A NEW CLASS OF ADAPTIVE DISCONTINUOUS PETROV–GALERKIN FINITE ELEMENT METHODS WITH APPLICATION TO SINGULARLY PERTURBED PROBLEMS

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Short course on DPG Method
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Lectures

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- Systematic choice of test norms. Robustness.
- Convergence proofs.

PETROV GALERKIN METHOD WITH OPTIMAL TEST FUNCTIONS

$$\Updownarrow$$

LEAST SQUARES (WITH A TWIST)

Least squares and optimal test functions

$$\left\{ \begin{array}{l} u \in U \\ b(u,v) = l(v) \quad v \in V \end{array} \right. \quad \Leftrightarrow \quad \begin{array}{l} Bu = l \quad B : U \to V' \\ < Bu, v >= b(u,v) \end{array}$$

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- **Least squares:** $U_h \subset U$,

$$\frac{1}{2}\|Bu_h - l\|^2_{V'} \to \min_{u_h \in U_h}$$

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$$R_V : V \to V', < R_V v, \delta v >= (v, \delta v)_V$$

is an *isometry*, $\|R_V v\|_{V'} = \|v\|_V$.

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- **Least squares reformulated:**

$$\frac{1}{2}\|Bu_h - l\|^2_{V'} = \frac{1}{2}\|R_V^{-1}(Bu_h - l)\|^2_V \to \min_{u_h \in U_h}$$

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$$(R_V^{-1}(Bu_h - l), R_V^{-1}B\delta u_h)_V = 0 \quad \delta u_h \in U_h$$

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$$(R_V^{-1}(Bu_h - l), R_V^{-1}B\delta u_h)_V = 0 \quad \delta u_h \in U_h$$

or

$$b(u_h, v_h) = l(v_h)$$

where

$$\begin{cases} v_h \in V \\ (v_h, \delta v)_V = b(\delta u_h, \delta v) \quad \delta v \in V \end{cases}$$

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- The energy norm of the FE error $u - u_h$ equals the residual and can be computed,

$$\|u-u_h\|_E = \|Bu - Bu_h\|_{V'} = \|l - Bu_h\|_{V'} = \|R_V^{-1}(l - Bu_h)\|_V = \|\psi\|_V$$

where the *error representation function* ψ comes from

$$\begin{cases} \psi \in V \\ (\psi, \delta v)_V = < l - Bu_h, \delta v >= l(\delta v) - b(u_h, \delta v), \quad \delta v \in V \end{cases}$$

(no need for a-posteriori error estimation, note the connection with implicit a-posteriori error estimation techniques...)

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Indeed,

$$\sup_u \sup_v \frac{|b(u,v)|}{\|u\| \, \|v\|_V} = \sup_v \sup_u \frac{|b(u,v)|}{\|u\| \, \|v\|_V} = \sup_v \frac{\|v\|_V}{\|v\|_V} = 1$$

implies

$$\sup_u \frac{\|u\|_E}{\|u\|} = 1 \quad \implies \quad \|u\|_E \le \|u\|$$

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  $$\|u\|_E = \|u\|_U$$

Also,

$$\inf_u \sup_v \frac{|b(u,v)|}{\|u\| \, \|v\|_V} = \inf_v \sup_u \frac{|b(u,v)|}{\|u\| \, \|v\|_V} = \inf_v \frac{\|v\|_V}{\|v\|_V} = 1$$

implies

$$\inf_u \frac{\|u\|_E}{\|u\|} = 1 \quad \implies \quad \|u\| \leq \|u\|_E$$

Petrov–Galerkin Method with Optimal Test Functions Abstract B^3 Framework (Repetitio Mater Studiorum Est)

Abstract Variational Problem

$$\begin{cases} u \in U \\ b(u,v) = l(v) \quad \forall v \in V \end{cases} \Leftrightarrow \begin{array}{l} Bu = l \quad B : U \to V' \\ < Bu, v >= b(u,v) \, \forall v \in V \end{array}$$

where

- U, V are Hilbert spaces,
- $b(u,v)$ is a continuous bilinear form on $U \times V$,

$$|b(u,v)| \le M \|u\|_U \, \|v\|_V$$

that satisfies the inf-sup condition ($\Leftrightarrow B$ is bounded below),

$$\inf_{\|u\|_U = 1} \sup_{\|v\|_V = 1} |b(u,v)| =: \gamma > 0$$

- $l \in V'$ represents the load and satisfies the compatibility condition $l(v) = 0, \forall v \in V_0$ where

$$V_0 := \{ v \in V \ : \ b(u,v) = 0 \quad \forall u \in U \}$$

Banach Closed Range and Babuška Theorems

Let $b(u,v), u \in U, v \in V$ be a continuous bilinear form, $|b(u,v)| \leq M\|u\|_U\|v\|_V$, $l \in V'$. Consider the variational problem,

$$\begin{cases} u \in \tilde{u}_D + U \\ b(u,v) = l(v), \quad \forall v \in V \end{cases}$$

The inf-sup condition

$$\sup_{v \in V} \frac{|b(u,v)|}{\|v\|_V} \geq \gamma \|u\|_U$$

implies stability

$$\|u\|_U \leq \gamma^{-1}\|l\|_{V'}$$

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$$\left\{ \begin{array}{l} u_{hp} \in \tilde{u}_D + U_{hp} \\ b(u_{hp}, v) = l(v), \quad \forall v \in V_{hp} \end{array} \right.$$

The discrete inf-sup condition

$$\sup_{v \in V_{hp}} \frac{|b(u_{hp}, v)|}{\|v\|_V} \ge \gamma_{hp}\|u_{hp}\|_U$$

implies discrete stability

$$\|u_{hp}\|_U \le \gamma_{hp}^{-1}\|l\|_{V'_{hp}}$$

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implies discrete stability

$$\|u_{hp}\|_U \leq \gamma_{hp}^{-1} \|l\|_{V'_{hp}}$$

and convergence

$$\|u - u_{hp}\|_U \leq \frac{M}{\gamma_{hp}} \inf_{w_{hp} \in \tilde{u}_D + U_{hp}} \|u - w_{hp}\|_U$$

Coercive Problems

If $U = V$, and the bilinear (sesquilinear) form is coercive,

$$b(u, u,) \geq \alpha \|u\|_U^2$$

Then both continuous and discrete stability constants are bounded below by α,

$$\gamma, \gamma_{hp} \geq \alpha \quad \implies \quad \frac{1}{\gamma_{hp}} \leq \frac{1}{\alpha}$$

Thus, for coercive problems, stability is guaranteed automatically.

