ASE 211 Homework 5

Due: 12:00 noon, Friday, October 6.

1. Consider the linear system given in problem P4.4 in the text. Apply three iterations of the Jacobi method, and three iterations of Gauss-Seidel to this problem (by hand), starting with an initial guess of zero.

$$\begin{aligned} 8x_1 + x_2 - x_3 = 8 \quad &\Rightarrow \quad x_1 = (8 - x_2 + x_3)/8 \\ -x_1 + 7x_2 - 2x_3 = 4 \quad &\Rightarrow \quad x_2 = (4 + x_1 + 2x_3)/7 \\ 2x_1 + x_2 + 9x_3 = 12 \quad &\Rightarrow \quad x_3 = (12 - 2x_1 - x_2)/9 \end{aligned}$$

Starting with $x_1 = x_2 = x_3 = 0$, Jacobi iteration gives

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 4/7 \\ x_3 &= 4/3 \end{aligned}$$

$$\begin{aligned} x_1 &= 1.095 \\ x_2 &= 1.095 \\ x_3 &= 1.048 \end{aligned}$$

$$\begin{aligned} x_1 &= .994 \\ x_2 &= 1.027 \\ x_3 &= .9683 \end{aligned}$$

Gauss-Seidel iteration gives

$$\begin{aligned} x_1 &= 1 \\ x_2 &= .7142 \\ x_3 &= 1.032 \end{aligned}$$

$$\begin{aligned} x_1 &= 1.040 \\ x_2 &= 1.015 \\ x_3 &= .9894 \end{aligned}$$

$$x_1 = .9968$$
$$x_2 = .9965$$
$$x_3 = 1.0011$$

2. Write an m-file which implements Gauss-Seidel iteration. The file should take as input the matrix A, right-hand side $\mathbf{b}$, an initial guess for the solution $\mathbf{x}$, the stopping tolerance ϵ and a limit on the maximum number of iterations *maxiter*. It should return the solution $\mathbf{x}$. Test your code on the following application.

In various types of applications (displacement problems, flow of heat through structures, computational fluid dynamics), we encounter matrices with very special types of structure. For example, the steady-state temperature u at 6 discrete points on a thin rod of length 1 subject to a heat source f and fixed temperature at the endpoints is approximated by the linear system

$$A\mathbf{u} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \frac{1}{36} \begin{bmatrix} f_1 + u_0 \\ f_2 \\ f_3 \\ f_4 + u_5 \end{bmatrix},$$

where u_0 and u_5 are the fixed temperatures at the two ends of the rod.

In general, if we model the temperature u at n discrete points along the rod and let u_i, $i = 0, \ldots, n-1$ denote the temperature at these points, then we solve a system of the form

$$A\mathbf{u} = \mathbf{f}$$

where $A_{i,i} = 2$, $i = 1, \ldots, n-2$, $A_{i,i-1} = A_{i-1,i} = -1$, $i = 2, \ldots, n-2$, all other entries in A are zero, and the vector $\mathbf{f}$ is given by

$$\frac{1}{n^2} \begin{bmatrix} f_1 + u_0 \\ f_2 \\ f_3 \\ \cdot \\ \cdot \\ \cdot \\ f_{n-3} \\ f_{n-2} + u_{n-1} \end{bmatrix}.$$

Compute the solution taking $f_i = 1$, $i = 1, \ldots, n-2$, $u_0 = 0$, $u_{n-1} = 1$, and take first $n = 10$, then $n = 50$. You should probably write a special m-file to set up the matrix A and right hand side $\mathbf{f}$.

Plot the solution u for both values of n.

```
function x=gauss_seidel(A,b,x,epsilon,itermax)
%
%  This function performs Gauss-Seidel iteration
%
%
n=size(A,1);
r=b-A*x;
%
%  compute norm of r
%  use built-in matlab function norm
%
rnorm=norm(r);
k=0;
while (rnorm > epsilon & k < itermax)
   for i=1:n
     x(i)=b(i);
     for j=1:i-1
       x(i)=x(i)-A(i,j)*x(j);
     end
     for j=i+1:n
       x(i)=x(i)-A(i,j)*x(j);
     end
     x(i)=x(i)/A(i,i);
   end
   r=b-A*x;
   k=k+1;
   rnorm=norm(r);
end
k
rnorm
```

```
function [A,f,x]=createA_f(n,u0,unm1)
% creates A and f for problem 2 in homework 5 and initializes x
%
```

```
for i=1:n-2
  A(i,i)=2;
  f(i)=1;
  x(i)=0;
end
for i=1:n-3
  A(i+1,i)=-1;
  A(i,i+1)=-1;
end
f(1)=f(1)+u0;
f(n-2)=f(n-2)+unm1;
for i=1:n-2
  f(i)=f(i)/n^2;
end
% change f and x to column vectors
f=f';
x=x';

>> n=10;
>> u0=0;
>> unm1=1;
>> [A,f,x]=createA_f(n,u0,unm1);
>> epsilon=.00001;
>> itermax=500;
>> x=gauss_seidel(A,f,x,epsilon,itermax);

k =

    65


rnorm =

   9.9823e-06

>> plot(x)
>> n=50;
>> clear A f x
>> [A,f,x]=createA_f(n,u0,unm1);
```

```
>> x=gauss_seidel(A,f,x,epsilon,itermax);

k =

   500


rnorm =

   3.2442e-04

>> plot(x)
>> diary
```

