \mathcal{B}^{\rho_i}$ denote the bases of neighborhoods in topologies generated by d and ρ_i , resp., then

$$\mathcal{B}^d \succ \mathcal{B}^{\rho_i}$$

which demonstrates that metric topology corresponding to d is *stronger* than both topologies corresponding to metrics ρ_i . We cannot draw any general conclusion about the relative strength of metric topologies corresponding to ρ_i , $i = 1, 2$.

5. Contraction Maps. Consider the following Initial-Value Problem (IVP):

$$\begin{cases} \frac{dq}{dt} = t^2 \ln(q(t)), & t > 0 \\ q(0) = 1 \end{cases}$$

- State Banach Contractive Map Theorem (3 points).

Answer:

Let (X, d) be a complete metric space. Let $D \subset X$ (then (D, d) is itself a metric space, too...), and $A : D \rightarrow D$ is a contraction, i.e.

$$d(A(f), A(g)) \leq k d(f, g), \quad \forall f, g \in D, \quad k < 1$$

Then function A has a unique fixed point in set D .

- Use the theorem to prove local existence and uniqueness of solution to the IVP, i.e. that there exists an interval $(0, T)$ in which the equation is satisfied. Provide a *concrete* value of T (17 points).

Solution:

The problem is equivalent to the solution of the integral equation:

$$q(t) = 1 + \int_0^t s^2 \ln(q(s)) ds$$

Consider the Chebyshev space $C[0, T]$ (with unknown T at this point...) and define the map A using the right-hand side of the equation above:

$$(Aq)(t) = 1 + \int_0^t s^2 \ln(q(s)) ds$$

First of all, we need to define a set $D \subset C[0, T]$ such that map A sets the set D into itself. Assume that $q(t)$ will vary in the box:

$$D = \{q \in C[0, T] : e^{-1} \leq q(t) \leq e, \quad 0 \leq t \leq T\} \quad (0.2)$$

(notice that the box includes the initial value $q = 1$). Then $-1 \leq \ln q(t) \leq 1$, i.e. $|\ln q(t)| \leq 1$. Consequently,

$$\left| \int_0^t s^2 \ln q(s) ds \right| \leq \int_0^t s^2 ds = \frac{t^3}{3},$$

and so,

$$|(Aq)(t) - 1| \leq \frac{T^3}{3}.$$

This gives two bounds for T . From the right:

$$(Aq)(t) \leq 1 + \frac{T^3}{3} \leq e \quad \Rightarrow \quad T \leq (3(e-1))^{1/3},$$

and from the left:

$$e^{-1} \leq 1 - \frac{T^3}{3} \leq (Aq)(t) \quad \Rightarrow \quad T \leq (3(1-e^{-1}))^{1/3}.$$

Now, map A must be a contraction. With flux $F(s, q) = s^2 \ln q$,

$$\left| \frac{\partial F}{\partial q} \right| = s^2 \left| \frac{1}{q} \right| \leq es$$

so, with q coming from box (0.2), the flux satisfies the Lipschitz condition:

$$|F(s, q_1) - F(s, q_2)| \leq es^2 |q_1 - q_2|.$$

This leads to the estimate;

$$|(Aq_1)(t) - (Aq_2)(t)| \leq \int_0^t es^2 ds \|q_1 - q_2\|_{C[0,T]} \leq \frac{e}{3} T^3 \|q_1 - q_2\|_{C[0,T]}.$$

Consequently, a sufficient condition for a contraction is

$$T < \left(\frac{3}{e}\right)^{1/3}.$$

In conclusion, the IVP will have a unique solution for

$$T < \min\{(3(e-1))^{1/3}, (3(1-e^{-1}))^{1/3}, \left(\frac{3}{e}\right)^{1/3}\}.$$

By the way, by inspection, $q = 1$ is the solution to the problem, for any T , so the estimates are very conservative.