Ritz and Bubnov-Galerkin Methods

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- then

$$\left\{ \begin{array}{l} u \in V \\ J(u) := \frac{1}{2} b(u,u) - l(u) \to \text{min} \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{l} u \in V \\ b(u,v) = l(v), \ v \in V \end{array} \right.$$

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- and, Bubnov-Galerkin method delivers the *best approximation error* in the energy norm,

$$\left\{ \begin{array}{l} u_h \in V_h \subset V \\ b(u_h, v_h) = l(v_h), \; v_h \in V_h \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{l} u_h \in V_h \\ \|u - u_h\|_E \to \text{min} \end{array} \right.$$

  where $\|v\|_E^2 = b(v,v)$.

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- You cannot do better ! (in energy norm...)

History of Discrete Stability by Demkowicz

1910 –– (Bubnov) Galerkin method

1954 –– numerical flux of P. Lax

1959 –– Petrov–Galerkin method

1964 –– Cea's lemma

1969 –– Mikhlin's asymptotic stability

1971 –– Babuska's theorem

1974 –– Brezzi's theory

1980 –– Barett and Morton use Petrov–Galerkin to symmetrize

1981 –– SUPG method of Brooks and Hughes, stabilized methods

1985 –– D and Oden use PG to change the norm of convergence

1986 –– Franca and Russo – bubble methods

1989 –– DPG method of Cockburn and Shu

2009 –– D and Gopalakrishnan – DPG method with optinal test functions

Energy Norm

The supremum in the inf-sup condition defines an equivalent, problem-dependent *energy (residual) norm*,

$$\|u\|_E := \sup_{\|v\|=1} |b(u,v)| = \|Bu\|_{V'}$$

For the energy norm, $M = \gamma = 1$. Recalling that the Riesz operator is an isometry form V into V', we may characterize the energy norm in an equivalent way as

$$\|u\|_E = \|v_u\|_V$$

where v_u is the solution of the variational problem,

$$\begin{cases} v_u \in V \\ (v_u, \delta v)_V = b(u, \delta v) \quad \forall \delta v \in V \end{cases}$$

Optimal Test Functions

Select your favorite trial basis functions: $e_j,\ j = 1, \ldots, N$. For each function e_j, introduce a corresponding *optimal test (basis) function* $\bar{e}_j \in V$ that realizes the supremum,

$$|b(e_j, \bar{e}_j)| = \sup_{\|v\|_V = 1} |b(e_j, v)|$$

i.e. it solves the variational problem,

$$\begin{cases} \bar{e}_j \in V \\ (\bar{e}_j, \delta v)_V = b(e_j, \delta v) \quad \forall \delta v \in V \end{cases}$$

Define the discrete test space as $\bar{V}_{hp} := \text{span}\{\bar{e}_j,\ j = 1, \ldots, N\} \subset V$. It follows from the construction of the optimal test functions that the *discrete* inf-sup constant

$$\inf_{\|u_{hp}\|_E = 1} \sup_{\|v_{hp}\| = 1} |b(u_{hp}, v_{hp})| = 1$$

The Best Approximation

Consequently, Babuška's Theorem

$$\|u - u_{hp}\|_E \le \frac{M}{\gamma_{hp}} \inf_{w_{hp} \in U_{hp}} \|u - w_{hp}\|_E$$

implies that

$$\|u - u_{hp}\|_E \le \inf_{w_{hp} \in U_{hp}} \|u - w_{hp}\|_E$$

i.e., the method delivers the *best approximation error* in the energy norm.

Stiffness Matrix Is Symmetric and Positive Definite

$$b(e_i, \bar{e}_j) = (\bar{e}_i, \bar{e}_j)_V = (\bar{e}_j, \bar{e}_i)_V = b(e_j, \bar{e}_i)$$

Energy Norm of FE Error $e_{hp} = u - u_{hp}$

can be computed *without* knowing the exact solution.

$$\left\{ \begin{array}{l} v_{e_{hp}} \in V \\ (v_{e_{hp}}, \delta v)_V = b(u - u_{hp}, \delta v) = l(\delta v) - b(u_{hp}, \delta v) \quad \forall \delta v \in V \end{array} \right.$$

We have then

$$\|e_{hp}\|_E = \|v_{e_{hp}}\|_V$$

We shall call $v_{e_{hp}}$ *the error representation function*

Note: No need for an a-posteriori error estimation.

Relation with Least Squares

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This is exactly our DPG method

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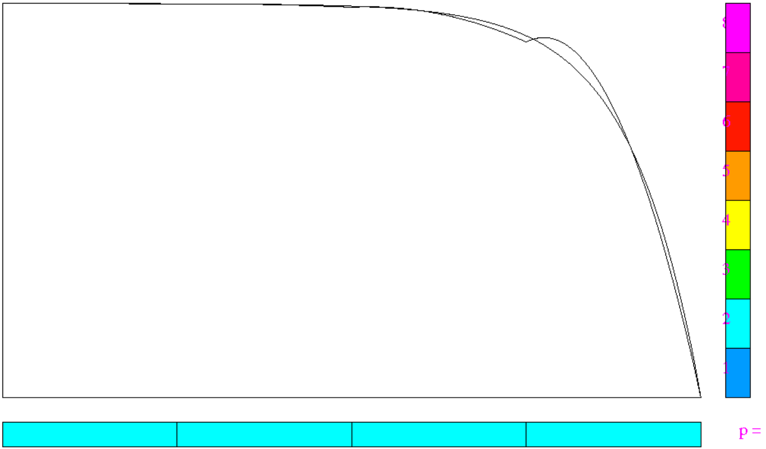
1D Convection–Dominated Diffusion

A reminder:

How does the usual Bubnov–Galerkin method perform for 1D Confusion ?

