The Shallow Water Equations

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The Shallow Water Equations (SWE)

What are they?

- The SWE are a system of hyperbolic/parabolic PDEs governing fluid flow in the oceans (sometimes), coastal regions (usually), estuaries (almost always), rivers and channels (almost always).
- The general characteristic of shallow water flows is that the vertical dimension is much smaller than the typical horizontal scale. In this case we can average over the depth to get rid of the vertical dimension.
- The SWE can be used to predict tides, storm surge levels and coastline changes from hurricanes, ocean currents, and to study dredging feasibility.
- SWE also arise in atmospheric flows and debris flows.

The SWE (Cont.)

How do they arise?

- The SWE are derived from the *Navier-Stokes equations*, which describe the motion of fluids.
- The Navier-Stokes equations are themselves derived from the equations for conservation of mass and linear momentum.

SWE Derivation Procedure

There are 4 basic steps:

1. Derive the Navier-Stokes equations from the conservation laws.
2. Ensemble average the Navier-Stokes equations to account for the turbulent nature of ocean flow. See [1, 3, 4] for details.
3. Specify boundary conditions for the Navier-Stokes equations for a water column.
4. Use the BCs to integrate the Navier-Stokes equations over depth.

In our derivation, we follow the presentation given in [1] closely, but we also use ideas in [2].

Conservation of Mass

Consider mass balance over a control volume Ω. Then

$$\underbrace{\frac{d}{dt}\int_{\Omega}\rho\, dV}_{\substack{\text{Time rate of change}\\ \text{of total mass in } \Omega}} = \underbrace{-\int_{\partial\Omega}(\rho\mathbf{v})\cdot\mathbf{n}\, dA}_{\substack{\text{Net mass flux across}\\ \text{boundary of } \Omega}},$$

where

- ρ is the fluid density (kg/m^3),
- $\mathbf{v} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ is the fluid velocity (m/s), and
- $\mathbf{n}$ is the outward unit normal vector on $\partial\Omega$.

Conservation of Mass: Differential Form

Applying Gauss's Theorem gives

$$\frac{d}{dt}\int_{\Omega} \rho \, dV = -\int_{\Omega} \nabla \cdot (\rho \mathbf{v}) \, dV.$$

Assuming that ρ is smooth, we can apply the Leibniz integral rule:

$$\int_{\Omega} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right] dV = 0.$$

Since Ω is arbitrary,

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0}$$

Conservation of Linear Momentum

Next, consider linear momentum balance over a control volume Ω. Then

$$\underbrace{\frac{d}{dt}\int_{\Omega}\rho\mathbf{v}\,dV}_{\substack{\text{Time rate of}\\ \text{change of total}\\ \text{momentum in }\Omega}} = \underbrace{-\int_{\partial\Omega}(\rho\mathbf{v})\mathbf{v}\cdot\mathbf{n}\,dA}_{\substack{\text{Net momentum flux}\\ \text{across boundary of }\Omega}} + \underbrace{\int_{\Omega}\rho\mathbf{b}\,dV}_{\substack{\text{Body forces}\\ \text{acting on }\Omega}} + \underbrace{\int_{\partial\Omega}\mathbf{Tn}\,dA}_{\substack{\text{External contact}\\ \text{forces acting}\\ \text{on }\partial\Omega}},$$

where

- $\mathbf{b}$ is the body force density per unit mass acting on the fluid (N/kg), and
- $\mathbf{T}$ is the Cauchy stress tensor (N/m^2). See [5, 6] for more details and an existence proof.

Conservation of Linear Momentum: Differential Form

Applying Gauss's Theorem again (and rearranging) gives

$$\frac{d}{dt}\int_\Omega \rho\mathbf{v}\,dV + \int_\Omega \nabla\cdot(\rho\mathbf{v}\mathbf{v})\,dV - \int_\Omega \rho\mathbf{b}\,dV - \int_\Omega \nabla\cdot\mathbf{T}\,dV = 0.$$

Assuming $\rho\mathbf{v}$ is smooth, we apply the Leibniz integral rule again:

$$\int_\Omega \left[\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla\cdot(\rho\mathbf{v}\mathbf{v}) - \rho\mathbf{b} - \nabla\cdot\mathbf{T}\right]\,dV = 0.$$

Since Ω is arbitrary,

$$\boxed{\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla\cdot(\rho\mathbf{v}\mathbf{v}) - \rho\mathbf{b} - \nabla\cdot\mathbf{T} = 0}$$

Conservation Laws: Differential Form

Combining the differential forms of the equations for conservation of mass and linear momentum, we have:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$
$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) = \rho \mathbf{b} + \nabla \cdot \mathbf{T}$$

To obtain the Navier-Stokes equations from these, we need to make some assumptions about our fluid (sea water), about the density ρ, and about the body forces $\mathbf{b}$ and stress tensor $\mathbf{T}$.

