A LINEAR DEGENERATE ELLIPTIC EQUATION ARISING FROM TWO-PHASE MIXTURES*

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Abstract. We consider the linear degenerate elliptic system of two first order equations $\mathbf{u} = -a(\phi)\nabla p$ and $\nabla \cdot (b(\phi)\mathbf{u}) + \phi p = \phi^{1/2} f$, where a and b satisfy $a(0) = b(0) = 0$ and are otherwise positive, and the *porosity* $\phi \geq 0$ may be zero on a set of positive measure. This model equation has a similar degeneracy to that arising in the equations describing the mechanical system modeling the dynamics of partially melted materials, e.g., in the Earth's mantle and in polar ice sheets and glaciers. In the context of mixture theory, ϕ represents the phase variable separating the solid one-phase ($\phi = 0$) and fluid-solid two phase ($\phi > 0$) regions. The equations should remain well-posed as ϕ vanishes so that the free boundary between the one and two phase regions need not be found explicitly. Two main problems arise. First, as ϕ vanishes, one equation is lost. Second, after we extract stability or energy bounds for the solution, we see that the *pressure* p is not controlled outside the support of ϕ. After an appropriate scaling of the pressure and velocity, we obtain a mixed system for which we can show existence and uniqueness of a solution over the entire domain, regardless of where ϕ vanishes. The key is to define the appropriate Hilbert space containing the velocity, which must have a well defined scaled divergence and normal trace. We then develop for the scaled problem a formal mixed finite element method based on lowest order Raviart-Thomas elements which is both stable and has an optimal convergence rate for sufficiently smooth solutions. Furthermore, we present a practical implementation that reduces to a stable, locally conservative, cell-centered finite difference method. We show some numerical results that verify that optimal rates of convergence and even superconvergence is attained for sufficiently regular solutions.

Key words. Degenerate Elliptic, Mixture Theory, Energy Bounds, Ice Sheets, Mantle Dynamics, Mixed Method

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1. Introduction.

Let $\Omega \subset \mathbb{R}^d$ be a domain, let $\phi : \Omega \to [0, \phi^*]$, $0 < \phi^* < \infty$, be a given differentiable function that we will call *porosity*, and let a and b lie in $C^1([0, \phi^*])$ with $a(0) = b(0) = 0$ and both positive on $(0, \phi^*]$. For the *velocity* $\mathbf{u}$ and the *pressure* p, we consider the linear degenerate elliptic boundary value problem

$$\mathbf{u} = -a(\phi)(\nabla p - \mathbf{g}) \quad \text{in } \Omega, \tag{1.1}$$

$$\nabla \cdot \big(b(\phi)\mathbf{u}\big) + \phi p = \phi^{1/2} f \quad \text{in } \Omega, \tag{1.2}$$

$$b(\phi)\mathbf{u} \cdot \nu = \phi^{1/2} g_{\mathrm{N}} \quad \text{on } \partial\Omega, \tag{1.3}$$

where $\mathbf{g}$ and f drive the system and Neumann boundary conditions have been applied for some g_{N} (we will also treat Robin boundary conditions (5.1)). The choice of scaling in (1.2)–(1.3) in terms of ϕ will become clear as we develop the ideas. The critical factor here is that ϕ may vanish on a set of positive measure. This leads to a loss of control on p (see (2.4) below), and a number of issues arise for numerical approximation. In fact p may be unbounded off the support of ϕ, which is difficult to approximate numerically. Moreover, if ϕ vanishes, say everywhere for simplicity, it

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appears that the first equation implies that $\mathbf{u} = 0$, but the second equation trivializes to $0 = 0$, leading to more numerical difficulties.

Degenerate elliptic equations have been approximated in many works, e.g., [12, 19, 4, 18, 8], using weighted Sobolev spaces and least squares techniques. But in these works, $\Omega = \{x : \phi(x) > 0\}$, so the degeneracies are isolated to $\partial\Omega$. It is important in some applications that ϕ may vanish on a set of positive measure.

The system (1.1)–(1.3) arises, for example, as a simplified mathematical model of mantle dynamics. Models of flow in the Earth's mantle [1, 16, 15, 14, 23] are based on a mixture of fluid melt and matrix solid. Both fluid and matrix phases are assumed to exist at each point of the domain. The porosity $\phi \geq 0$ represents the relative volume of fluid melt to the bulk volume, and this quantity is very small (a few percent) within the mantle. Fluid melt is believed to form between rock crystal boundaries [26], forming a porous medium, and so the interstitial fluid velocity $\mathbf{v}_\mathrm{f}$ is governed by a Darcy law in terms of the fluid pressure p_f, such as

$$\phi(\mathbf{v}_\mathrm{f} - \mathbf{v}_\mathrm{m}) = -\frac{K(\phi)}{\mu_\mathrm{f}}(\nabla p_\mathrm{f} - \rho\mathbf{g}), \tag{1.4}$$

for some (relative) permeability $K(\phi)$, viscosity μ_f, and density ρ. The matrix solid is deformable, and it is modeled as a highly viscous fluid governed by a Stokes equation. Conservation of mass, assuming constant and equal phase densities (or a Boussinesq approximation), gives the mixture equation

$$\nabla\cdot\big(\phi\mathbf{v}_\mathrm{f} + (1-\phi)\mathbf{v}_\mathrm{m}\big) = 0, \tag{1.5}$$

and a compaction relation is given as

$$\mu_\mathrm{m}\nabla\cdot\mathbf{v}_\mathrm{m} = \phi(p_\mathrm{f} - p_\mathrm{m}). \tag{1.6}$$

Because these systems have been combined using mixture theory, one obtains a single, two-phase model that is assumed to hold even when one of the phases disappears. Such models have advantages in numerical approximation, since the free boundary between the one and two phase regions need not be determined, and the equations remain unaltered in a time-dependent problem when a phase disappears or forms in some region of the domain. A similar model arises in modeling two-phase flow within a non-deformable porous medium [6, 17, 7, 10] and in the modeling of partially melted ice, e.g., in glacier dynamics [13, 5, 24].

Nevertheless, the model (1.4)–(1.6) gives rise to a degenerate system when the fluid melt disappears. The Stokes part is well-posed, since there is always matrix rock present at each point of space (i.e., $\phi \leq \phi^* < 1$). Thus we ignore the matrix part of the problem. Assuming a relative permeability of the form $K(\phi) = \mu_\mathrm{f}\,\phi\,a(\phi)$ and setting $b(\phi) = \phi$, we then extract the simplified mathematical model (1.1)–(1.3) with $\mathbf{u} = \mathbf{v}_\mathrm{f} - \mathbf{v}_\mathrm{m}$, and $p = p_\mathrm{f}$. In the true system, $\phi^{-1/2}f$ would represent the matrix pressure, and perhaps one would set $a(\phi) = \phi^{1+2\theta}$ for some $\theta \geq 0$. The techniques developed in this paper will be applied to the simulation of the mechanics of mantle dynamics in [23] with Professor Marc A. Hesse.

2. A-priori estimates and a change of dependent variables. Let $(\cdot,\cdot)$ denote the $L^2(\Omega)$ inner-product or possibly duality pairing, and $\langle\cdot,\cdot\rangle$ denote the $L^2(\partial\Omega)$ inner-product or duality pairing. We proceed formally by assuming that there is a sufficiently smooth solution to the degenerate system (1.1)–(1.3) and that ϕ is reasonable. It will be convenient to define

$$c(\phi) = \sqrt{b(\phi)/a(\phi)} \quad\text{and}\quad d(\phi) = \sqrt{a(\phi)\,b(\phi)}.$$

After an integration by parts we obtain the weak form

$$(c(\phi)^2\mathbf{u},\boldsymbol{\psi}) - \big(p,\nabla\cdot(b(\phi)\boldsymbol{\psi})\big) = (b(\phi)\mathbf{g},\boldsymbol{\psi}), \tag{2.1}$$

$$\big(\nabla\cdot(b(\phi)\mathbf{u}),w\big) + (\phi p,w) = (\phi^{1/2}f,w), \tag{2.2}$$

where the test function $\boldsymbol{\psi}$ satisfies the homogeneous boundary condition (1.3) with $g_{\mathrm{N}} = 0$ on $\partial\Omega$.

Suppose also that we can extend g_{N} to $\mathbf{u}_{\mathrm{N}}$ in Ω such that

$$b(\phi)\mathbf{u}_{\mathrm{N}}\cdot\nu = \phi^{1/2}g_{\mathrm{N}} \quad \text{on } \partial\Omega, \tag{2.3}$$

Taking $\boldsymbol{\psi} = \mathbf{u} - \mathbf{u}_{\mathrm{N}}$ and $w = p$, and also taking $w = \phi^{-1}\nabla\cdot(b(\phi)\mathbf{u})$, in (2.1)–(2.2) results in the a-priori energy estimates

$$\begin{aligned}&\|c(\phi)\mathbf{u}\| + \|\phi^{1/2}p\| + \|\phi^{-1/2}\nabla\cdot(b(\phi)\mathbf{u})\|\\ &\quad\le C\big\{\|f\| + \|d(\phi)\mathbf{g}\| + \|c(\phi)\mathbf{u}_{\mathrm{N}}\| + \|\phi^{-1/2}\nabla\cdot(b(\phi)\mathbf{u}_{\mathrm{N}})\|\big\},\end{aligned} \tag{2.4}$$

in terms of the norm $\|\cdot\|_{k,\omega}$ of the Hilbert space $H^k(\omega)$, where $\|\cdot\|_\omega = \|\cdot\|_{0,\omega}$ and $\|\cdot\| = \|\cdot\|_\Omega$. Assuming the data are given so that the right-hand side is bounded, as ϕ vanishes, we potentially lose control of p (and possibly $\mathbf{u}$). This makes sense, since p is the fluid pressure and there is no fluid phase. Nevertheless, we wish to have a well-posed two-phase mixture even as one phase disappears. We do this by making a change of dependent variables.

Let the *scaled velocity* and *pressure* be defined as

$$\mathbf{v} = c(\phi)\mathbf{u} = \sqrt{b(\phi)/a(\phi)}\,\mathbf{u}, \tag{2.5}$$

$$q = \phi^{1/2}p, \tag{2.6}$$

respectively, since we have control of these quantities. The system (1.1)–(1.3) becomes

$$\mathbf{v} = -d(\phi)\big(\nabla(\phi^{-1/2}q) - \mathbf{g}\big) \quad \text{in } \Omega, \tag{2.7}$$

$$\nabla\cdot(d(\phi)\mathbf{v}) + \phi^{1/2}q = \phi^{1/2}f \quad \text{in } \Omega, \tag{2.8}$$

$$d(\phi)\mathbf{v}\cdot\nu = \phi^{1/2}g_{\mathrm{N}} \quad \text{on } \partial\Omega. \tag{2.9}$$

If we divide the second equation by $\phi^{1/2}$, this system is antisymmetric, since the formal adjoint of $-d(\phi)\nabla(\phi^{-1/2}(\cdot))$ is $\phi^{-1/2}\nabla\cdot(d(\phi)(\cdot))$. Now the energy estimates read more simply as

$$\begin{aligned}&\|\mathbf{v}\| + \|q\| + \|\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v})\|\\ &\quad\le C\big\{\|f\| + \|d(\phi)\mathbf{g}\| + \|\mathbf{v}_{\mathrm{N}}\| + \|\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_{\mathrm{N}})\|\big\},\end{aligned} \tag{2.10}$$

where $\mathbf{v}_{\mathrm{N}} = c(\phi)\mathbf{u}_{\mathrm{N}}$.