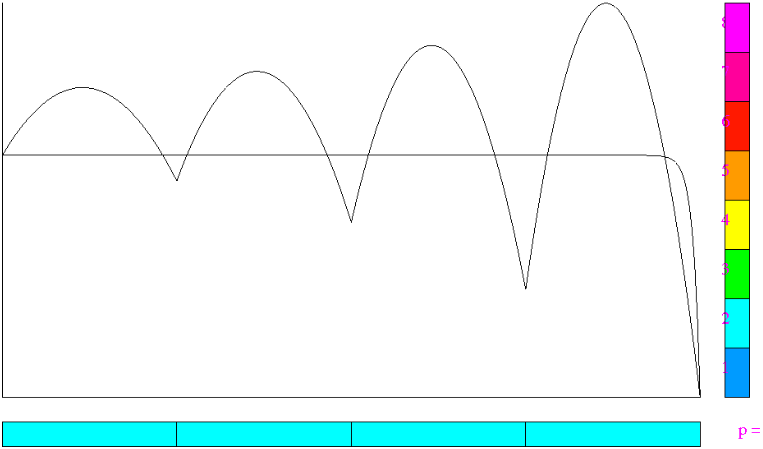
$$\begin{cases} -\epsilon u'' + u' = 0 & \text{in } (0,1) \\ u(0) = 1,\ u(1) = 0 \end{cases}$$

Bubnov-Galerkin Method



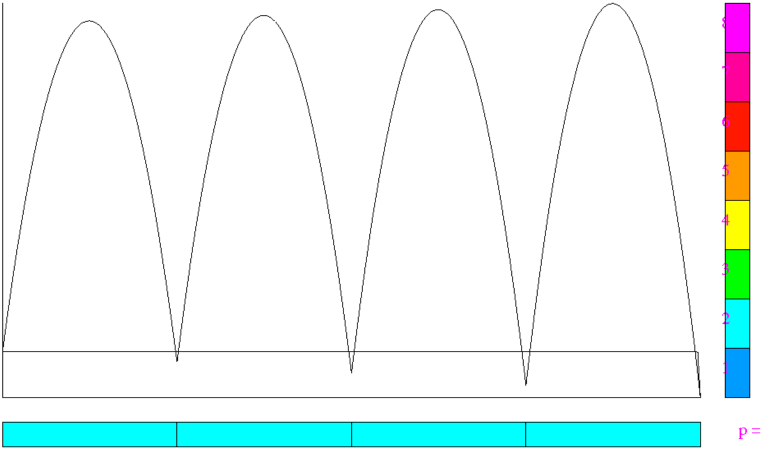
$$\epsilon = 10^{-1}$$

Bubnov-Galerkin Method



$$\epsilon = 10^{-2}$$

Bubnov-Galerkin Method



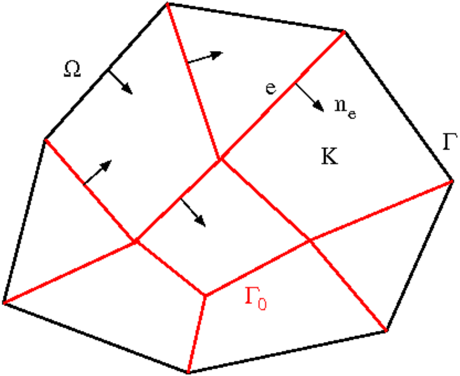
$$\epsilon = 10^{-3}$$

Ultraweak Variational Formulation and DPG Method for 2D Confusion Problem

2D Convection-Dominated Diffusion

$$\left\{\begin{aligned} \frac{1}{\epsilon}\boldsymbol{\sigma} - \boldsymbol{\nabla} u &= 0 && \text{in } \Omega \\ -\mathrm{div}(\boldsymbol{\sigma} - \boldsymbol{\beta} u) &= f && \text{in } \Omega \\ u &= u_0 && \text{on } \partial\Omega \end{aligned}\right.$$

DPG Method



Elements: K
Edges: e
Skeleton: $\Gamma_h = \bigcup_K \partial K$
Internal skeleton: $\Gamma_h^0 = \Gamma_h - \partial\Omega$

Take an element K. Multiply the equations with test functions $\boldsymbol{\tau} \in \boldsymbol{H}(\mathrm{div}, K), v \in H^1(K)$:

$$\begin{cases} \frac{1}{\epsilon}\boldsymbol{\sigma} \cdot \boldsymbol{\tau} - \boldsymbol{\nabla} u \cdot \boldsymbol{\tau} & = 0 \\ -\mathrm{div}(\boldsymbol{\sigma} - \boldsymbol{\beta} u) v & = f v \end{cases}$$

DPG Method

Integrate over the element K:

$$\begin{cases} \int_K \frac{1}{\epsilon}\boldsymbol{\sigma} \cdot \boldsymbol{\tau} - \boldsymbol{\nabla} u \cdot \boldsymbol{\tau} & = 0 \\ -\int_K \mathrm{div}(\boldsymbol{\sigma} - \boldsymbol{\beta} u) v & = f v \end{cases}$$

Integrate by parts (relax) *both* equations:

$$\left\{ \begin{array}{ll} \int_K \frac{1}{\epsilon}\boldsymbol{\sigma} \cdot \boldsymbol{\tau} + \int_K u \, \mathrm{div}\boldsymbol{\tau} - \int_{\partial K} u \, \tau_n & = 0 \\ \int_K (\boldsymbol{\sigma} - \boldsymbol{\beta} u) \cdot \boldsymbol{\nabla} v - \int_{\partial K} q \, \mathrm{sgn}(\boldsymbol{n}) \, v & = \int_K f v \end{array} \right.$$

where $q = (\boldsymbol{\sigma} - \boldsymbol{\beta} u) \cdot \boldsymbol{n}_e$ and

$$\mathrm{sgn}(\boldsymbol{n}) = \left\{ \begin{array}{rl} 1 & \text{if } \boldsymbol{n} = \boldsymbol{n}_e \\ -1 & \text{if } \boldsymbol{n} = -\boldsymbol{n}_e \end{array} \right.$$

DPG Method

Declare traces and fluxes to be independent unknowns:

$$\left\{ \begin{aligned} \int_K \frac{1}{\epsilon}\boldsymbol{\sigma}\cdot\boldsymbol{\tau} + \int_K u\,\mathrm{div}\boldsymbol{\tau} - \int_{\partial K} \hat{u}\,\tau_n &= 0 \\ -\int_K (\boldsymbol{\sigma} - \boldsymbol{\beta} u)\cdot\boldsymbol{\nabla} v + \int_{\partial K} \hat{q}\,\mathrm{sgn}(\boldsymbol{n}) v &= \int_K f v \end{aligned} \right.$$

Use BC to eliminate known traces

$$\begin{cases} \int_K \frac{1}{\epsilon}\boldsymbol{\sigma}\cdot\boldsymbol{\tau} + \int_K u\,\mathsf{div}\boldsymbol{\tau} - \int_{\partial K-\partial\Omega} \hat{u}\,\tau_n & = \int_{\partial K\cap\partial\Omega} u_0\,\tau_n \\ -\int_K(\boldsymbol{\sigma}-\boldsymbol{\beta}u)\cdot\boldsymbol{\nabla}v + \int_{\partial K}\hat{q}\,\mathsf{sgn}(\boldsymbol{n})v & = \int_K fv \end{cases}$$

