1. Let $T$ be a compact operator from a Hilbert space $U$ into a Hilbert space $V$.

   (a) Define the notion of compact operators.

   (b) Show that $T^*T$ and $TT^*$ are compact, self-adjoint, positive semi-definite operators from $U$ (resp.) into itself.

   (c) Prove that all eigenvalues of a self-adjoint operator are real.

   (d) Prove that $T^*T$ and $TT^*$ have identical non-negative eigenvalues and derive a relation between the corresponding eigenspaces.

(25 points)

(a) See the book

(b) Composition of a compact and continuous operator (in any order) is compact. We have,

$$ (T^*Tu, u)_U = (Tu, Tu)_V = (u, T^*Tu) \geq 0 $$

and the same argument holds for $TT^*$.

(c) Let $\lambda \neq 0$ be an eigenvalue of a self-adjoint operator $A$ from a Hilbert space $U$ into itself, and $u$ the corresponding eigenvector. We have,

$$ \lambda(u, u) = (\lambda u, u) = (Au, u) = (u, Au) = (u, \lambda u) = \bar{\lambda}(u, u). $$

Hence

$$ (\lambda - \bar{\lambda})(u, u) = 0 \quad \Rightarrow \quad \lambda - \bar{\lambda} = 0 \quad \Rightarrow \quad \lambda \in \mathbb{R}. $$

(d) Let $(\lambda, u)$ be an eigenpair for $T^*T$, $\lambda \neq 0$,

$$ T^*Tu = \lambda u. $$

Apply $T$ to both sides of the equation to get,

$$ TT^*Tu = \lambda Tu. $$
which proves that \((\lambda, Tu)\) is an eigenpair for \(TT^*\). Conversely, if \((\lambda, v)\) is an eigenpair for \(TT^*\) then \((\lambda, T^*v)\) is an eigenpair for \(T^*T\). Let

\[
U_\lambda := \mathcal{N}(\lambda - T^*T), \quad V_\lambda := \mathcal{N}(\lambda - TT^*)
\]

be the eigenspaces corresponding to \(\lambda\). The first property above proves that \(T\) sets \(U_\lambda\) into \(V_\lambda\),

\[
T(U_\lambda) \subset V_\lambda.
\]

Let \(v \in V_\lambda\), i.e.

\[
TT^*v = \lambda v \quad \Rightarrow \quad v = T(\lambda^{-1}T^*v),
\]

i.e. there exists an \(u \in U_\lambda\), namely, \(u = \lambda^{-1}T^*v\) such that \(v = Tu\). In other words,

\[
T(U_\lambda) = V_\lambda.
\]

By the same argument,

\[
T^*(V_\lambda) = U_\lambda.
\]

Finally,

\[
T^*Tu = 0 \quad \Rightarrow \quad T^*v = 0 \quad \text{for} \quad v = Tu \quad \Rightarrow \quad TT^*v = 0.
\]

i.e. \(T(\mathcal{N}(T^*T)) \subset \mathcal{N}(TT^*)\). Similarly, \(T^*(\mathcal{N}(TT^*)) \subset \mathcal{N}(T^*T)\).

Note that \(\mathcal{N}(T^*T) = \mathcal{N}(T)\) and \(\mathcal{N}(TT^*) = \mathcal{N}(T^*)\).
2. (a) Define discrete, residual and continuous spectrum for an operator $A : U \ni D(A) \rightarrow U$ where $U$ is a Hilbert space.

(b) Determine spectrum of operator $A$ where

$$
U = L^2(\mathbb{R}) \quad D(A) = H^1(\mathbb{R}) \quad Au = \frac{du}{dx} + u
$$

*Hint:* Use Fourier transform.

(25 points)

This is a slight modification of the example discussed in the book for $Au = u'$. Direct computations using Fourier transform reveal that there is neither point nor residual spectrum. The continuous spectrum consists of the line $\lambda = 1 + i\xi, \xi \in \mathbb{R}$. 
3. Consider a first order operator $A$ in $L^2(0,1)$,

$$D(A) = \{ u \in H^1(0,1) : u(0) = u(1) = 0 \} \quad Au = u' - 2u$$

where the derivative is understood in the sense of distributions.

(a) Define a closed operator and prove that operator $A$ is closed. You may use the fact that pointwise value $u(x), x \in [0,1]$ of $u \in H^1(0,1)$ represents a continuous functional.

(b) Determine the adjoint operator $A^*$ and its null space.

(c) Prove that operator $A$ is bounded below in $L^2(0,1)$.

(d) Discuss the well posedness of the problem:

$$u \in D(A), \quad Au = f$$

with an appropriate right-hand side $f$.