3. The space $H_{\phi,d}(\mathrm{div})$. We are led to define the space

$$H_{\phi,d}(\mathrm{div};\Omega) = \big\{\mathbf{v}\in(L^2(\Omega))^d \,:\, \phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v})\in L^2(\Omega)\big\}, \tag{3.1}$$

wherein we must assume that ϕ is well enough behaved to support the definition. The natural conditions seem to be that

$$\phi^{-1/2}d(\phi)\in L^\infty(\Omega) \quad\text{and}\quad \phi^{-1/2}\nabla d(\phi)\in(L^\infty(\Omega))^d. \tag{3.2}$$

The meaning is then clear: To define $H_{\phi,d}(\mathrm{div})$, we interpret

$$\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}) = \phi^{-1/2}\nabla d(\phi)\cdot\mathbf{v} + \phi^{-1/2}d(\phi)\nabla\cdot\mathbf{v}, \tag{3.3}$$

and so we simply require that $\phi^{-1/2}d(\phi)$ times the weak divergence of $\mathbf{v}$ lies in L^2. To ensure that the formal adjoint operator $-d(\phi)\nabla(\phi^{-1/2}(\cdot))$ is well defined, we also ask that

$$\phi^{-3/2}d(\phi)\nabla\phi \in (L^\infty(\Omega))^d. \tag{3.4}$$

We remark that our conditions (3.2) and (3.4) are equivalent to the requirement that $\phi^{-1/2}d(\phi)\in W^{1,\infty}(\Omega)$ and $\phi^{-1/2}\nabla d(\phi)\in (L^\infty(\Omega))^d$.

LEMMA 3.1. *If* (3.2) *and* (3.4) *holds, then* $H_{\phi,d}(\mathrm{div};\Omega)$ *is a Hilbert space with the inner-product*

$$(\mathbf{u},\mathbf{v})_{H_{\phi,d}(\mathrm{div})} = (\mathbf{u},\mathbf{v}) + \big(\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{u}), \phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v})\big). \tag{3.5}$$

Moreover, $H(\mathrm{div};\Omega) = \big\{\mathbf{v}\in(L^2(\Omega))^d : \nabla\cdot\mathbf{v}\in L^2(\Omega)\big\} \subset H_{\phi,d}(\mathrm{div};\Omega)$.

Proof. It is clear that $H_{\phi,d}(\mathrm{div})$ is a linear space and that $(\cdot,\cdot)_{H_{\phi,d}(\mathrm{div})}$ is an inner-product. We must show that the space is complete. Let $\{\mathbf{u}_n\}_{n=1}^\infty\subset H_{\phi,d}(\mathrm{div})$ be a Cauchy sequence, which is to say that

$$\|\mathbf{u}_m-\mathbf{u}_n\|^2_{H_{\phi,d}(\mathrm{div})} = \|\mathbf{u}_m-\mathbf{u}_n\|^2 + \|\phi^{-1/2}\nabla\cdot(d(\phi)(\mathbf{u}_m-\mathbf{u}_n))\|^2 \longrightarrow 0$$

as $m,n\to\infty$. As a consequence, there is $\mathbf{u}\in(L^2)^d$ such that $\mathbf{u}_n\to\mathbf{u}$ as $n\to\infty$, and there is $\xi\in L^2$ such that $\xi_n = \phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{u}_n)\to\xi$ as $n\to\infty$.

To obtain strong convergence of the full divergence term, take a test function $\psi\in C_0^\infty(\Omega)$ and compute

$$\begin{aligned}\big(\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{u}_n),\psi\big) &= -\big(\mathbf{u}_n, d(\phi)\nabla(\phi^{-1/2}\psi)\big)\\ &= \tfrac12(\mathbf{u}_n,\phi^{-3/2}d(\phi)\nabla\phi\,\psi) - (\mathbf{u}_n,\phi^{-1/2}d(\phi)\nabla\psi)\\ &\longrightarrow \tfrac12(\mathbf{u},\phi^{-3/2}d(\phi)\nabla\phi\,\psi) - (\mathbf{u},\phi^{-1/2}d(\phi)\nabla\psi)\\ &= \big(\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{u}),\psi\big)\end{aligned}$$

and conclude that $\xi_n = \phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{u}_n)$ converges weakly to $\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{u})$ in L^2. Therefore $\xi = \phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{u})$ and the proof of completeness is finished.

For any $\mathbf{v}\in H(\mathrm{div})$, both $\mathbf{v}\in(L^2)^d$ and $\nabla\cdot\mathbf{v}\in L^2$, and so (3.3) implies that $\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v})\in L^2$, and the final assertion of the lemma holds as well. ◻

We wish to apply the Neumann boundary condition. Define the normal trace operator $\gamma_{\phi,d}: H_{\phi,d}(\mathrm{div};\Omega)\to H^{-1/2}(\partial\Omega)$ using the integration by parts formula

$$\langle\gamma_{\phi,d}(\mathbf{v}), w\rangle = \big(\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}), w\big) + \big(\mathbf{v}, d(\phi)\nabla(\phi^{-1/2}w)\big), \tag{3.6}$$

wherein $w\in H^{1/2}(\partial\Omega)$ has been extended to $w\in H^1(\Omega)$. Note that (3.2) and (3.4) imply that the operator is well-defined by the right-hand side, and that we can interpret $\gamma_{\phi,d}(\mathbf{v}) = \phi^{-1/2}d(\phi)\mathbf{v}\cdot\nu$.

LEMMA 3.2. *If* (3.2) *and* (3.4) *hold, then the normal trace operator* $\gamma_{\phi,d}: H_{\phi,d}(\mathrm{div};\Omega)\to H^{-1/2}(\partial\Omega)$ *is well defined by* (3.6) *and there is a constant* $C>0$ *such that*

$$\|\gamma_{\phi,d}(\mathbf{v})\|_{-1/2,\partial\Omega} = \|\phi^{-1/2}d(\phi)\mathbf{v}\cdot\nu\|_{-1/2,\partial\Omega} \le C\|\mathbf{v}\|_{H_{\phi,d}(\mathrm{div})} \tag{3.7}$$

for any $\mathbf{v} \in H_{\phi,d}(\mathrm{div};\Omega)$.

Finally, we apply the homogeneous boundary condition to define

$$H_{\phi,d,0}(\mathrm{div};\Omega) = \big\{\mathbf{v} \in H_{\phi,d}(\mathrm{div};\Omega) \;:\; \gamma_{\phi,d}(\mathbf{v}) = \phi^{-1/2} d(\phi)\mathbf{v}\cdot\nu = 0 \text{ on } \partial\Omega\big\}, \tag{3.8}$$

and we let the image of the normal trace operator be denoted by

$$H^{-1/2}_{\phi,d}(\partial\Omega) = \gamma_{\phi,d}\big(H_{\phi,d}(\mathrm{div};\Omega)\big) \subset H^{-1/2}(\partial\Omega). \tag{3.9}$$

4. A scaled weak formulation and unique existence of the solution. Replacing (2.3) is the requirement of a function $\mathbf{v}_\mathrm{N} \in H_{\phi,d}(\mathrm{div};\Omega)$ such that

$$\gamma_{\phi,d}(\mathbf{v}_\mathrm{N}) = \phi^{-1/2} d(\phi)\mathbf{v}_\mathrm{N}\cdot\nu = g_\mathrm{N}, \tag{4.1}$$

which can be found provided that $g_\mathrm{N} \in H^{-1/2}_{\phi,d}(\partial\Omega)$. In place of the weak system (2.1)–(2.2), we test (2.7) and $\phi^{-1/2}$ times (2.8) to obtain our scaled weak formulation: Find $\mathbf{v} \in H_{\phi,d,0}(\mathrm{div};\Omega) + \mathbf{v}_\mathrm{N}$ and $q \in L^2(\Omega)$ such that

$$(\mathbf{v},\boldsymbol{\psi}) - \big(q, \phi^{-1/2}\nabla\cdot(d(\phi)\boldsymbol{\psi})\big) = (d(\phi)\mathbf{g},\boldsymbol{\psi}) \quad \forall\boldsymbol{\psi} \in H_{\phi,d,0}(\mathrm{div};\Omega), \tag{4.2}$$

$$\big(\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}), w\big) + (q,w) = (f,w) \qquad \forall w \in L^2(\Omega). \tag{4.3}$$

We require that $f \in L^2(\Omega)$ and $d(\phi)\mathbf{g} \in (L^2(\Omega))^d$.

We saw the a-priori energy estimates (2.10) for (4.2)–(4.3), which imply that if there is a solution to the problem, then it is unique. To prove existence of a solution, we let $\delta \geq 0$ and we define the stabilized bilinear form $a_\delta : \big(H_{\phi,d,0}(\mathrm{div}) \times L^2\big) \times \big(H_{\phi,d,0}(\mathrm{div}) \times L^2\big) \to \mathbb{R}$ by

$$\begin{aligned} a_\delta\big((\mathbf{v}_0,q),(\boldsymbol{\psi},w)\big) &\\ = (\mathbf{v}_0,\boldsymbol{\psi}) - \big(q,\phi^{-1/2}\nabla\cdot(d(\phi)\boldsymbol{\psi})\big) &+ \big(\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_0), w\big) + (q,w) \\ + \delta\big\{\big(\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_0), \phi^{-1/2}\nabla\cdot(d(\phi)\boldsymbol{\psi})\big) &+ \big(q,\phi^{-1/2}\nabla\cdot(d(\phi)\boldsymbol{\psi})\big)\big\} \end{aligned} \tag{4.4}$$

and the linear form $b_\delta : H_{\phi,d,0}(\mathrm{div}) \times L^2 \to \mathbb{R}$ by

$$\begin{aligned} b_\delta(\boldsymbol{\psi},w) = (d(\phi)\mathbf{g},\boldsymbol{\psi}) - (\mathbf{v}_\mathrm{N},\boldsymbol{\psi}) &+ \big(f - \phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_\mathrm{N}), w\big) \\ &+ \delta\big\{\big(f - \phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_\mathrm{N}), \phi^{-1/2}\nabla\cdot(d(\phi)\boldsymbol{\psi})\big)\big\}. \end{aligned} \tag{4.5}$$

These two forms are clearly continuous, i.e., bounded.

We now have the problem: Find $(\mathbf{v}_0,q) \in H_{\phi,d,0}(\mathrm{div};\Omega) \times L^2(\Omega)$ such that

$$a_\delta\big((\mathbf{v}_0,q),(\boldsymbol{\psi},w)\big) = b_\delta(\boldsymbol{\psi},w) \quad \forall(\boldsymbol{\psi},w) \in H_{\phi,d,0}(\mathrm{div};\Omega) \times L^2(\Omega). \tag{4.6}$$

With $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_\mathrm{N}$ and $\delta = 0$, this problem is (4.2)–(4.3). In fact,

$$\begin{aligned} &a_\delta\big((\mathbf{v}_0,q),(\boldsymbol{\psi},w)\big) - b_\delta(\boldsymbol{\psi},w) \\ &\quad = a_0\big((\mathbf{v}_0,q),(\boldsymbol{\psi}, w + \delta\phi^{-1/2}\nabla\cdot(d(\phi)\boldsymbol{\psi}))\big) - b_0\big(\boldsymbol{\psi}, w + \delta\phi^{-1/2}\nabla\cdot(d(\phi)\boldsymbol{\psi})\big), \end{aligned}$$

with $w + \delta\phi^{-1/2}\nabla\cdot(d(\phi)\boldsymbol{\psi})$ being arbitrary in L^2. Therefore, the problems (4.6) are all equivalent for any $\delta \geq 0$.