Trace and Flux Spaces

$$\begin{aligned}
\Gamma_h &:= \bigcup_K \partial K \quad \text{(skeleton)} \\
\Gamma_h^0 &:= \Gamma_h - \partial\Omega \quad \text{(internal skeleton)} \\
\tilde{H}^{1/2}(\Gamma_h^0) &:= \{V|_{\Gamma_h^0} \,:\, V \in H_0^1(\Omega)\} \\
&\quad \text{with the minimum extension norm:} \\
\|v\|_{\tilde{H}^{1/2}(\Gamma_h^0)} &:= \inf\{\|V\|_{H^1} \,:\, V|_{\Gamma_h^0} = v\} \\
H^{-1/2}(\Gamma_h) &:= \{\sigma_n|_{\Gamma_h} \,:\, \boldsymbol{\sigma} \in \boldsymbol{H}(\text{div}, \Omega)\} \\
&\quad \text{with the minimum extension norm:} \\
\|\sigma_n\|_{H^{-1/2}(\Gamma_h)} &:= \inf\{\|\boldsymbol{\sigma}\|_{H(\text{div},\Omega)} \,:\, \boldsymbol{\sigma n}|_{\Gamma_h} = \sigma_n\}
\end{aligned}$$

DPG Method, a summary

$$\begin{cases} \int_K \frac{1}{\epsilon}\boldsymbol{\sigma}\cdot\boldsymbol{\tau} + \int_K u\,\mathrm{div}\boldsymbol{\tau} - \int_{\partial K - \partial\Omega} \hat{u}\,\tau_n & = \int_{\partial K\cap\partial\Omega} u_0\,\tau_n \\ -\int_K \boldsymbol{\sigma}\cdot\boldsymbol{\nabla} v + \int_{\partial K} \hat{q}\,\mathrm{sgn}(\boldsymbol{n})v & = \int_K fv \end{cases}$$

Main points:

- Both equations have been integrated by parts (relaxed).
- Traces $\hat{u} \sim u$ and fluxes $\hat{q} \sim (\boldsymbol{\sigma} - \boldsymbol{\beta}u)\cdot\boldsymbol{n}_e$ are independent unknowns (DPG is a hybrid method).
- Boundary conditions have been built in.
- Test functions are discontinuous (come from "broken" Sobolev spaces). This is critical to enable the idea of using optimal test functions.

Functional Setting

Group variables:
Solution $\boldsymbol{U} = (u, \boldsymbol{\sigma}, \hat{u}, \hat{q})$:

$$\begin{aligned} & u, \sigma_1, \sigma_2 \in L^2(\Omega_h) \\ & \hat{u} \in \tilde{H}^{1/2}(\Gamma_h^0) \\ & \hat{q} \in H^{-1/2}(\Gamma_h) \end{aligned}$$

Test function $\boldsymbol{V} = (\boldsymbol{\tau}, v)$:

$$\begin{aligned} & \boldsymbol{\tau} \in \boldsymbol{H}(\mathrm{div}, \Omega_h) \\ & v \in H^1(\Omega_h) \end{aligned}$$

Variational problem:

$$b(\boldsymbol{U}, \boldsymbol{V}) = l(\boldsymbol{V}), \quad \forall \boldsymbol{V}$$

DPG Method, abstract notation

$$\begin{cases} \frac{1}{\epsilon}(\boldsymbol{\sigma},\boldsymbol{\tau})_\Omega + (u, \mathrm{div}\boldsymbol{\tau})_{\Omega_h} - < \hat{u}, \tau_n >_{\Gamma_h^0} & = < u_0, \tau_n >_{\partial\Omega} \\ -(\boldsymbol{\sigma}, \boldsymbol{\nabla} v)_{\Omega_h} - < \hat{q}, v >_{\Gamma_h} & = (f, v)_\Omega \end{cases}$$

$$\begin{aligned} b((u,\boldsymbol{\sigma},\hat{u},\hat{q}),(\boldsymbol{\tau},v)) &= (u, \mathrm{div}\boldsymbol{\tau} + \boldsymbol{\beta}\cdot\boldsymbol{\nabla} v)_{\Omega_h} + (\boldsymbol{\sigma}, \tfrac{1}{\epsilon}\boldsymbol{\tau} - \boldsymbol{\nabla} v)_{\Omega_h} \\ &\quad - < \hat{u}, \tau_n >_{\Gamma_h^0} - < \hat{q}, v >_{\Gamma_h} \end{aligned}$$

DPG Method with Optimal Test Functions

Punchlines

Punchlines

- If the test norm is localizable, i.e.

$$(v, \delta v)_V = \sum_K (v, \delta v)_{V_K}$$

where $(v, \delta v)_{V_K}$ defines an inner product for test functions over element K,

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- then the determination of the optimal test functions is done locally. Given trial functions e_i, we compute on the fly corresponding optimal test functions $\hat{e}_i$ by solving element variational problems,

$$\begin{cases} \hat{e}_i \in V(K) \\ (\hat{e}_i, \delta v)_V = b(e_i, \delta v), \quad \forall \delta v \in V(K) \end{cases}$$

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- Solution of the local problem above can still be only approximated using an "enriched space" and standard Bubnov-Galerkin method.

Challenges

- If the optimal test functions are not well approximated, some nice properties are lost.

*Crucial for h-refinements

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- How do we prove that the continuous (hybrid) problem is well-posed ? $\surd$

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Challenges

- If the optimal test functions are not well approximated, some nice properties are lost.
- How do we prove that the continuous (hybrid) problem is well-posed ? $\surd$
- How do we prove that the stability of the continuous problem is mesh independent* ? $\surd$
- How do we choose the test norm so the method delivers results (is robust) in a norm we want? $\surd$

*Crucial for h-refinements

Relevant progress

- Nov 09 Proved: mesh independence for any 1D problem
- Dec 09 Proved: robustness for 1D confusion with special weighted test norm
- Jan 10 Developed: 1D and 2D hp-adaptive codes for the confusion problem and broke solvability records; $\epsilon = 10^{-11}$ for 1D, and $\epsilon = 10^{-7}$ for 2D problems.
- Mar 10 Discovered: concept of optimal and (practical) quasi-optimal test norm.
- Jul 10 Solved: 1D Burgers and N-S eqns with $\epsilon = 10^{-11}$ and $\epsilon = 10^{-10}$.
- Aug 10 Proved: mesh independence and well-posedness (but not robustness) for nD confusion.
- Jun 11 Developed a strategy for constructing robust DPG methods, and proved robustness for nD confusion.