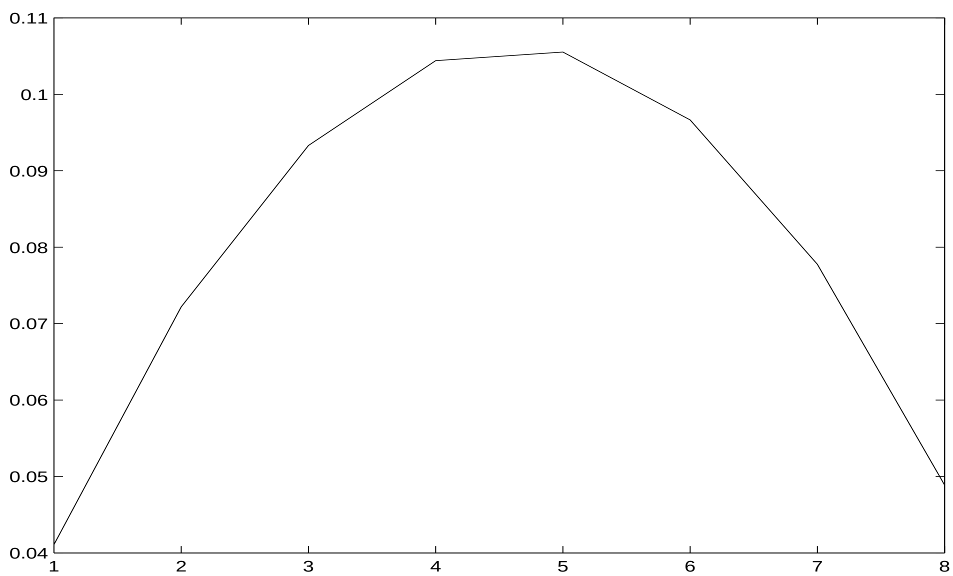

Figure 1: Plot of the solution for $n = 10$

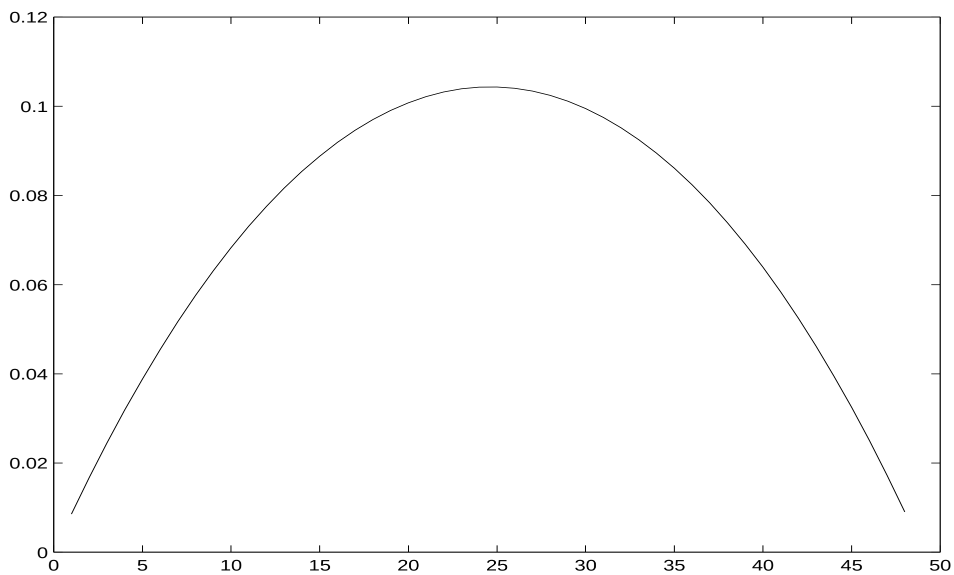

Figure 2: Plot of the solution for $n = 50$