We claim that for any $\delta \in (0,2)$, the bilinear form a_δ is coercive. To see this, simply compute that

$$\begin{aligned} a_\delta\big((\mathbf{v}_0,q),(\mathbf{v}_0,q)\big) &= \|\mathbf{v}_0\|^2 + \|q\|^2 + \delta\|\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_0)\|^2 + \delta\big(q,\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_0)\big) \\ &\ge \|\mathbf{v}_0\|^2 + \big(1-\tfrac{1}{2}\delta\big)\|q\|^2 + \tfrac{1}{2}\delta\|\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_0)\|^2 \\ &\ge \tfrac{1}{2}\delta\|\mathbf{v}_0\|_{H_{\phi,d}(\mathrm{div})} + \big(1-\tfrac{1}{2}\delta\big)\|q\|^2. \end{aligned}$$

Therefore, we can apply the Lax-Milgram Theorem to conclude that (4.6) has a unique solution for $\delta \in (0,2)$. By the equivalence of the weak problems, we then have a solution for any $\delta \ge 0$, and in particular one for our problem (4.6) with $\delta = 0$, i.e., (4.2)–(4.3) (which we already know is unique).

THEOREM 4.1. *Let* (3.2) *and* (3.4) *hold,* $f \in L^2(\Omega)$, $d(\phi)\mathbf{g} \in (L^2(\Omega))^d$, *and* $g_\mathrm{N} \in H^{-1/2}_{\phi,d}(\partial\Omega)$. *If* $\mathbf{v}_\mathrm{N}$ *is defined by* (4.1), *then there is a unique solution to the problem* (4.2)–(4.3), *and the energy estimates* (2.10) *and* (2.4) *hold.*

5. Some extensions of the results. We can handle Dirichlet and Robin boundary conditions, and in some cases, we can show that $p \in L^2(\Omega)$.

5.1. Dirichlet and Robin boundary conditions. Instead of Neumann boundary conditions (1.3), we could impose Dirichlet or Robin boundary conditions of the form

$$\phi p - \kappa^2 b(\phi)\mathbf{u}\cdot\nu = \phi^{1/2} g_\mathrm{R} \quad \text{on } \partial\Omega, \tag{5.1}$$

where $\kappa \ge 0$ is a bounded function and g_R is given. The scaled version is

$$q - \kappa^2\phi^{-1/2} d(\phi)\mathbf{v}\cdot\nu = g_\mathrm{R} \quad \text{on } \partial\Omega, \tag{5.2}$$

and the scaled weak form is: Find $\mathbf{v} \in H_{\phi,d}(\mathrm{div};\Omega)$ and $q \in L^2(\Omega)$ such that

$$\begin{aligned} &(\mathbf{v},\boldsymbol{\psi}) - \big(q,\phi^{-1/2}\nabla\cdot(d(\phi)\boldsymbol{\psi})\big) + \langle\kappa^2\phi^{-1}(d(\phi))^2\mathbf{v}\cdot\nu,\boldsymbol{\psi}\cdot\nu\rangle \\ &\qquad = (d(\phi)\mathbf{g},\boldsymbol{\psi}) - \langle g_\mathrm{R},\phi^{-1/2}d(\phi)\boldsymbol{\psi}\cdot\nu\rangle \qquad \forall\boldsymbol{\psi} \in H_{\phi,d}(\mathrm{div};\Omega), \end{aligned} \tag{5.3}$$

$$\big(\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}),w\big) + (q,w) = (f,w) \qquad \forall w \in L^2(\Omega). \tag{5.4}$$

We require that $g_\mathrm{R} \in H^{1/2}(\partial\Omega)$ (actually, we could require merely that g_R is in the dual space $(H^{-1/2}_{\phi,d}(\partial\Omega))^*$) and, as before, $f \in L^2(\Omega)$ and $d(\phi)\mathbf{g} \in (L^2(\Omega))^d$. Using the Trace Lemma 3.2, the a-priori energy estimates are

$$\begin{aligned} &\|\mathbf{v}\| + \|q\| + \|\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v})\| + \|\kappa\phi^{-1/2}d(\phi)\mathbf{v}\cdot\nu\|_{\partial\Omega} \\ &\qquad \le C\big\{\|f\| + \|d(\phi)\mathbf{g}\| + \|g_\mathrm{R}\|_{1/2,\partial\Omega}\big\}, \end{aligned} \tag{5.5}$$

and the analogue to Theorem 4.1 can be proved in a similar way. We need to modify a_δ by extending it to $\big(H_{\phi,d}(\mathrm{div})\times L^2\big)\times\big(H_{\phi,d}(\mathrm{div})\times L^2\big)$ and adding a term, obtaining

$$\tilde{a}_\delta\big((\mathbf{v},q),(\boldsymbol{\psi},w)\big) = a_\delta\big((\mathbf{v},q),(\boldsymbol{\psi},w)\big) + \langle\kappa^2\phi^{-1}(d(\phi))^2\mathbf{v}\cdot\nu,\boldsymbol{\psi}\cdot\nu\rangle.$$

We also redefine b_δ as $\tilde{b}_\delta : H_{\phi,d}(\mathrm{div}) \times L^2 \to \mathbb{R}$ such that

$$\tilde{b}_\delta(\boldsymbol{\psi},w) = (d(\phi)\mathbf{g},\boldsymbol{\psi}) + (f,w) - \langle g_\mathrm{R},\phi^{-1/2}d(\phi)\boldsymbol{\psi}\cdot\nu\rangle + \delta\big(f,\phi^{-1/2}\nabla\cdot(d(\phi)\boldsymbol{\psi})\big).$$

THEOREM 5.1. *Let* (3.2) *and* (3.4) *hold,* $f \in L^2(\Omega)$, $d(\phi)\mathbf{g} \in (L^2(\Omega))^d$, *and* $g_\mathrm{R} \in H^{1/2}(\partial\Omega)$. *Then there is a unique solution to the problem* (5.3)–(5.4), *and the energy estimates* (5.5) *hold.*

5.2. A condition for the pressure to be in L^2. In some cases the solution is more regular than implied by (2.4). Proceeding formally from (1.1)–(1.2), we have the single equation

$$-\nabla\cdot\big((d(\phi))^2\nabla p\big)+\phi p=\phi^{1/2}f-\nabla\cdot\big((d(\phi))^2\mathbf{g}\big). \tag{5.6}$$

We multiply by $\phi^{-1}p$ and integrate by parts using the homogeneous Neumann boundary condition (1.3) to see that

$$\big((d(\phi))^2\nabla p,\nabla(\phi^{-1}p)\big)+\|p\|^2=(\phi^{-1/2}f,p)+\big((d(\phi))^2\mathbf{g},\nabla(\phi^{-1}p)\big).$$

After formally expanding the derivative terms, we obtain

$$\begin{aligned}
&\big(\phi^{-1}(d(\phi))^2\nabla p,\nabla p\big)+\|p\|^2\\
&\quad=(\phi^{-1/2}f,p)+\big(\phi^{-1}(d(\phi))^2\mathbf{g},\nabla p\big)-\big(\phi^{-2}(d(\phi))^2\nabla\phi\cdot\mathbf{g},p\big)\\
&\qquad+\big(\phi^{-2}(d(\phi))^2\nabla\phi\cdot\nabla p,p\big)\\
&\quad\le\epsilon\big\{\|p\|^2+\|\phi^{-1/2}d(\phi)\nabla p\|^2\big\}+\|\phi^{-3/2}d(\phi)\nabla\phi\|_{L^\infty(\Omega)}\|\phi^{-1/2}d(\phi)\nabla p\|\,\|p\|\\
&\qquad+C_\epsilon\big\{\|\phi^{-1/2}f\|^2+\|\phi^{-1/2}d(\phi)\mathbf{g}\|^2+\|\phi^{-2}(d(\phi))^2\nabla\phi\cdot\mathbf{g}\|^2\big\}
\end{aligned}$$

for any $\epsilon>0$, and so

$$\|\phi^{-1/2}d(\phi)\nabla p\|+\|p\|\le C\big\{\|\phi^{-1/2}f\|+\|\phi^{-1/2}d(\phi)\mathbf{g}\|+\|\phi^{-2}(d(\phi))^2\nabla\phi\cdot\mathbf{g}\|\big\}, \tag{5.7}$$

provided that $\|\phi^{-3/2}d(\phi)\nabla\phi\|_{L^\infty(\Omega)}<2$. For greater generality, we have chosen to work with the scaled pressure. However, it is interesting to note that p may in fact be stable in many cases, and then it is only the loss of an equation that is problematic for numerical approximation.

6. Mixed finite element methods. We now discuss discrete versions of our scaled systems (4.2)–(4.3) for Neumann and (5.3)–(5.4) for Dirichlet and Robin boundary conditions. We will assume that Ω is a rectangle ($d=2$) or brick ($d=3$), and impose a rectangular finite element mesh $\mathcal{T}_h$ over the domain of maximal spacing h. Let $\mathcal{E}_h$ denote the set of element edges ($d=2$) or faces ($d=3$). We use the notation $|E|$ for the measure (area or volume) of $E\in\mathcal{T}_h$ and $|e|$ for the measure (length or area) of $e\in\mathcal{E}_h$. Let $\mathbb{P}_n$ be the set of polynomials of total degree n and let $\mathbb{P}_{n_1,n_2,n_3}$ (omit n_3 if $d=2$) be the set of polynomials of degree n_i in x_i for each $i=1,\dots,d$.

We will approximate $(\mathbf{v},q)$ in the lowest order Raviart-Thomas (RT_0) finite element space $\mathbf{V}_h\times W_h$ [20, 9, 21]. On an element $E\in\mathcal{T}_h$, $\mathbf{V}_h(E)=\mathbb{P}_{1,0,0}\times\mathbb{P}_{0,1,0}\times\mathbb{P}_{0,0,1}$ (omit the last component if $d=2$) and $W_h(E)=\mathbb{P}_0$. We also need the space $\mathbf{V}_{h,0}=\{\mathbf{v}_h\in\mathbf{V}_h:\mathbf{v}_h\cdot\nu=0\text{ on }\partial\Omega\}$.

Later we will make use of the usual projection operators associated with RT_0. Let $\mathcal{P}_{W_h}=\hat{\cdot}:L^2(\Omega)\to W_h$ denote the $L^2(\Omega)$ projection operator, which projects a function into the space of piecewise constant functions. Moreover, let $\pi:H(\mathrm{div};\Omega)\cap L^{2+\epsilon}(\Omega)\to\mathbf{V}_h$ (any $\epsilon>0$) denote the standard Raviart-Thomas or Fortin operator that preserves element average divergence and edge normal fluxes [20, 9, 21].

To simplify the treatment of boundary conditions, when using Neumann conditions, let $\beta_{\mathrm{N}}=1$ and $\beta_{\mathrm{R}}=1-\beta_{\mathrm{N}}=0$, and when using Robin conditions, $\beta_{\mathrm{N}}=0$ and $\beta_{\mathrm{R}}=1$. Also let $\tilde{\mathbf{V}}_h=\beta_{\mathrm{N}}\mathbf{V}_{h,0}+\beta_{\mathrm{R}}\mathbf{V}_h$.