Sample test norm(s)

Mathematician's test norm:

$$\|(v,\boldsymbol{\tau})\|_1^2 := \|v\|^2 + \|\nabla v\|^2 + \|\boldsymbol{\tau}\|^2 + \|\mathrm{div}\boldsymbol{\tau}\|^2$$

Weighted norm:

$$\|(v,\boldsymbol{\tau})\|_2^2 := \|v\|_w^2 + \|\nabla v\|_w^2 + \|\boldsymbol{\tau}\|_w^2 + \|\mathrm{div}\boldsymbol{\tau}\|_w^2$$

Quasi-optimal test norm:

$$\|(v,\boldsymbol{\tau})\|_3^2 := \|v\|^2 + \|\frac{1}{\epsilon}\boldsymbol{\tau} + \nabla v\|^2 + \|\mathrm{div}\boldsymbol{\tau} - \boldsymbol{\beta}\cdot\nabla v\|^2$$

Weighted norm revisited:

$$\|(v,\boldsymbol{\tau})\|_4^2 := \epsilon\|v\|^2 + \|\boldsymbol{\beta}\cdot\nabla v\|_w^2 + \epsilon\|\nabla v\|^2 + \|\boldsymbol{\tau}\|^2 + \|\mathrm{div}\boldsymbol{\tau}\|_w^2$$

Error Representation Function

Residual equals energy norm of the error:

$$\|u - u_h\|_E^2 = \|Bu_h - l\|_{V'}^2 = \|\underbrace{R_V^{-1}(Bu_h - l)}_{:=\psi}\|_V^2 = \sum_K \|\psi_K\|_{V_K}^2$$

where the *element error representation function* ψ_K is determined by solving,

$$\begin{cases} \psi_K \in V_K \\ (\psi_K, \delta v)_{V_K} = b(u - u_h, \delta v) = l(\delta v) - b(u_h, \delta v), \quad \delta v \in V_K \end{cases}$$

Lectures

- Petrov-Galerkin Method with Optimal Test Functions.
- Ultraweak variational formulation and the DPG method for convection-dominated diffusion.
- 1D analysis. Adaptivity.
- Wave propagation as an example of a complex-valued problem.
- Systematic choice of test norms. Robustness.
- Convergence proofs.

1D Confusion Problem:

Ultraweak variational formulation and the DPG method
1D analysis and adaptivity

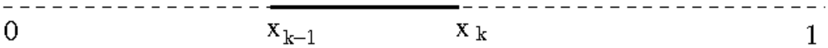
1D Confusion

1D model problem:

$$\begin{cases} u(0) = u_0, & u(1) = 0 \\ \frac{1}{\epsilon}\sigma - u' & = 0 \\ -\sigma' + u' & = f \end{cases}$$

DPG Method for 1D Confusion

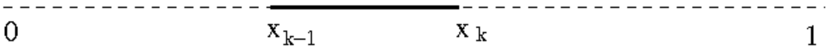
Pick an element:



Multiply the equations with test functions:

$$\begin{aligned} \tfrac{1}{\epsilon}\sigma\tau - u'\tau &= 0 \\ -\sigma' v + u' v &= f v \end{aligned}$$

Pick an element:

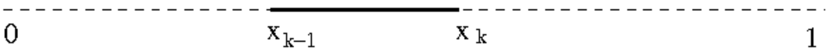


Integrate over the element:

$$
\begin{aligned}
\frac{1}{\epsilon}\int_{x_{k-1}}^{x_k}\sigma\tau - \int_{x_{k-1}}^{x_k}u'\tau &= 0\\
-\int_{x_{k-1}}^{x_k}\sigma' v + \int_{x_{k-1}}^{x_k}u' v &= \int_{x_{k-1}}^{x_k} f v
\end{aligned}
$$

DPG Method for 1D Confusion

Pick an element:

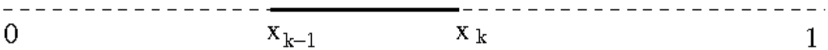


Integrate by parts:

$$
\begin{aligned}
\frac{1}{\epsilon}\int_{x_{k-1}}^{x_k}\sigma\tau+\int_{x_{k-1}}^{x_k}u\tau'-[u\tau]|_{x_{k-1}}^{x_k} &= 0\\
\int_{x_{k-1}}^{x_k}\sigma v'-[\sigma v]|_{x_{k-1}}^{x_k}-\int_{x_{k-1}}^{x_k}uv'+[uv]|_{x_{k-1}}^{x_k} &= \int_{x_{k-1}}^{x_k}fv
\end{aligned}
$$

DPG Method for 1D Confusion

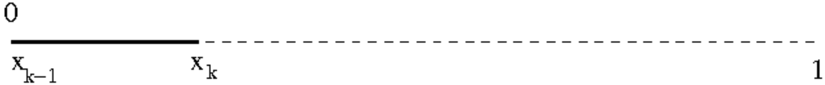
Pick an element:



Declare fluxes to be independent unknowns:

$$\begin{aligned}
\frac{1}{\epsilon}\int_{x_{k-1}}^{x_k} \sigma\tau + \int_{x_{k-1}}^{x_k} u\tau' - [\hat{u}\tau]\big|_{x_{k-1}}^{x_k} &= 0 \\
\int_{x_{k-1}}^{x_k} \sigma v' - [\hat{\sigma} v]\big|_{x_{k-1}}^{x_k} - \int_{x_{k-1}}^{x_k} u v' + [\hat{u} v]\big|_{x_{k-1}}^{x_k} &= \int_{x_{k-1}}^{x_k} f v
\end{aligned}$$

Pick an element:



For elements adjacent to the boundary use the BC's to move known fluxes to the RHS:

$$
\begin{aligned}
\frac{1}{\epsilon}\int_{x_0}^{x_1}\sigma\tau + \int_{x_0}^{x_1} u\tau' - \hat{u}(x_1)\tau(x_1) &= -u_0\tau(0) \\
\int_{x_0}^{x_1}\sigma v' - [\hat{\sigma} v]\big|_{x_0}^{x_1} - \int_{x_0}^{x_1} u v' + \hat{u}(x_1)v(x_1) &= \int_{x_0}^{x_1} f v + u_0 v(x_0)
\end{aligned}
$$