(25 points)

See book for definitions.

**Answers:**

(a) Let $u_n \in D(A)$ and $(u_n, Au_n) \to (u, v)$, i.e. $u_n \to u$, $Au_n \to v$, all convergence understood in the $L^2$-sense. Consequently, $u'_n \to v + 2u$. By definition,

$$\int u'_n \phi = -\int u_n \phi' \quad \forall \phi \in C_0^\infty(0,1)$$

Passing to the limit on both sides, we get

$$\int (v + 2u) \phi = -\int u \phi' \quad \forall \phi \in C_0^\infty(0,1)$$

which proves that $v + 2u = u'$ in the sense of distributions.

Thus $u' = v + 2u \in L^2(0,1)$ and, therefore, $u_n \to u$ also in $H^1(0,1)$ which in turn implies that $u(0) = u(1) = 0$. Consequently, $u \in D(A)$, and $v = u' - 2u = Au$ as required.

(b) Integration by part argument gives:

$$D(A^*) = H^1(0,1) \quad A^*v = -v' - 2v$$

$$\mathcal{N}(A^*) = \mathbb{R}e^{-2x} := \{ ce^{-2x} : c \in \mathbb{R} \}$$
(c) We have,

\[ \|Au\|^2 = \int_0^1 (u' - 2u)^2 = \int_0^1 (u')^2 - 4 \int_0^1 uu' + 4 \int_0^1 u^2 = \int_0^1 (u')^2 \geq (P + 4) \|u\|^2 \]

where \( P \) is the Poincaré constant. Note that

\[ \int_0^1 uu' = \int_0^1 (u^2)' = u^2 \bigg|_0^1 = 0 \quad \text{for } u \in D(A). \]

(d) For every right hand side \( f \in L^2(0, 1) \) that satisfies the compatibility condition:

\[ \int_0^1 f(x)e^{-2x} \, dx = 0, \]

the problem has a unique solution that depends continuously upon the data

\[ \|u\| \leq \alpha^{-1}\|f\| \]

where \( \alpha \) is the boundedness below constant.
4. Consider the “ultraweak” variational formulation of the previous problem,

\[
\begin{cases}
u \in U := L^2(0,1) \\
\int_0^1 u A^* v \, dx = \int_0^1 fv \, dx \quad \forall v \in V := H^1(0,1)
\end{cases}
\]

where \( A^* \) denotes the formal adjoint of \( A \), \( A^* v = -v' - 2v \).

(a) Define operator \( B : U \to V' \) and its conjugate corresponding to the bilinear form \( b(u,v) \).

(b) Use Babuška-Nečas Theorem and results from the previous problem to investigate the well-posedness of the problem.

**Hint:** Can you relate the inf-sup constant for this problem with the boundedness below (Friedrichs) constant of operator \( A \) from the previous problem? (25 points)

There are two operators associated with the bilinear form:

\[
B : L^2(0,1) \to (H^1(0,1))'
\]

\[
B' : H^1(0,1) \to L^2(0,1) \sim (L^2(0,1))'
\]

Due to reflexivity of Hilbert space, operator \( B' \) can be identified with the transpose of \( B \). The whole point of this exercise is to realize that transpose \( B' \) coincides with the adjoint \( A^* \) discussed in the previous problem. Direct application of Cauchy-Schwartz inequality shows that both forms: \( b(u,v) \) and \( l(v) \) are continuous. Finally, the Closed Range Theorem for continuous operators implies that

\[
\gamma = \inf_{u \in L^2} \sup_{v \in H^1} \left| \int_0^1 u(-v' - 2v) \right| = \inf_{[v] \in H^1(N'(B'))} \sup_{u \in L^2} \left| \int_0^1 u(-v' - 2v) \right|.
\]

Since \( B' = A^* \), the right-hand side coincides with the boundedness below (Friedrichs) constant for the quotient operator corresponding to \( A^* \). But, by the Closed Range Theorem for closed operators, this constant is equal exactly to constant \( \alpha \) discussed in the previous problem. Consequently, there is no need to prove anything new. Application of Babuška-Nečas Theorem implies that, for any right-hand side \( f \) satisfying the compatibility condition, we have a unique solution that depends continuously upon the data. The subtle difference between the strong and (ultraweak) variational formulations is the regularity. The ultraweak formulation may accommodate “distributional loads”:

\[
l \in (H^1(0,1))'
\]

with the continuous dependence upon data modified accordingly:

\[
\| u \| \leq \gamma^{-1} \| l \|_{(H^1(0,1))'}.
\]