6.1. A formal method based on $\mathbf{RT}_0$. The formal mixed finite element method for Neumann (4.2)–(4.3) or Robin (5.3)–(5.4) boundary conditions is: Find $\mathbf{v}_h \in \tilde{\mathbf{V}}_h + \beta_{\mathrm{N}}\mathbf{v}_{\mathrm{N}}$ and $q_h \in W_h$ such that

$$(6.1)\quad (\mathbf{v}_h, \boldsymbol{\psi}) - \big(q_h, \phi^{-1/2}\nabla\cdot(d(\phi)\boldsymbol{\psi})\big) + \beta_{\mathrm{R}}\langle \kappa^2\phi^{-1}(d(\phi))^2\mathbf{v}_h\cdot\nu, \boldsymbol{\psi}\cdot\nu\rangle = (d(\phi)\mathbf{g}, \boldsymbol{\psi}) - \beta_{\mathrm{R}}\langle g_{\mathrm{R}}, \phi^{-1/2}d(\phi)\boldsymbol{\psi}\cdot\nu\rangle \qquad \forall \boldsymbol{\psi}\in\tilde{\mathbf{V}}_h,$$

$$(6.2)\quad \big(\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_h), w\big) + (q_h, w) = (f, w) \qquad \forall w \in W_h.$$

When using Neumann boundary conditions, we might choose to find $\mathbf{v}_h \in \mathbf{V}_{h,0} + \pi\mathbf{v}_{\mathrm{N}}$ instead. To show unique solvability and stability of the Robin case, we require a discrete version of the Trace Lemma 3.2.

LEMMA 6.1. *If (3.2) and (3.4) hold and the finite element mesh $\mathcal{T}_h$ is quasi-uniform, then there is a constant $C > 0$ such that*

$$(6.3)\qquad \|\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_h)\| \le C\big\{\|\mathbf{v}_h\| + \|\mathcal{P}_{W_h}[\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_h)]\|\big\}$$

for any $\mathbf{v}_h \in \mathbf{V}_h$. Moreover, for some possibly different constant $C > 0$,

$$(6.4)\qquad \|\gamma_{\phi,d}(\mathbf{v}_h)\|_{-1/2,\partial\Omega} = \|\phi^{-1/2}d(\phi)\mathbf{v}_h\cdot\nu\|_{-1/2,\partial\Omega} \le C\big\{\|\mathbf{v}_h\| + \|\mathcal{P}_{W_h}[\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_h)]\|\big\}$$

for any $\mathbf{v}_h \in \mathbf{V}_h$.

Proof. The triangle inequality gives that

$$\|\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_h)\| \le \|\mathcal{P}_{W_h}[\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_h)]\| + \|\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_h) - \mathcal{P}_{W_h}[\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_h)]\|.$$

We compute that

$$\begin{aligned}
&\|\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_h) - \mathcal{P}_{W_h}[\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_h)]\| \\
&\quad = \|\phi^{-1/2}\nabla d(\phi)\cdot\mathbf{v}_h - \mathcal{P}_{W_h}[\phi^{-1/2}\nabla d(\phi)\cdot\mathbf{v}_h] \\
&\qquad\qquad + \phi^{-1/2}d(\phi)\nabla\cdot\mathbf{v}_h - \mathcal{P}_{W_h}[\phi^{-1/2}d(\phi)\nabla\cdot\mathbf{v}_h]\| \\
&\quad \le 2\|\phi^{-1/2}\nabla d(\phi)\cdot\mathbf{v}_h\| + \|\phi^{-1/2}d(\phi)\nabla\cdot\mathbf{v}_h - \mathcal{P}_{W_h}[\phi^{-1/2}d(\phi)\nabla\cdot\mathbf{v}_h]\| \\
&\quad \le C\|\mathbf{v}_h\| + \big\|\big(\phi^{-1/2}d(\phi) - \mathcal{P}_{W_h}[\phi^{-1/2}d(\phi)]\big)\nabla\cdot\mathbf{v}_h\big\|,
\end{aligned}$$

since $\phi^{-1/2}\nabla d(\phi) \in (L^\infty(\Omega))^d$ by assumption (3.2). Further for the last term,

$$\begin{aligned}
&\big\|\big(\phi^{-1/2}d(\phi) - \mathcal{P}_{W_h}[\phi^{-1/2}d(\phi)]\big)\nabla\cdot\mathbf{v}_h\big\| \\
&\quad \le \|\phi^{-1/2}d(\phi) - \mathcal{P}_{W_h}[\phi^{-1/2}d(\phi)]\|_{L^\infty(\Omega)}\|\nabla\cdot\mathbf{v}_h\| \\
&\quad \le Ch\|\phi^{-1/2}d(\phi)\|_{W^{1,\infty}(\Omega)}\|\nabla\cdot\mathbf{v}_h\| \\
&\quad \le C\|\phi^{-1/2}d(\phi)\|_{W^{1,\infty}(\Omega)}\|\mathbf{v}_h\|,
\end{aligned}$$

using [11] for the approximation of the L^2-projection in L^∞ and an inverse estimate, since the finite element mesh $\mathcal{T}_h$ is assumed to be quasi-uniform. Because $\phi^{-1/2}d(\phi) \in W^{1,\infty}(\Omega)$ by assumptions (3.2) and (3.4), the first result (6.3) is established. The discrete trace bound (6.4) then follows directly from the Trace Lemma 3.2. ∎

Substituting into (6.1)–(6.2) the discrete solution $\boldsymbol{\psi} = \mathbf{v}_h - \beta_{\mathrm{N}}\mathbf{v}_{\mathrm{N}} \in \tilde{\mathbf{V}}_h$ and $w = q_h + \mathcal{P}_{W_h}[\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_h)] \in W_h$ shows the stability result

$$\begin{aligned}
&\|\mathbf{v}_h\| + \|q_h\| + \|\mathcal{P}_{W_h}[\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_h)]\| + \beta_{\mathrm{R}}\|\kappa\phi^{-1/2}d(\phi)\mathbf{v}_h\cdot\nu\|_{\partial\Omega}\\
&\quad\le C\big\{\|f\| + \|d(\phi)\mathbf{g}\|\\
&\qquad + \beta_{\mathrm{R}}\|g_{\mathrm{R}}\|_{1/2,\partial\Omega} + \beta_{\mathrm{N}}\big(\|\mathbf{v}_{\mathrm{N}}\| + \|\mathcal{P}_{W_h}[\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_{\mathrm{N}})]\|\big)\big\}.
\end{aligned}\tag{6.5}$$

Uniqueness of the solution is therefore established, and existence follows because the discrete system over a basis is a square linear system.

In fact, the formal method gives rise to a linear system of saddle point form (see (6.11) below), which can be difficult to solve. Moreover, it is perhaps not completely clear how to evaluate the divergence terms involving division by ϕ when ϕ vanishes.

6.2. A practical method based on $\mathbf{RT}_0$. We modify the formal method to make it easier to implement and solve, and also to obtain local mass conservation. Denote the local average of ϕ over element $E \in \mathcal{T}_h$ by

$$\phi_E = \frac{1}{|E|}\int_E \phi\,dx = \hat{\phi}|_E.\tag{6.6}$$

We present the practical method for Neumann ($\beta_{\mathrm{N}} = 1$) and Robin ($\beta_{\mathrm{R}} = 1$) boundary conditions. The practical method is: Find $\mathbf{v}_h \in \tilde{\mathbf{V}}_h + \beta_{\mathrm{N}}\mathbf{v}_{\mathrm{N}}$ and $q_h \in W_h$ such that

$$\begin{aligned}
&(\mathbf{v}_h,\boldsymbol{\psi})_Q - \big(q_h,\hat{\phi}^{-1/2}\nabla\cdot(d(\phi)\boldsymbol{\psi})\big) + \beta_{\mathrm{R}}\langle\kappa^2\phi^{-1}(d(\phi))^2\mathbf{v}_h\cdot\nu,\boldsymbol{\psi}\cdot\nu\rangle\\
&\qquad = (d(\phi)\mathbf{g},\boldsymbol{\psi}) - \beta_{\mathrm{R}}\langle g_{\mathrm{R}},\phi^{-1/2}d(\phi)\boldsymbol{\psi}\cdot\nu\rangle \qquad \forall\boldsymbol{\psi}\in\tilde{\mathbf{V}}_h,
\end{aligned}\tag{6.7}$$

$$\big(\hat{\phi}^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_h),w\big) + (q_h,w) = (\hat{\phi}^{-1/2}\phi^{1/2}f,w) \qquad \forall w\in W_h,\tag{6.8}$$

with the caveat that we set the divergence terms to zero on an element E that would otherwise include $\hat{\phi}^{-1/2}$ when $\hat{\phi}|_E = \phi_E = 0$, and in that case, $\hat{\phi}^{-1/2}\phi^{1/2}f = f$. For simplicity of implementation, we have approximated the first integral in (6.7) by a trapezoidal quadrature rule $(\cdot,\cdot)_Q$. This approximation is not required to define the method, but it will lead to a much simpler cell-centered finite difference method [22, 3], as we will see below.

We easily recover the discrete pressure $p_h \in W_h$ by setting for all $E \in \mathcal{T}_h$

$$p_h|_E = \begin{cases} 0 & \text{if } \phi_E = 0,\\ \phi_E^{-1/2}q_h|_E & \text{if } \phi_E \neq 0,\end{cases}\tag{6.9}$$

and the discrete velocity $\mathbf{u}_h \in \tilde{\mathbf{V}}_h + \beta_{\mathrm{N}}\mathbf{v}_{\mathrm{N}}$ is defined by setting for all $e \in \mathcal{E}_h$

$$\mathbf{u}_h\cdot\nu|_e = \begin{cases} 0 & \text{if } b_e \equiv \displaystyle\int_e b(\phi)\,ds = 0,\\ b_e^{-1}\displaystyle\int_e d(\phi)\,ds\,\mathbf{v}_h\cdot\nu|_e & \text{if } b_e \neq 0,\end{cases}\tag{6.10}$$

so that $\pi(b(\phi)\mathbf{u}_h) = \pi(d(\phi)\mathbf{v}_h)$.