DPG Method for 1D Confusion

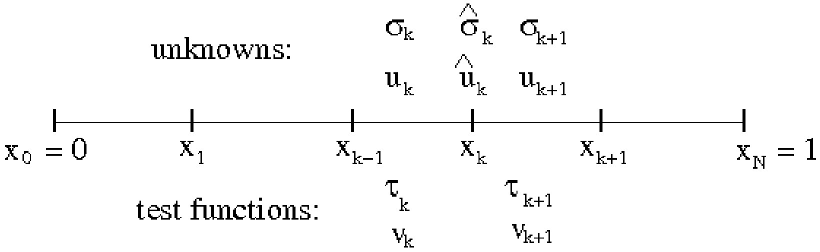
Pick an element:

Sum up over elements:

$$\frac{1}{\epsilon}\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}\sigma\tau + \int_{x_j}^{x_0} u\tau' - \sum_{j=1}^{N-1}\hat{u}(x_j)[\tau(x_j)] - \hat{u}(x_N)\tau(x_N) = -u_0\tau(0)$$

$$\int_{x_0}^{x_1}\sigma v' - [\hat{\sigma} v]|_{x_1}^{x_0} - \int_{x_1}^{x_0} uv' + \hat{u}(x_1)v(x_1) = \int_{x_{k-1}}^{x_k} fv + u_0$$

DPG Variational Formulation



For each $k = 1, \ldots, N$,

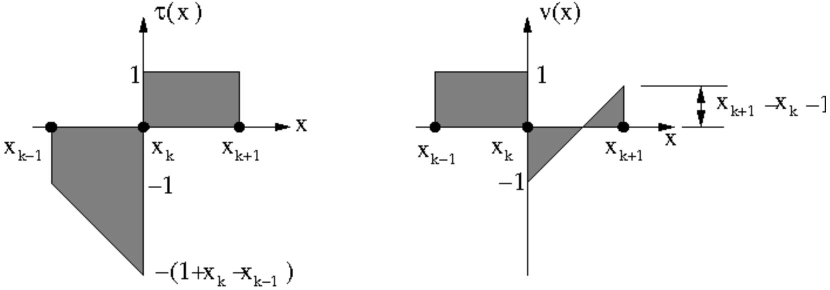
$$\begin{cases} \frac{1}{\epsilon}\int_{x_{k-1}}^{x_k} \sigma_k \tau + \int_{x_{k-1}}^{x_k} u_k \tau' - (\hat{u}\tau)|_{x_{k-1}}^{x_k} = 0 \\ \int_{x_{k-1}}^{x_k} \sigma_k v' + (\hat{\sigma} v)|_{x_{k-1}}^{x_k} - \int_{x_{k-1}}^{x_k} u_k v' + (\hat{u} v)|_{x_{k-1}}^{x_k} = \int_{x_{k-1}}^{x_k} f v \end{cases}$$

for every optimal test function τ, v. For $k = 1$, $\hat{u}(0) = u_0$ is known and is moved to the right-hand side. Similarly, $\hat{u}(1) = 0$ in the last equation for $k = N$.

Optimal Test Functions

$$\begin{cases} (\tau_k, \delta\tau)_k &= \frac{1}{\epsilon}\int_{x_{k-1}}^{x_k} \sigma_k \delta\tau + \int_{x_{k-1}}^{x_k} u_k \delta\tau' - (\hat{u}\delta\tau)|_{x_{k-1}}^{x_k} & \forall\delta\tau \\ (v_k, \delta v)_k &= \int_{x_{k-1}}^{x_k} \sigma_k v' - (\hat{\sigma}\delta v)|_{x_{k-1}}^{x_k} - \int_{x_{k-1}}^{x_k} u_k v' + (\hat{u}\delta v)|_{x_{k-1}}^{x_k} & \forall\delta v \end{cases}$$

where $(\cdot,\cdot)_k$ is an inner product for k-th element.



Optimal test function corresponding to flux $\hat{u}(x_k) = 1$ and
$(u, v)_k = \int_{x_{k-1}}^{x_k} u'v'\,dx + u(x_k)v(x_k)$

Approximate Optimal Test Functions

Practical approach:
Solve for the optimal test functions in an *enriched space:*

$$\mathcal{P}^{p+\Delta p}(K)$$

with a globally defined Δp.
Warning:
This should not be confused with using $\mathcal{P}^{p+\Delta p}(K)$ for the test space. The optimal test functions constitute only a proper subspace of $\mathcal{P}^{p+\Delta p}(K)$

Globally and Locally Optimal Test Functions in 1D (Issue: mesh independence)

Global Test Functions

Formulation with continuous test functions:

$$
\left\{
\begin{array}{lll}
\sigma, u \in L^2(0,1),\ \hat{\sigma}(0), \hat{\sigma}(1) \in \mathbb{R} & & \\
\frac{1}{\epsilon} \int_0^1 \sigma\tau + \int_0^1 u\tau' & = u_0\tau(0) & \forall \tau \in H^1(0,1) \\
\int_0^1 \sigma v' + [\hat{\sigma} v]|_0^1 - \int_0^1 u v' & = u_0 v(0) & \forall v \in H^1(0,1)
\end{array}
\right.
$$

requires no interelement fluxes but leads to a global problem for the optimal test functions:

$$
\left\{
\begin{array}{ll}
\tau, v \in H^1(0,1) & \\
\int_0^1 \tau'\delta\tau' + \tau\delta\tau = \frac{1}{\epsilon}\int_0^1 \sigma\delta\tau + \int_0^1 u\delta\tau' & \forall \delta\tau \in H^1(0,1) \\
\int_0^1 v'\delta v' + v\delta v = \int_0^1 \sigma\delta v' + [\hat{\sigma}\delta v]|_0^1 - \int_0^1 u\delta v' & \forall \delta v \in H^1(0,1)
\end{array}
\right.
$$

The resulting stiffness matrix is full but the resulting energy norm is mesh independent!

Q: A relation between the globally and locally optimal test functions ?