Sea water: Properties and Assumptions

- It is incompressible. This means that ρ does not depend on p. It does not necessarily mean that ρ is constant! In ocean modeling, ρ depends on the salinity and temperature of the sea water.
- Salinity and temperature are assumed to be constant throughout our domain, so we can just take ρ as a constant. So we can simplify the equations:

$$\nabla \cdot \mathbf{v} = 0,$$

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla \cdot (\rho\mathbf{v}\mathbf{v}) = \rho\mathbf{b} + \nabla \cdot \mathbf{T}.$$

- Sea water is a Newtonian fluid. This affects the form of $\mathbf{T}$.

Body Forces and Stresses in the Momentum Equation

We know that gravity is one body force, so

$$\rho \mathbf{b} = \rho \mathbf{g} + \rho \mathbf{b}_{\text{others}},$$

where

- $\mathbf{g}$ is the acceleration due to gravity (m/s^2), and
- $\mathbf{b}_{\text{others}}$ are other body forces (e.g. the Coriolis force in rotating reference frames) (N/kg). We will neglect for now.

For a Newtonian fluid,

$$\mathbf{T} = -p\mathbf{I} + \bar{\mathbf{T}}$$

where p is the pressure (Pa) and $\bar{\mathbf{T}}$ is a matrix of stress terms.

The Navier-Stokes Equations

So our final form of the Navier-Stokes equations in 3D are:

$$\nabla \cdot \mathbf{v} = 0,$$

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla \cdot (\rho\mathbf{v}\mathbf{v}) = -\nabla p + \rho\mathbf{g} + \nabla \cdot \bar{\mathbf{T}},$$

The Navier-Stokes Equations

Written out:

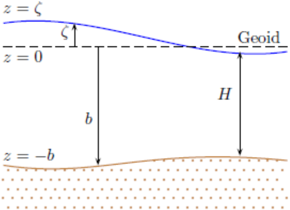
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (1)$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho uw)}{\partial z} = \frac{\partial(\tau_{xx} - p)}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \quad (2)$$

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho v^2)}{\partial y} + \frac{\partial(\rho vw)}{\partial z} = \frac{\partial \tau_x y}{\partial x} + \frac{\partial(\tau_{yy} - p)}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \quad (3)$$

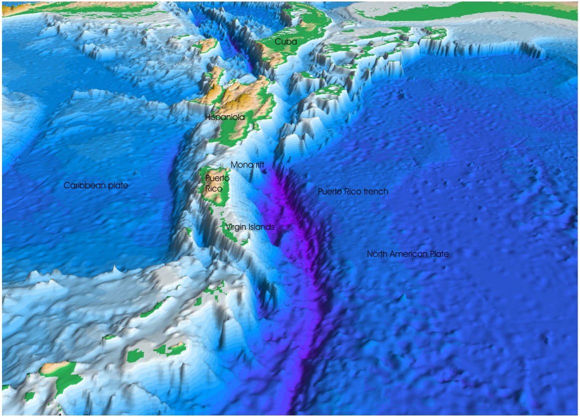
$$\frac{\partial(\rho w)}{\partial t} + \frac{\partial(\rho uw)}{\partial x} + \frac{\partial(\rho vw)}{\partial y} + \frac{\partial(\rho w^2)}{\partial z} = -\rho g + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial(\tau_{zz} - p)}{\partial z} \quad (4)$$

A Typical Water Column



- $\zeta = \zeta(t, x, y)$ is the elevation (m) of the *free surface* relative to the *geoid*.
- $b = b(x, y)$ is the bathymetry (m), measured positive downward from the geoid.
- $H = H(t, x, y)$ is the total depth (m) of the water column. Note that $H = \zeta + b$.

A Typical Bathymetric Profile



Bathymetry of the Atlantic Trench. Image courtesy USGS.

Boundary Conditions

We have the following BCs:

1. At the bottom $(z = -b)$
    - No slip: $u = v = 0$
    - No normal flow:
    $$u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y} + w = 0 \tag{5}$$
    - Bottom shear stress:
    $$\tau_{bx} = \tau_{xx}\frac{\partial b}{\partial x} + \tau_{xy}\frac{\partial b}{\partial y} + \tau_{xz} \tag{6}$$
    where τ_{bx} is specified bottom friction (similarly for y direction).
2. At the free surface $(z = \zeta)$
    - No relative normal flow:
    $$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} - w = 0 \tag{7}$$
    - $p = 0$ (done in [2])
    - Surface shear stress:
    $$\tau_{sx} = -\tau_{xx}\frac{\partial \zeta}{\partial x} - \tau_{xy}\frac{\partial \zeta}{\partial y} + \tau_{xz} \tag{8}$$

z-momentum Equation

Before we integrate over depth, we can examine the momentum equation for vertical velocity. By a scaling argument, all of the terms except the pressure derivative and the gravity term are small.
Then the z-momentum equation collapses to

$$\frac{\partial p}{\partial z} = \rho g$$

implying that

$$p = \rho g(\zeta - z).$$

This is the *hydrostatic pressure distribution*. Then

$$\frac{\partial p}{\partial x} = \rho g \frac{\partial \zeta}{\partial x} \tag{9}$$

with similar form for $\frac{\partial p}{\partial y}$.