We now discuss in detail implementation of the method, restricted to Robin boundary conditions for simplicity of exposition. The degrees of freedom for $\mathbf{V}_h$ are the constant normal values on the edges or faces, and the degrees of freedom of W_h are the average values over the elements. Let bases be defined, respectively, by

$$\big\{\mathbf{v}_e\,:\,\mathbf{v}_e\cdot\nu|_f = \delta_{e,f}\ \forall e,f\in\mathcal{E}_h\big\}\ \text{ and }\ \big\{w_E\,:\,w_E|_F = \delta_{E,F}\ \forall E,F\in\mathcal{T}_h\big\},$$

where $\delta_{i,j}$ is the Kronecker delta function for indices i and j. Then the linear system corresponding to the method (6.7)–(6.8) is

$$\begin{pmatrix} A & -B \\ B^T & C \end{pmatrix} \begin{pmatrix} v \\ q \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}, \tag{6.11}$$

wherein v and q represent the degrees of freedom for $\mathbf{v}$ and q, respectively. Because of the quadrature rule, it is easy to compute

$$\begin{aligned} A_{e,f} &= (\mathbf{v}_e, \mathbf{v}_f)_Q + \langle \kappa^2 \phi^{-1} (d(\phi))^2 \mathbf{v}_e \cdot \nu, \mathbf{v}_f \cdot \nu \rangle \\ &= \Big(\frac{1}{2} |E_e| + \int_{e \cap \partial\Omega} \kappa^2 \phi^{-1} (d(\phi))^2 \, ds \Big) \delta_{e,f}, \end{aligned} \tag{6.12}$$

$$C_{E,F} = (w_E, w_F) = |E| \, \delta_{E,F}, \tag{6.13}$$

where E_e is the support of $\mathbf{v}_e$ (i.e., the one or two elements adjacent to e). Moreover,

$$\begin{aligned} a_e &= (d(\phi)\mathbf{g}, \mathbf{v}_e) - \langle g_{\mathrm{R}}, \phi^{-1/2} d(\phi) \mathbf{v}_e \cdot \nu \rangle \\ &= \int_{E_e} d(\phi) \mathbf{g} \cdot \mathbf{v}_e \, dx - \mathbf{v}_e \cdot \nu \int_{e \cap \partial\Omega} g_{\mathrm{R}} \phi^{-1/2} d(\phi) \, ds, \end{aligned} \tag{6.14}$$

$$b_E = (\hat{\phi}^{-1/2} \phi^{1/2} f, w_E) = \begin{cases} \displaystyle\int_E f \, dx & \text{if } \phi_E = 0, \\ \phi_E^{-1/2} \displaystyle\int_E \phi^{1/2} f \, dx & \text{if } \phi_E \neq 0. \end{cases} \tag{6.15}$$

The matrix B remains, but it is now clear how it is defined because we have avoided division by ϕ when $\phi = 0$. In terms of our projection operators,

$$B_{e,E} = \big(\hat{\phi}^{-1/2} \nabla \cdot (d(\phi) \mathbf{v}_e), w_E \big) = \langle \phi_E^{-1/2} d(\phi) \mathbf{v}_e \cdot \nu, w_E \rangle_{\partial E};$$

that is,

$$B_{e,E} = \begin{cases} 0 & \text{if } e \not\subset \partial E \text{ or } \phi_E = 0, \\ \phi_E^{-1/2} \mathbf{v}_e \cdot \nu_E \displaystyle\int_e d(\phi) \, dx & \text{if } e \subset \partial E \text{ and } \phi_E \neq 0. \end{cases} \tag{6.16}$$

The Schur complement of (6.11) for q is

$$v = A^{-1}(Bq + a), \tag{6.17}$$

$$(B^T A^{-1} B + C) \, q = b - B^T A^{-1} a. \tag{6.18}$$

The matrix of the second equation can be formed easily as a 5- or 7-point finite difference stencil for $d = 2$ or 3, respectively, because A and C are diagonal and positive definite [22, 3]. Thus we solve the second equation (6.18) relatively efficiently as a cell-centered finite difference method for q, and then form v using q and the first equation (6.17).

We remark that, in practice, our problem is a subproblem of a larger system that determines ϕ, so ϕ will be represented in a discrete space Φ_h. We might chose the space used by the first author and Chen [2] to approximate the solution to a nondegenerate second order elliptic problem using a nonconforming method. The nonconforming method is equivalent to a mixed method based on RT_0. The solution variable is approximated with degrees of freedom defined by element average values

ϕ_E $\forall E \in \mathcal{T}_h$ and face (or edge) average values $\phi_e = \dfrac{1}{|e|}\displaystyle\int_e \phi\, ds$ $\forall e \in \mathcal{E}_h$. The space (when $d = 3$) is

$$\Phi_h = \big\{\phi : \phi|_E \in \mathbb{P}_{2,0,0} + \mathbb{P}_{0,2,0} + \mathbb{P}_{0,0,2}\ \forall\, E \in \mathcal{T}_h \text{ and } \phi_e \text{ is unique } \forall e \in \mathcal{E}_h\big\}.$$

In this case, we could define the method using only the degrees of freedom of $\phi \in \Phi_h$. To do so, we would simply replace ϕ by ϕ_e in (6.12), the second integral of (6.14), and (6.16), and replace ϕ by ϕ_E in the first integral of (6.14) and (6.15).

6.3. Local mass conservation of the practical method. The equation (6.8) implies that mass is conserved locally by the practical method. We simply take the test function $\phi_E^{1/2} w_E \in W_h$ to see that

$$\int_E \nabla\cdot(d(\phi)\mathbf{v}_h)\,dx + \int_E \phi_E^{1/2} q_h\,dx = \int_E \phi^{1/2} f\,dx, \tag{6.19}$$

which is the same as

$$\int_E \nabla\cdot\pi(d(\phi)\mathbf{v}_h)\,dx + \int_E \phi\,\phi_E^{-1/2} q_h\,dx = \int_E \phi^{1/2} f\,dx.$$

Since (6.10) defines $\mathbf{u}_h \in \mathbf{V}_h$ such that $\pi(d(\phi)\mathbf{v}_h) = \pi(b(\phi)\mathbf{u}_h)$ and (6.9) defines $p_h \in W_h$, we have

$$\int_E \nabla\cdot(b(\phi)\mathbf{u}_h)\,dx + \int_E \phi p_h\,dx = \int_E \phi^{1/2} f\,dx, \tag{6.20}$$

which is the mass conservation equation (1.2) integrated over E.

6.4. Solvability and stability of the practical method. We recognize that $(\boldsymbol{\psi},\boldsymbol{\psi})_Q^{1/2}$ is a norm on $\mathbf{V}_h$ equivalent to $\|\boldsymbol{\psi}\|$ with bounds independent of h [25]. Substituting $\boldsymbol{\psi} = \mathbf{v}_h - \beta_{\rm N}\mathbf{v}_{\rm N} \in \tilde{\mathbf{V}}_h$ and $w = q_h + \mathcal{P}_{W_h}[\hat\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_h)] \in W_h$ into (6.7)–(6.8) gives

$$\begin{aligned}
&\|\mathbf{v}_h\|^2 + \|q_h\|^2 + \|\mathcal{P}_{W_h}[\hat\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_h)]\|^2 + \beta_{\rm R}\|\kappa\phi^{-1/2}d(\phi)\mathbf{v}_h\cdot\nu\|_{\partial\Omega}^2\\
&\quad\le C\big\{\|\hat\phi^{-1/2}\phi^{1/2} f\|^2 + \|d(\phi)\mathbf{g}\|^2 + \beta_{\rm R}|\langle g_{\rm R}, \phi^{-1/2}d(\phi)\mathbf{v}_h\cdot\nu\rangle|^2\\
&\qquad + \beta_{\rm N}\big(\|\mathbf{v}_{\rm N}\|^2 + \|\mathcal{P}_{W_h}[\hat\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_{\rm N})]\|^2\big)\big\}.
\end{aligned}$$

Uniqueness and, therefore, existence of the solution is established.

Unfortunately, we do not have a Discrete Trace Lemma involving $\hat\phi$. In many cases, we obtain the stability bound

$$\begin{aligned}
&\|\mathbf{v}_h\| + \|q_h\| + \|\mathcal{P}_{W_h}[\hat\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_h)]\| + \beta_{\rm R}\|\kappa\phi^{-1/2}d(\phi)\mathbf{v}_h\cdot\nu\|_{\partial\Omega}\\
&\quad\le C\big\{\|\hat\phi^{-1/2}\phi^{1/2} f\| + \|d(\phi)\mathbf{g}\| + \beta_{\rm R}\|g_{\rm R}\|_{0,\partial\Omega}\\
&\qquad + \beta_{\rm N}\big(\|\mathbf{v}_{\rm N}\| + \|\mathcal{P}_{W_h}[\hat\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_{\rm N})]\|\big)\big\}.
\end{aligned} \tag{6.21}$$

This holds when we use Neumann boundary conditions ($\beta_{\rm N} = 1$, $\beta_{\rm R} = 0$), and also when we have either homogeneous Robin conditions, i.e., $g_{\rm R} = 0$, or uniform Robin conditions, i.e., $\kappa \ge \kappa_* > 0$ for some constant κ_*.

In the case of nonhomogeneous, nonuniform Robin boundary conditions, we modify the treatment of $g_{\rm R}$ in (6.7) when $\beta_{\rm R} = 1$. We base the method on a discrete version of the definition of the normal trace (3.6), to wit

$$\langle g_{\rm R}, \phi^{-1/2} d(\phi)\mathbf{v}_h\cdot\nu\rangle = \big(\hat g_{\rm R}, \hat\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_h)\big) + \big(d(\phi)\nabla(\phi^{-1/2}g_{\rm R}),\mathbf{v}_h\big),$$

The modified practical method uses (6.8) combined with

$$\begin{aligned}(\mathbf{v}_h,\boldsymbol{\psi})_Q &- \big(q_h,\hat\phi^{-1/2}\nabla\cdot(d(\phi)\boldsymbol{\psi})\big) + \langle\kappa^2\phi^{-1}(d(\phi))^2\mathbf{v}_h\cdot\nu,\boldsymbol{\psi}\cdot\nu\rangle\\ &= (d(\phi)\mathbf{g},\boldsymbol{\psi}) - \big(\hat g_{\rm R},\hat\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_h)\big) - \big(d(\phi)\nabla(\phi^{-1/2}g_{\rm R}),\mathbf{v}_h\big)\ \forall\boldsymbol{\psi}\in\mathbf{V}_h,\end{aligned} \tag{6.22}$$

wherein we need any $H^1(\Omega)$-extension of $g_{\rm R}$ to the interior. In this case, we obtain the stability estimate (6.21) with the term $\beta_{\rm R}\|g_{\rm R}\|_{0,\partial\Omega}$ replaced by $\beta_{\rm R}\|g_{\rm R}\|_1$.

7. An analysis of the error of the formal method. A full analysis of the error of the practical method is an open question, and is complicated by the fact that ϕ may vanish at points of an element E, but ϕ_E may not vanish. Such an analysis is beyond the scope of this paper. However, we can give an idea of the errors to be expected by analyzing the formal system that treats the integrals and ϕ exactly, i.e., (6.1)–(6.2). For simplicity of exposition, we continue the discussion for the Robin system ($\beta_N = 0$).

Since $\mathbf{V}_h \subset H(\mathrm{div}) \subset H_{\phi,d}(\mathrm{div})$, we can take the difference of the true weak formulation (5.3)–(5.4) with discrete test functions and (6.1)–(6.2), which leads to the system

$$\begin{aligned}(\mathbf{v}-\mathbf{v}_h,\boldsymbol{\psi}) &- \big(q-q_h,\phi^{-1/2}\nabla\cdot(d(\phi)\boldsymbol{\psi})\big)\\ &+ \langle\kappa^2\phi^{-1}(d(\phi))^2(\mathbf{v}-\mathbf{v}_h)\cdot\nu,\boldsymbol{\psi}\cdot\nu\rangle = 0\quad\forall\boldsymbol{\psi}\in\mathbf{V}_h,\end{aligned} \tag{7.1}$$

$$\big(\phi^{-1/2}\nabla\cdot(d(\phi)(\mathbf{v}-\mathbf{v}_h)),w\big) + (q-q_h,w) = 0\quad\forall w\in W_h. \tag{7.2}$$

We modify this system by introducing our two projection operators to see that

$$\begin{aligned}&(\pi\mathbf{v}-\mathbf{v}_h,\boldsymbol{\psi}) - \big(\hat q-q_h,\mathcal{P}_{W_h}[\phi^{-1/2}\nabla\cdot(d(\phi)\boldsymbol{\psi})]\big) + \langle\kappa^2\phi^{-1}(d(\phi))^2(\pi\mathbf{v}-\mathbf{v}_h)\cdot\nu,\boldsymbol{\psi}\cdot\nu\rangle\\ &\quad= -(\mathbf{v}-\pi\mathbf{v},\boldsymbol{\psi}) + \big(q-\hat q,\phi^{-1/2}\nabla\cdot(d(\phi)\boldsymbol{\psi})\big) - \langle\kappa^2\phi^{-1}(d(\phi))^2(\mathbf{v}-\pi\mathbf{v})\cdot\nu,\boldsymbol{\psi}\cdot\nu\rangle,\\ &\big(\mathcal{P}_{W_h}[\phi^{-1/2}\nabla\cdot(d(\phi)(\pi\mathbf{v}-\mathbf{v}_h))],w\big) + (\hat q-q_h,w)\\ &\quad= -\big(\mathcal{P}_{W_h}[\phi^{-1/2}\nabla\cdot(d(\phi)(\mathbf{v}-\pi\mathbf{v}))],w\big).\end{aligned}$$