Globally Optimal Test Functions

$$\begin{cases} -\tau'' + \tau &= \frac{1}{\epsilon}\sigma - u' \qquad \text{in } \mathcal{D}'(0,1) \\ -v'' + v &= (-\sigma - u)' \quad \text{in } \mathcal{D}'(0,1) \end{cases}$$

Globally Optimal Test Functions

Equivalently,

$$\begin{cases} -\tau'' + \tau &= \frac{1}{\epsilon}\sigma - u' & \text{in } (x_{k-1}, x_k),\ k = 1, \ldots, N \\ [\tau' - u] &= 0 & \text{at } x_k,\ k = 1, \ldots, N-1 \\ \\ -v'' + v &= (-\sigma - u)' & \text{in } (x_{k-1}, x_k),\ k = 1, \ldots, N \\ [v' - \sigma + u] &= 0 & \text{at } x_k,\ k = 1, \ldots, N-1 \end{cases}$$

Globally Optimal Test Functions

With boundary conditions,

$$
\begin{cases}
-\tau'' + \tau &= \frac{1}{\epsilon}\sigma - u' & \text{in } (x_{k-1}, x_k),\ k = 1, \ldots, N \\
[\tau' - u] &= 0 & \text{at } x_k,\ k = 1, \ldots, N-1 \\
\tau' - u &= 0 & \text{at } x_0, x_N \\
\\
-v'' + v &= (-\sigma - u)' & \text{in } (x_{k-1}, x_k),\ k = 1, \ldots, N \\
[v' - \sigma + u] &= 0 & \text{at } x_k,\ k = 1, \ldots, N-1 \\
v' - \sigma + u &= \hat{\sigma} & \text{at } x_0, x_N
\end{cases}
$$

Globally Optimal Test Functions

Multiply with discontinuous test functions $\delta\tau, \delta v$ and integrate over individual elements,

$$\begin{cases} \int_{x_{k_1}}^{x_k} (-\tau'' + \tau)\delta\tau & = \int_{x_{k_1}}^{x_k} (\frac{1}{\epsilon}\sigma - u')\delta\tau \\ \int_{x_{k_1}}^{x_k} (-v'' + v)\delta v & = \int_{x_{k_1}}^{x_k} (-\sigma + u)'\delta v \end{cases}$$

Globally Optimal Test Functions

Integrate by parts,

$$\begin{cases} \int_{x_{k_1}}^{x_k} \tau'\delta\tau' + \tau\delta\tau & = \int_{x_{k_1}}^{x_k} \frac{1}{\epsilon}\sigma\delta\tau + u\delta\tau' + (\tau' - u)\delta\tau|_{x_{k-1}}^{x_k} \\ \int_{x_{k_1}}^{x_k} v'\delta v' + v\delta v & = \int_{x_{k_1}}^{x_k} (\sigma - u)\delta v' + (v' - \sigma + u)\delta v|_{x_{k-1}}^{x_k} \end{cases}$$

Globally Optimal Test Functions

Sum up over elements using interface and boundary conditions

$$
\left\{
\begin{array}{ll}
\sum_{k=1}^{N} \int_{x_{k_1}}^{x_k} \tau' \delta\tau' + \tau\delta\tau & = \sum_{k=1}^{N} \int_{x_{k_1}}^{x_k} \frac{1}{\epsilon}\sigma\delta\tau + u\delta\tau' \\
 & + \sum_{k=1}^{N-1} (\tau' - u)[\delta\tau](x_k) \\
\sum_{k=1}^{N} \int_{x_{k_1}}^{x_k} v'\delta v' + v\delta v & = \sum_{k=1}^{N} \int_{x_{k_1}}^{x_k} (\sigma - u)\delta v' \\
 & + \sum_{k=1}^{N-1} (v' - \sigma + u)[\delta v](x_k) + (\hat{\sigma}\delta v)|_0^1
\end{array}
\right.
$$

Globally Optimal Test Functions

Conclusion:
The globally optimal test function corresponding to an hp trial shape function $(\sigma, u, \hat{\sigma}(0), \hat{\sigma}(1))$ is a linear combination of the corresponding locally optimal test function corresponding to the same trial function and locally optimal test functions corresponding to fluxes $(\tau' - u), (v' - \sigma + u)$ at interelement boundaries x_k.

Remark: The result is true for any 1D problem but it does not generalize to multidimensions where the globally optimal test functions *can only be approximated* with the locally optimal ones.

Mesh Independence

Theorem
Test space corresponding to formulation with globally conforming test functions is contained in the DPG test space. Consequently, the FE solutions corresponding to both formulations are identical. Part of the energy norm corresponding to the DPG formulation and unknowns $(\sigma, u, \hat{\sigma}(0), \hat{\sigma}(1))$ coincides with the energy norm corresponding to the globally optimal test functions and, therefore, is mesh independent

Continuity of Error Representation Function

A related result:

Theorem

The error representation function corresponding to the DPG formulation is globally conforming (continuous).

(A great check for the control of round off error...)

Continuity of Error Representation Function

Notation:

$$\begin{array}{rl} \mathcal{U} = (\boldsymbol{\sigma}, \boldsymbol{u}, \hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{u}}) & \text{exact solution} \\ \mathcal{U}_{hp} & \text{approximate solution} \\ (x_{k-1}, x_k),\ (x_k, x_{k+1}) & \text{neighboring elements} \\ (\tau_{\hat{u}_k}, v_{\hat{u}_k}) & \text{optimal test function corresponding to flux } \hat{u}_k \end{array}$$

Orthogonality condition for the error function $\mathcal{E}_{hp} := \mathcal{U} - \mathcal{U}_{hp}$:

$$\begin{gathered} b(\mathcal{U} - \mathcal{U}_{hp}, (\tau_{\hat{u}_k}, v_{\hat{u}_k})) \\ = b_k(\mathcal{U} - \mathcal{U}_{hp}, (\tau_{\hat{u}_k}, v_{\hat{u}_k},)) + b_{k+1}(\mathcal{U} - \mathcal{U}_{hp}, (\tau_{\hat{u}_k}, v_{\hat{u}_k})) = 0 \end{gathered}$$

where b_k denotes k-th element contribution to the global bilinear form.

Continuity of Error Representation Function

Let (ϕ_k, ψ_k) be the error representation function for the k-th element,

$$((\phi_k, \psi_k), (\delta\phi, \delta\psi))_k = b_k(\mathcal{E}_{hp}, (\delta\phi, \delta\psi)), \quad \forall(\delta\phi, \delta\psi)$$

The error orthogonality condition implies then

$$\begin{gathered}((\phi_k, \psi_k), (\tau_{\hat{u}_k}, v_{\hat{u}_k}))_k + ((\phi_{k+1}, \psi_{k+1}), (\tau_{\hat{u}_k}, v_{\hat{u}_k}))_{k+1} \\ = b(\mathcal{U} - \mathcal{U}_{hp}, (\tau_{\hat{u}_k}), v_{\hat{u}_k}) = 0\end{gathered}$$