The 2D SWE: Continuity Equation

We now integrate the *continuity equation* $\nabla \cdot \mathbf{v} = 0$ from $z = -b$ to $z = \zeta$. Since both b and ζ depend on t, x, and y, we apply the Leibniz integral rule:

$$\begin{aligned}
0 &= \int_{-b}^{\zeta} \nabla \cdot \mathbf{v}\, dz \\
&= \int_{-b}^{\zeta} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dz + w\big|_{z=\zeta} - w\big|_{z=-b} \\
&= \frac{\partial}{\partial x} \int_{-b}^{\zeta} u\, dz + \frac{\partial}{\partial y} \int_{-b}^{\zeta} v\, dz - \left(u\big|_{z=\zeta} \frac{\partial \zeta}{\partial x} + u\big|_{z=-b} \frac{\partial b}{\partial x} \right) \\
&\quad - \left(v\big|_{z=\zeta} \frac{\partial \zeta}{\partial y} + v\big|_{z=-b} \frac{\partial b}{\partial y} \right) + w\big|_{z=\zeta} - w\big|_{z=-b}
\end{aligned}$$

The Continuity Equation (Cont.)

Defining depth-averaged velocities as

$$\bar{u} = \frac{1}{H}\int_{-b}^{\zeta} u\,dz, \qquad \bar{v} = \frac{1}{H}\int_{-b}^{\zeta} v\,dz,$$

we can use our BCs to get rid of the boundary terms. So the depth-averaged continuity equation is

$$\boxed{\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\bar{u}) + \frac{\partial}{\partial y}(H\bar{v}) = 0} \tag{10}$$

LHS of the x- and y-Momentum Equations

If we integrate the left-hand side of the x-momentum equation over depth, we get:

$$\int_{-b}^{\zeta}[\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}u^2 + \frac{\partial}{\partial y}(uv) + \frac{\partial}{\partial z}(uw)]\,dz$$
$$= \frac{\partial}{\partial t}(H\bar{u}) + \frac{\partial}{\partial x}(H\bar{u}^2) + \frac{\partial}{\partial y}(H\bar{u}\bar{v}) + \begin{Bmatrix}\text{Diff. adv.}\\ \text{terms}\end{Bmatrix} \quad (11)$$

The differential advection terms account for the fact that the average of the product of two functions is not the product of the averages.
We get a similar result for the left-hand side of the y-momentum equation.

RHS of x- and y-Momentum Equations

Integrating over depth gives us

$$\begin{cases} -\rho g H \frac{\partial \zeta}{\partial x} + \tau_{sx} - \tau_{bx} + \frac{\partial}{\partial x} \int_{-b}^{\zeta} \tau_{xx} + \frac{\partial}{\partial y} \int_{-b}^{\zeta} \tau_{xy} \\ -\rho g H \frac{\partial \zeta}{\partial y} + \tau_{sy} - \tau_{by} + \frac{\partial}{\partial x} \int_{-b}^{\zeta} \tau_{xy} + \frac{\partial}{\partial y} \int_{-b}^{\zeta} \tau_{yy} \end{cases} \tag{12}$$

At long last...

Combining the depth-integrated continuity equation with the LHS and RHS of the depth-integrated x- and y-momentum equations, the 2D (nonlinear) SWE in conservative form are:

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\bar{u}) + \frac{\partial}{\partial y}(H\bar{v}) = 0$$

$$\frac{\partial}{\partial t}(H\bar{u}) + \frac{\partial}{\partial x}\left(H\bar{u}^2\right) + \frac{\partial}{\partial y}(H\bar{u}\bar{v}) = -gH\frac{\partial \zeta}{\partial x} + \frac{1}{\rho}[\tau_{sx} - \tau_{bx} + F_x]$$

$$\frac{\partial}{\partial t}(H\bar{v}) + \frac{\partial}{\partial x}(H\bar{u}\bar{v}) + \frac{\partial}{\partial y}\left(H\bar{v}^2\right) = -gH\frac{\partial \zeta}{\partial y} + \frac{1}{\rho}[\tau_{sy} - \tau_{by} + F_y]$$

The surface stress, bottom friction, and F_x and F_y must still determined on a case-by-case basis.

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