Assuming that $\mathbf{v}$ is sufficiently regular to compute $\pi\mathbf{v}$, the test functions $\boldsymbol{\psi} = \pi\mathbf{v} - \mathbf{v}_h \in \mathbf{V}_h$ and $w = \hat q - q_h + \mathcal{P}_{W_h}[\phi^{-1/2}\nabla\cdot(d(\phi)(\pi\mathbf{v}-\mathbf{v}_h))] \in W_h$, lead us to

$$\begin{aligned}&\|\pi\mathbf{v}-\mathbf{v}_h\|^2 + \|\hat q-q_h\|^2\\ &\qquad+ \|\mathcal{P}_{W_h}[\phi^{-1/2}\nabla\cdot(d(\phi)(\pi\mathbf{v}-\mathbf{v}_h))]\|^2 + \|\kappa\phi^{-1/2}d(\phi)(\pi\mathbf{v}-\mathbf{v}_h)\cdot\nu\|^2_{\partial\Omega}\\ &\quad\le C\big\{\|\mathbf{v}-\pi\mathbf{v}\|^2 + \|q-\hat q\|^2\\ &\qquad+ \|\mathcal{P}_{W_h}[\phi^{-1/2}\nabla\cdot(d(\phi)(\mathbf{v}-\pi\mathbf{v}))]\|^2 + \|\kappa\phi^{-1/2}d(\phi)(\mathbf{v}-\pi\mathbf{v})\cdot\nu\|^2_{\partial\Omega}\\ &\qquad+ \big|\big(q-\hat q,\phi^{-1/2}\nabla\cdot(d(\phi)(\pi\mathbf{v}-\mathbf{v}_h))\big)\big|\big\}.\end{aligned}$$

The last term on the right-hand side is troublesome. It arises because we could not substitute the test function $\phi^{-1/2}\nabla\cdot(d(\phi)(\pi\mathbf{v}-\mathbf{v}_h)) \notin W_h$ for w.

We continue by estimating

$$\begin{aligned}
&\big|\big(q-\hat{q},\phi^{-1/2}\nabla\cdot(d(\phi)(\pi\mathbf{v}-\mathbf{v}_h))\big)\big|\\
&\quad=\big|\big(q-\hat{q},(I-\mathcal{P}_{W_h})\big[\phi^{-1/2}\nabla d(\phi)\cdot(\pi\mathbf{v}-\mathbf{v}_h)+\phi^{-1/2}d(\phi)\nabla\cdot(\pi\mathbf{v}-\mathbf{v}_h)\big]\big)\big|\\
&\quad\le C\,\|q-\hat{q}\|\,\big\{\|\pi\mathbf{v}-\mathbf{v}_h\|+\|(I-\mathcal{P}_{W_h})[\phi^{-1/2}d(\phi)\nabla\cdot(\pi\mathbf{v}-\mathbf{v}_h)]\|\big\},
\end{aligned}$$

and, since $\nabla\cdot(\pi\mathbf{v}-\mathbf{v}_h)\in W_h$,

$$\begin{aligned}
&\|(I-\mathcal{P}_{W_h})[\phi^{-1/2}d(\phi)\nabla\cdot(\pi\mathbf{v}-\mathbf{v}_h)]\|\\
&\quad=\|[(I-\mathcal{P}_{W_h})\phi^{-1/2}d(\phi)]\nabla\cdot(\pi\mathbf{v}-\mathbf{v}_h)\|\\
&\quad\le\|(I-\mathcal{P}_{W_h})\phi^{-1/2}d(\phi)\|_{L^\infty(\Omega)}\|\nabla\cdot(\pi\mathbf{v}-\mathbf{v}_h)\|\\
&\quad\le Ch\,\|\phi^{-1/2}d(\phi)\|_{W^{1,\infty}(\Omega)}\|\nabla\cdot(\pi\mathbf{v}-\mathbf{v}_h)\|\\
&\quad\le C\|\pi\mathbf{v}-\mathbf{v}_h\|,
\end{aligned}$$

using [11] again for the approximation of the L^2-projection in L^∞ and an inverse estimate, assuming that the finite element mesh $\mathcal{T}_h$ is quasi-uniform.

Similar estimates can be shown to hold for the system using Neumann boundary conditions. In this case, the test function $\boldsymbol{\psi}=\pi\mathbf{v}-\mathbf{v}_h+\mathbf{v}_{\mathrm{N}}-\pi\mathbf{v}_{\mathrm{N}}\in\mathbf{V}_{h,0}$ is required.

THEOREM 7.1. *Let* (3.2) *and* (3.4) *hold,* $f\in L^2(\Omega)$, $d(\phi)\mathbf{g}\in(L^2(\Omega))^d$, *and assume that the finite element mesh* $\mathcal{T}_h$ *is quasi-uniform. Let* $(\mathbf{v},q)$ *be either the solution to* (4.2)–(4.3) *with* $\mathbf{v}_N\in H_{\phi,d}(\mathrm{div};\Omega)$ *(and set* $\beta_{\mathrm{N}}=1$, $\beta_{\mathrm{R}}=0$*) or the solution to* (5.3)–(5.4) *with* $g_{\mathrm{R}}\in H^{1/2}(\partial\Omega)$ *(and set* $\beta_{\mathrm{N}}=0$, $\beta_{\mathrm{R}}=1$*). Let* $(\mathbf{v}_h,q_h)$ *be the solution to the formal method* (6.1)–(6.2)*. Assume that* $\mathbf{v},\beta_{\mathrm{N}}\mathbf{v}_{\mathrm{N}}\in H(\mathrm{div};\Omega)\cap L^{2+\epsilon}(\Omega)$ *for some* $\epsilon>0$*. Then*

$$\begin{aligned}
&\|\mathbf{v}-\mathbf{v}_h\|+\|q-q_h\|+\|\mathcal{P}_{W_h}[\phi^{-1/2}\nabla\cdot(d(\phi)(\mathbf{v}-\mathbf{v}_h))]\|\\
&\qquad+\beta_{\mathrm{R}}\|\kappa\phi^{-1/2}d(\phi)(\mathbf{v}-\mathbf{v}_h)\cdot\nu\|_{\partial\Omega}\\
&\quad\le C\big\{\|\mathbf{v}-\pi\mathbf{v}\|+\|q-\hat{q}\|+\|\mathcal{P}_{W_h}[\phi^{-1/2}\nabla\cdot(d(\phi)(\mathbf{v}-\pi\mathbf{v}))]\|\\
&\qquad+\beta_{\mathrm{R}}\|\kappa\phi^{-1/2}d(\phi)(\mathbf{v}-\pi\mathbf{v})\cdot\nu\|_{\partial\Omega}+\beta_{\mathrm{N}}\|\mathbf{v}_{\mathrm{N}}-\pi\mathbf{v}_{\mathrm{N}}\|_{H_{\phi,d}(\mathrm{div};\Omega)}\big\}\\
&\quad\le C\big\{\|\mathbf{v}-\pi\mathbf{v}\|_{H(\mathrm{div};\Omega)}+\|q-\hat{q}\|\\
&\qquad+\beta_{\mathrm{R}}\|\kappa(\mathbf{v}-\pi\mathbf{v})\cdot\nu\|_{\partial\Omega}+\beta_{\mathrm{N}}\|\mathbf{v}_{\mathrm{N}}-\pi\mathbf{v}_{\mathrm{N}}\|_{H(\mathrm{div};\Omega)}\big\}.
\end{aligned}\tag{7.3}$$

If the solution is sufficiently regular, the approximation is of order $\mathcal{O}(h)$. We might expect that the practical method (6.7)–(6.8) that we proposed also achieves $\mathcal{O}(h)$ convergence. Moreover, we might expect to see superconvergence of order $\mathcal{O}(h^2)$ for sufficiently regular solutions [25]. We see evidence of these assertions in the numerical results presented below.

8. Some closed form solutions in one dimension. Before presenting numerical results, it is instructive to consider a few closed form solutions to the problem in one dimension. Let $\Omega=(-1,1)$, $a(\phi)=b(\phi)=\phi$ (so $d(\phi)=\phi$, $c(\phi)=1$), $\mathbf{g}=0$, and $\tilde{f}=\phi^{-1/2}f$, and reduce the mixed system (1.1)–(1.3) to

$$-(\phi^2p')'+\phi p=\phi\tilde{f},\quad -1<x<1,\tag{8.1}$$

$$\phi^{3/2}(-1)\,p'(-1)=\phi^{3/2}(1)\,p'(1)=0.\tag{8.2}$$

This is a Sturm-Liouville problem. By our energy estimates (2.4), we require that $u=v=-\phi p'\in L^2(-1,1)$ when $\phi^{1/2}\tilde{f}\in L^2(-1,1)$.

For $\alpha > 0$, let us simplify to the porosity

$$\phi(x) = \begin{cases} 0, & x < 0, \\ x^\alpha, & x > 0. \end{cases} \tag{8.3}$$

The conditions (3.2) and (3.4) hold if and only if $\alpha \geq 2$. Now (8.1)–(8.2) becomes

$$-x^\alpha\, p'' - 2\alpha\, x^{\alpha-1}\, p' + p = \tilde{f}, \quad 0 < x < 1, \quad \text{and} \quad p'(1) = 0.$$

When $\alpha = 2$, we have the Euler equation

$$-x^2\, p'' - 4x\, p' + p = \tilde{f}, \quad 0 < x < 1. \tag{8.4}$$

In this case the Euler exponents satisfy $r(r-1) + 4r - 1 = 0$, and so

$$r_1 = \frac{-3 + \sqrt{13}}{2} \approx 0.3 > 0 \quad \text{and} \quad r_2 = \frac{-3 - \sqrt{13}}{2} \approx -3.3 < 0, \tag{8.5}$$

and the solution to the homogeneous equation is $p_{\text{hom}}(x) = c_1 x_1^r + c_2 x^{r_2}$. The boundary condition and the requirement that $u = -\phi p' \in L^2(0,1)$ shows that the solution is unique. Variation of parameters gives the nonhomogeneous solution as

$$p(x) = \frac{-1}{r_1 - r_2}\biggl\{ x^{r_1}\biggl(\int \frac{\tilde{f}(y)}{y^{r_1+1}}\, dy + c_1 \biggr) - x^{r_2}\biggl(\int \frac{\tilde{f}(y)}{y^{r_2+1}}\, dy + c_2 \biggr)\biggr\}.$$

If we restrict to

$$\tilde{f}(x) = x^\beta, \quad 0 < x < 1,$$

then, provided $\beta \neq r_1, r_2$ and $0 < x < 1$, we have the closed form solutions

$$p(x) = \frac{-x^\beta}{(\beta - r_1)(\beta - r_2)} + C_1\, x^{r_1} + C_2\, x^{r_2}.$$

To get $f = \phi^{1/2} \tilde{f} = x^{1+\beta} \in L^2(0,1)$, restrict to $\beta > -3/2$. Then $u = -\phi p' \in L^2(0,1)$ implies that $C_2 = 0$, and the boundary condition determines C_1. The solution is

$$q(x) = x\, p(x) \quad \text{and} \quad p(x) = \begin{cases} 0, & -1 < x \leq 0, \\ \dfrac{\beta x^{r_1} - r_1 x^\beta}{r_1(\beta - r_1)(\beta - r_2)}, & 0 < x < 1, \end{cases} \tag{8.6}$$

$$v(x) = u(x) = \begin{cases} 0, & -1 < x \leq 0, \\ \dfrac{-\beta(x^{r_1+1} - x^{\beta+1})}{(\beta - r_1)(\beta - r_2)}, & 0 < x < 1. \end{cases} \tag{8.7}$$

9. Some numerical results. In this section we test the convergence of our proposed numerical scheme (6.7)–(6.8), using Dirichlet boundary conditions, but *without* using the stabilizing variant (6.22). We fix the domain $\Omega = (-1,1)^2$ and use a uniform rectangular mesh of $n = 1/h$ elements in each coordinate direction.

We implement the tests in terms of manufactured solutions in which closed form expressions for ϕ and p are given, and from these f and Dirichlet boundary conditions (i.e., $\kappa = 0$) are computed. In all tests, we take $\mathbf{g} = 0$ and $a(\phi) = b(\phi) = \phi$ (so $d(\phi) = \phi$, $c(\phi) = 1$, and $\mathbf{u} = \mathbf{v}$).

We use discrete L^2-norms to measure the relative errors. For p and q, we use the midpoint quadrature rule applied to the L^2-norm, which gives an approximation to $\|\hat{q} - q_h\|$ and $\|\hat{p} - p_h\|$. For $\mathbf{v}$, we use the trapezoidal rule applied to the $(L^2)^2$-norm, which is effectively like the norm $\|\pi\mathbf{v} - \mathbf{v}_h\|$. In both cases, these are norms for which superconvergence might be expected.

TABLE 9.1

Euler's equation. Shown are the relative discrete L^2-norm errors of q, p, and $\mathbf{u}$ for various number of elements $n \times n$ and for four values of β. The convergence rate corresponds to a superconvergent approximation, restricted by the regularity of the true solution.

		scaled pressure q		pressure p		velocity u	
β	n	error	rate	err	rate	err	rate
0.5	32	0.002043	—	0.006756	—	0.007438	—
	64	0.000642	1.669	0.004341	0.638	0.002387	1.640
	128	0.000199	1.691	0.002724	0.672	0.000754	1.663
	256	0.000061	1.709	0.001681	0.697	0.000235	1.681
	512	0.000018	1.723	0.001024	0.716	0.000073	1.695
-0.5	32	0.001913	—	0.040343	—	0.013276	—
	64	0.000802	1.254	0.039971	0.013	0.006749	0.976
	128	0.000358	1.166	0.039289	0.025	0.003426	0.978
	256	0.000167	1.101	0.038474	0.030	0.001731	0.985
	512	0.000080	1.060	0.037617	0.033	0.000872	0.990
-1.0	32	0.006379	—	0.155115	—	0.015402	—
	64	0.004849	0.396	0.164987	-0.089	0.010550	0.546
	128	0.003526	0.460	0.170768	-0.050	0.007338	0.524
	256	0.002521	0.484	0.173955	-0.027	0.005142	0.513
	512	0.001790	0.494	0.175645	-0.014	0.003618	0.507
-1.5	32	0.060245	—	0.273779	—	0.004816	—
	64	0.059596	0.016	0.278083	-0.023	0.003470	0.473
	128	0.058620	0.024	0.279416	-0.007	0.002856	0.281
	256	0.057593	0.026	0.279819	-0.002	0.002507	0.188
	512	0.056590	0.025	0.279939	-0.001	0.002281	0.137

9.1. A simple Euler's equation in one dimension. For a given ϕ, it is difficult to determine the exact regularity of the solution. We begin with a test case corresponding to our closed form solution (8.6)–(8.7) of the Euler equation (8.4). In this case, it is easy to see that in terms of the potential singularity near $x = 0$, $q \sim \mathbf{u} \sim |x|^{1.3} + |x|^{1+\beta}$ and $p \sim |x|^{0.3} + |x|^{\beta}$, and so, approximately,

$$q, u \in H^{\min(1.8, 3/2+\beta)} \quad \text{and} \quad p \in H^{\min(0.8, 1/2+\beta)}.$$

Since $\phi(x) = x^2$, $0 < x < 1$, we have that $\phi^{-1/2}\phi' = 2 \in L^\infty(0,1)$ and our conditions (3.2) and (3.4) on ϕ are satisfied.

We consider four values of β (which is the parameter in the source function $\tilde{f} = x^\beta$ (or $f = x^{1+\beta}$), $0 < x < 1$), $\beta = 1/2$, $-1/2$, -1, and $-3/2$. The numerical results are presented in Table 9.1. Based on the regularity of the solution, if the solution exhibited superconvergence, we would expect the order of convergence for q and u to be $\mathcal{O}(h^{1.8})$ for $\beta = 1/2$, $\mathcal{O}(h^1)$ for $\beta = -1/2$, $\mathcal{O}(h^{1/2})$ for $\beta = -1$, and no convergence for $\beta = -3/2$. These rates are seen, approximately, in the numerical results. Moreover, the order of convergence for p should be $\mathcal{O}(h^{0.8})$ for $\beta = 1/2$ and no convergence for the other values of β, which we also see approximately.

We remark that the convergence rate is slightly better if instead of using (6.15), we simply set $b_E = (f, w_E)$. This, of course, would lead to a loss of strict local mass conservation.

TABLE 9.2

Smooth p two-dimensional test. Shown are the relative discrete L^2-norm errors of q, p, and $\mathbf{u}$ for various number of elements $n \times n$ and for three values of α defining ϕ. The convergence rate is better than expected for low values of α.

		scaled pressure q		pressure p		velocity $\mathbf{u}$	
α	n	error	rate	err	rate	err	rate
2	32	0.012878	—	0.020996	—	0.029391	—
	64	0.003260	1.982	0.007574	1.471	0.009392	1.646
	128	0.000825	1.983	0.002655	1.512	0.002791	1.751
	256	0.000209	1.979	0.000924	1.523	0.000795	1.811
	512	0.000054	1.966	0.000322	1.521	0.000221	1.849
1	32	0.007507	—	0.008594	—	0.023786	—
	64	0.001929	1.961	0.002941	1.547	0.007442	1.676
	128	0.000493	1.966	0.001001	1.555	0.002182	1.770
	256	0.000127	1.955	0.000343	1.545	0.000616	1.824
	512	0.000034	1.924	0.000119	1.533	0.000170	1.858
0.25	32	0.007443	—	0.009351	—	0.019810	—
	64	0.004953	0.588	0.006521	0.520	0.008355	1.246
	128	0.003549	0.481	0.004687	0.476	0.004913	0.766
	256	0.002528	0.490	0.003348	0.485	0.003429	0.519
	512	0.001788	0.500	0.002380	0.493	0.002469	0.474
0.125	32	0.066864	—	0.082809	—	0.048566	—
	64	0.053265	0.328	0.065477	0.339	0.038811	0.323
	128	0.042347	0.331	0.051784	0.338	0.032259	0.267
	256	0.033806	0.325	0.041165	0.331	0.026911	0.262
	512	0.027147	0.317	0.032935	0.322	0.022434	0.263

9.2. A smooth solution test in two-dimensions. For the next series of tests, we assume that $p = \cos(6xy^2)$ is smooth and that ϕ is given by

$$\phi = \begin{cases} 0 & x \leq -3/4 \text{ or } y \leq -3/4, \\ (x+3/4)^\alpha (y+3/4)^{2\alpha} & \text{otherwise.} \end{cases} \tag{9.1}$$

We note that $\phi^{-1/2}\nabla\phi$ is in $\big(L^\infty((-1,1)^2)\big)^2$ if and only if $\alpha \geq 2$. Nevertheless, we consider the four values $\alpha = 2$, 1, 1/4, and 1/8. Results are given in Table 9.2. We see good convergence for this problem for $\alpha = 2$ and 1, and some degradation for the smaller values of α. It appears that condition (3.2) (the second part of which is precisely the condition (3.4) when $d(\phi) = \phi$) may be overly restrictive for convergence. In fact, it may be enough that $\phi^{-1/2}\nabla\phi \in \big(L^2((-1,1)^2)\big)^2$, which is true here if and only if $\alpha > 1$.

We depict the solution p and q in Figure 9.1. Although p was chosen to be smooth, we have displayed $p = 0$ in the one-phase region, since it is ill-defined there. Therefore, p is not smooth on the boundary between the one and two phase regions $\mathcal{B} = \{x = -3/4, y \geq -3/4\} \cup \{x \geq -3/4, y = -3/4\}$. We also display the scaled pressure q, which is well behaved for $\alpha = 2$ and degenerates near $\mathcal{B}$ as α decreases (i.e., as $\phi^{1/2}\nabla\phi$ loses its regularity).

The reader should note that $\mathcal{B}$ lies on a grid line. If we take an odd number of elements, we will avoid this. Results are shown in Table 9.3. When $\alpha = 2$, we see similar errors and rates of convergence as for the case of $\mathcal{B}$ being resolved by the grid

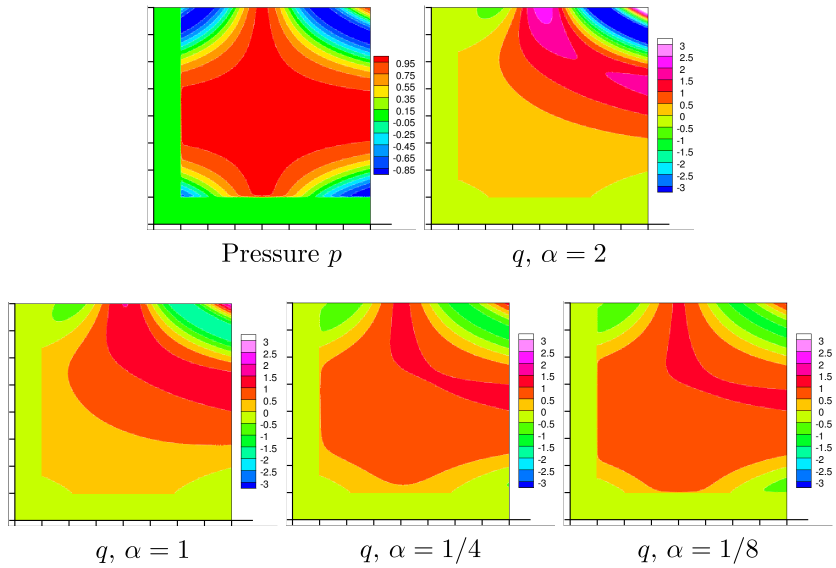


FIG. 9.1. *Smooth p two-dimensional test. Shown are the pressure p and scaled pressure q for various four values of α defining ϕ. The pressure is smooth, except on the boundary of the support of ϕ (i.e., $x = -3/4$ or $y = -3/4$. The scaled pressure becomes less regular as α decreases near the boundary.*

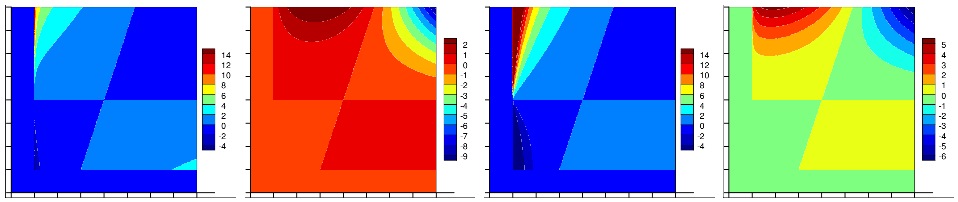
TABLE 9.3
Smooth p two-dimensional test. Shown are the relative discrete L^2-norm errors of q, p, and $\mathbf{u}$ for various odd numbers of elements $n \times n$ and for $\alpha = 2$ and 0.25 defining ϕ. The convergence is similar to the case of grids that resolve the boundary between the one and two phase regions when $\alpha = 2$, but not for $\alpha = 1/4$.

		scaled pressure q		pressure p		velocity $\mathbf{u}$	
α	n	error	rate	err	rate	err	rate
2	33	0.012137	—	0.021447	—	0.028001	—
	65	0.003171	1.980	0.007832	1.486	0.009146	1.651
	129	0.000817	1.979	0.002769	1.517	0.002753	1.752
	257	0.000210	1.971	0.000969	1.523	0.000790	1.811
	513	0.000055	1.938	0.000339	1.520	0.000220	1.850
0.25	33	0.031315	—	0.047229	—	0.039155	—
	65	0.020907	0.596	0.029933	0.673	0.024967	0.664
	129	0.014105	0.574	0.019492	0.626	0.017147	0.548
	257	0.009588	0.560	0.012969	0.591	0.012062	0.510
	513	0.006566	0.548	0.008778	0.565	0.008545	0.499

in Table 9.2. However, the errors are worse for the more challenging case of $\alpha = 1/4$, although the convergence rates seem to settle to about the same values.

9.3. A nonsmooth solution test in two-dimensions. For the final series of tests, we assume that ϕ is again given by (9.1), but we impose the nonsmooth pressure solution

$$p = y(y - 3x)(x + 3/4)^{\beta}, \quad \beta = -1/4 \text{ or } -3/4. \tag{9.2}$$



p, $\beta = -1/4$ q, $\beta = -1/4$ p, $\beta = -3/4$ q, $\beta = -3/4$

FIG. 9.2. *Nonsmooth p two-dimensional test. Shown are the pressure p and scaled pressure q for various four values of α defining ϕ. The pressure is smooth, except on the boundary of the support of ϕ (i.e., $x = -3/4$ or $y = -3/4$. The scaled pressure becomes less regular as α decreases near the boundary.*

TABLE 9.4
Nonsmooth p two-dimensional test. Shown are the relative discrete L^2-norm errors of q, p, and $\mathbf{u}$ for various odd numbers of elements $n \times n$ and for $\beta = -1/4$ and $-3/4$ defining p in (9.2).

		scaled pressure q		pressure p		velocity $\mathbf{u}$	
β	n	error	rate	err	rate	err	rate
$-1/4$	33	0.005050	—	0.045199	—	0.002885	—
	65	0.002193	1.231	0.034160	0.413	0.000786	1.918
	129	0.000944	1.230	0.027326	0.326	0.000211	1.919
	257	0.000402	1.239	0.022448	0.285	0.000056	1.925
	513	0.000171	1.237	0.018661	0.267	0.000015	1.906
$-3/4$	33	0.004155	—	0.193534	—	0.004991	—
	65	0.002554	0.718	0.184637	0.069	0.002113	1.268
	129	0.001608	0.675	0.179129	0.044	0.000935	1.190
	257	0.000991	0.702	0.175644	0.029	0.000432	1.120
	513	0.000601	0.724	0.173380	0.019	0.000210	1.044

This pressure and the scaled pressure $q = \phi^{1/2}p$ are depicted in Fig. 9.2, where one can see clearly the degeneracy in p near $x = -3/4$ and that q is better behaved. In the case $\beta = -1/4$, q and $\mathbf{u}$ lie in $H^{1.25}$ and are relatively smooth, whereas when $\beta = -3/4$, q and $\mathbf{u}$ lie only in $H^{0.75}$. We use grids that do *not* resolve the interface between the one and two-phase regions. The discrete errors and convergence rates are shown in Table 9.4. The scaled pressure converges as expected, and the velocity seems to be converging a bit better than expected. The pressure barely converges at all.

10. Summary and conclusions. We considered a two phase mixture of matrix and fluid melt, which can degenerate as the porosity ϕ vanishes. Energy estimates suggested that the pressure p is uncontrolled; moreover, an equation is lost when ϕ vanishes, making it difficult to handle the equations numerically.

We changed variables to a scaled set that remain bounded in the energy estimates. To formulate a well posed mixed weak problem in the scaled variables, we defined precisely the Hilbert space $H_{\phi,d}(\mathrm{div})$ within which the scaled pressure resides. The key hypotheses were that $\phi^{-1/2}d(\phi) \in W^{1,\infty}(\Omega)$ and $\phi^{-1/2}\nabla d(\phi) \in (L^\infty(\Omega))^d$. Moreover, a normal trace operator was defined to handle boundary conditions. Existence and uniqueness of a solution to the weak formulation was obtained from the Lax-Milgram Theorem.

We defined a theoretically simple, formal mixed finite element method based on lowest order Raviart-Thomas spaces. The method is stable, and an error analysis showed optimal rates of convergence for sufficiently smooth solutions. We modified the method to make it more practical in implementation, producing a cell-centered finite difference method that is stable and locally mass conservative.

In a simple case in one dimension, the equations reduce to an Euler equation for which a closed form solution was computed. Numerical tests of this problem showed that the practical method achieved optimal rates of convergence with respect to the regularity of the solution; in fact, superconvergence of the velocity and scaled pressure were observed. Convergence of the true pressure was relatively poor.

A numerical test for a two-dimensional problem using $d(\phi) = \phi$ also exhibited superconvergence. To see optimal convergence rates in this test, it was necessary that $\phi^{1/2}\nabla\phi \in L^2$, which is weaker than being in L^∞. Moreover, meshes that did not match the boundary of the one-phase region showed no degradation of results from cases with meshes that match this boundary.

To summarize in brief, we developed a model, linear, degenerate elliptic problem related to the simulation of certain two-phase mixtures of partially melted materials. When the porosity ϕ is reasonable but perhaps degenerate on a set of positive measure, we developed a stable and convergent formal mixed finite element method. We also developed an easy to implement, locally mass conservative, and stable cell-centered finite difference method.

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REFERENCES

[1] E. Aharonov, J. A. Whitehead, P. B. Kelemen, and M. Spiegelman, *Channeling instability of upwelling melt in the mantle*, J. Geophysical Research, 100 (1995), pp. 20,433–20,450.

[2] T. Arbogast and Z. Chen, *On the implementation of mixed methods as nonconforming methods for second order elliptic problems*, Math. Comp., 64 (1995), pp. 943–972.

[3] T. Arbogast, M. F. Wheeler, and I. Yotov, *Mixed finite elements for elliptic problems with tensor coefficients as cell-centered finite differences*, SIAM J. Numer. Anal., 34 (1997), pp. 828–852.

[4] D. Arroyo, A. Bespalov, and N. Heuer, *On the finite element method for elliptic problems with degenerate and singular coefficients*, Math. Comp., 76 (2007), pp. 509–537.

[5] A. Aschwanden, E. Bueler, C. Khroulev, and H. Blatter, *An enthalpy formulation for glaciers and ice sheets*, J. Glaciology, 58 (2012), pp. 441–457.

[6] J. Bear, *Dynamics of Fluids in Porous Media*, Dover, New York, 1972.

[7] J. Bear and A. H.-D. Cheng, *Modeling Groundwater Flow and Contaminant Transport*, Springer, New York, 2010.

[8] S. Bidwell, M. E. Hassell, and C. R. Westphal, *A weighted least squares finite element method for elliptic problems with degenerate and singular coefficients*, Math. Comp., 82 (2013), pp. 673–688.

[9] F. Brezzi and M. Fortin, *Mixed and hybrid finite element methods*, Springer-Verlag, New York, 1991.

[10] Z. Chen, G. Huan, and Y. Ma, *Computational Methods for Multiphase Flows in Porous Media*, vol. 2 of Computational Science and Engineering Series, SIAM, Philadelphia, 2006.

[11] J. Douglas, Jr., T. Dupont, and L. Wajlbin, *The stability in L^q of the L^2-projection into finite element function spaces*, Numer. Math., 23 (1975), pp. 193–197.

[12] K. Eriksson and V. Thomée, *Galerkin methods for singular boundary value problems in one space dimension*, Math. Comp., 42 (1984), pp. 345–367.

[13] A. C. Fowler, *On the transport of moisture in polythermal glaciers*, Geophys. Astrophys. Fluid Dynamics, 28 (1984), pp. 99–140.

[14] M. A. Hesse, A. R. Schiemenz, Y. Liang, and E. M. Parmentier, *Compaction-dissolution waves in an upwelling mantle column*, Geophysical J. Int., 187 (2011), pp. 1057–1075.

[15] I. J. Hewitt and A. C. Fowler, *Partial melting in an upwelling mantle column*, Proc. R. Soc. A, 464 (2008), pp. 2467–2491.

[16] R. F. Katz, *Magma dynamics with the enthalpy method: Benchmark solutions and magmatic focusing at mid-ocean ridges*, J. Petrology, 49 (2008), pp. 2099–2121.

[17] L. W. Lake, *Enhanced Oil Recovery*, Prentice Hall, Englewood Cliffs, New Jersey, 1989.

[18] H. Li, *A-priori analysis and the finite element method for a class of degenerate elliptic equations*, Math. Comp., 78 (2009), pp. 713–737.

[19] D. Marini and P. Pietra, *Mixed finite element approximation of a degenerate elliptic problem*, Numer. Math. 71 (1995, 71 (1995), pp. 225–236.

[20] R. A. Raviart and J. M. Thomas, *A mixed finite element method for 2nd order elliptic problems*, in Mathematical Aspects of Finite Element Methods, I. Galligani and E. Magenes, eds., no. 606 in Lecture Notes in Math., Springer-Verlag, New York, 1977, pp. 292–315.

[21] J. E. Roberts and J.-M. Thomas, *Mixed and hybrid methods*, in Handbook of Numerical Analysis, P. G. Ciarlet and J. L. Lions, eds., vol. 2, Elsevier Science Publishers B.V. (North-Holland), Amsterdam, 1991, ch. Finite Element Methods (Part 1), pp. 523–639.

[22] T. F. Russell and M. F. Wheeler, *Finite element and finite difference methods for continuous flows in porous media*, in The Mathematics of Reservoir Simulation, R. E. Ewing, ed., no. 1 in Frontiers in Applied Mathematics, SIAM, Philadelphia, 1983, pp. 35–106, Ch. II.

[23] A. L. Taicher, T. Arbogast, and M. A. Hesse, *A mixed framework for two-phase Darcy-Stokes mixtures*, In preparation, (2014).

[24] C. J. van der Veen, *Fundamentals of Glacier Dynamics*, CRC Press, second ed., 2013.

[25] A. Weiser and M. F. Wheeler, *On convergence of block-centered finite-differences for elliptic problems*, SIAM. J. Numer. Anal., 25 (1988), pp. 351–375.

[26] W. Zhu, G. Gaetani, F. Fusseis, L. Montesi, and F. D. Carlo, *Microtomography of partiallty molten rocks: three dimensional melt distribution in mantle peridotite*, Science, 332 (2011), pp. 88–91.