On the other side, it follows from the definition of optimal test functions that

$$((\tau_{\hat{u}_k}, v_{\hat{u}_k}), (\delta\phi, \delta\psi))_k = \delta\psi(x_k), \quad \forall(\delta\phi, \delta\psi)$$

and

$$((\tau_{\hat{u}_k}, v_{\hat{u}_k}), (\delta\phi, \delta\psi))_{k+1} = -\delta\psi(x_k), \quad \forall(\delta\phi, \delta\psi)$$

Continuity of Error Representation Function

Selecting $(\delta\phi, \delta\psi) = (\phi_k, \psi_k)$ and (ϕ_{k+1}, ψ_{k+1}) above, and summing up the two equations, we get

$$\psi_k(x_k) - \psi_{k+1}(x_k) = 0$$

In the same way we prove continuity of ϕ.
Important consequence: solution of the global problem

$$\begin{cases} (\phi, \psi) \in H^1(0,1) \\ ((\phi, \psi), (\delta\phi, \delta\psi)) = b(\mathcal{E}_{hp}, (\delta\phi, \delta\psi)) \quad \forall(\delta\phi, \delta\psi) \in H^1(0,1) \end{cases}$$

where $(\phi, \psi) = \sum_{k=1}^{N} (\phi, \psi)_k$, leads to the same error representation function.
Conclusion: If (ϕ, ψ) is mesh independent then so is the energy norm of the FE error. Consequently, *both* h and p-refinements must lead to the decrease of the energy error.

Robustness

Consider the spectral (one element) case and two norms for test functions

$$\|v\|_1^2 = \int_0^1 |v'|^2 w(x)\,dx + |v(1)|^2$$
$$\|v\|_2^2 = \int_0^1 (|v'|^2 + |v|^2) w(x)\,dx$$

where $w(x)$ is a weight function. Under appropriate conditions on $w(x)$, the two norms are equivalent with order 1 equivalence constants. The energy norm corresponding to the first norm can be computed analytically

$$\|(\sigma, \hat{\sigma}(0), \hat{\sigma}(1), u)\|_E^2 =$$

$$\|\frac{1}{\epsilon}\int_x^1 \sigma + u\|_{L^2_{1/w}}^2 + |\frac{1}{\epsilon}\int_0^1 \sigma|^2 + \|\sigma - u - \hat{\sigma}(0)\|_{L^2_{1/w}}^2 + |\hat{\sigma}(0) - \hat{\sigma}(1)|^2$$

The second test norm is localizable.

Robustness

Theorem
Let

$$w(x) = \mathsf{max}\{x, \epsilon\}$$

Then there exists an order one constant C such that:

$$\|\sigma\|_{L^2}, \|u\|_{L^2} \leq C\|(\sigma, \hat{\sigma}(0), \hat{\sigma}(1), u)\|_E$$

By the equivalence of the two test norms, the result holds also for the energy norm corresponding to the localizable test norm.

Convergence Analysis

$$\|\sigma - \sigma_{hp}\|_{L^2}, \quad \|u - u_{hp}\|_{L^2} \lesssim \|(\sigma - \sigma_{hp}, u - u_{hp}, \hat{\sigma} - \hat{\sigma}_{hp}, \hat{u} - \hat{u}_{hp})\|_E$$

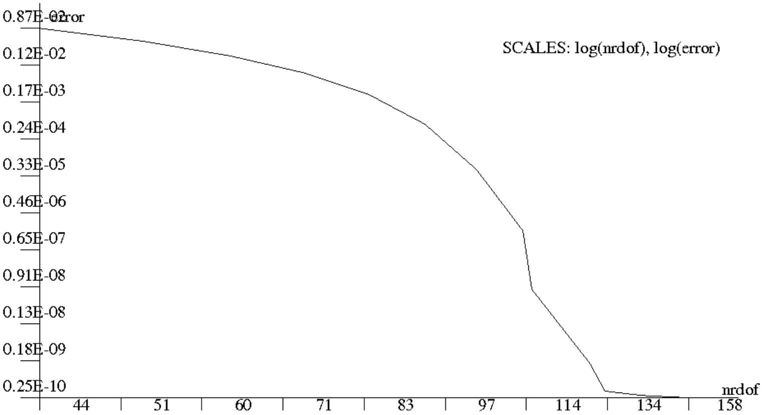
$$= \underbrace{\inf_{(\sigma_{hp}, u_{hp}, \hat{\sigma}_{hp} \hat{u}_{hp})} \|(\sigma - \sigma_{hp}, u - u_{hp}, \hat{\sigma} - \hat{\sigma}_{hp}, \hat{u} - \hat{u}_{hp})\|_E}_{\text{estimate needed}}$$

A “Greedy Poor Man” hp Algorithm

Set $\alpha = 0.5$
Do while $\alpha < 0.1$
  Solve the problem on the current mesh
  For each element K in the mesh
    Compute element error contribution e_K
  end of loop through elements
  For each element K in the mesh
    if $e_K > \alpha^2 \max_K e_K$ then
      if new $h \leq \epsilon$ then
        h-refine the element
      elseif new $p \leq p_{max}$ then
        p-refine the element
      endif
    endif
  end of loop through elements
if (new N_{dof} = old N_{dof}) reset $\epsilon = \epsilon/2$
end of loop through mesh refinements

Convergence History for $\epsilon = 10^{-3}$

$\Delta p = 4$, $p_{max} = 4$

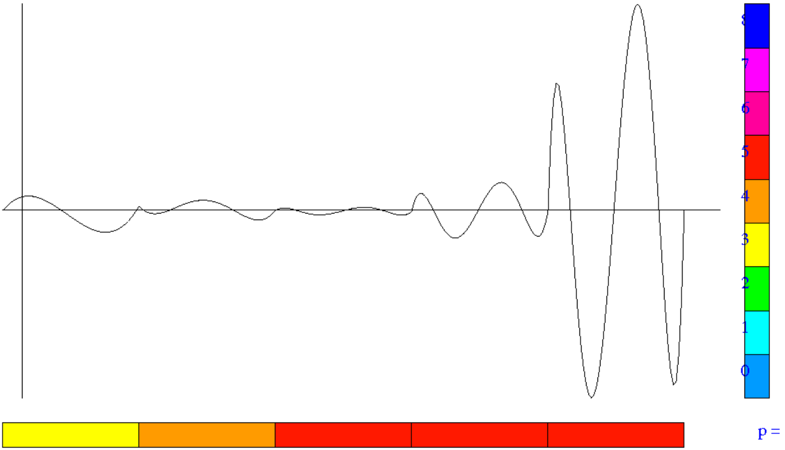


Resolution of boundary layer for $\epsilon = 10^{-3}$

$\Delta p = 4$, $p_{max} = 4$

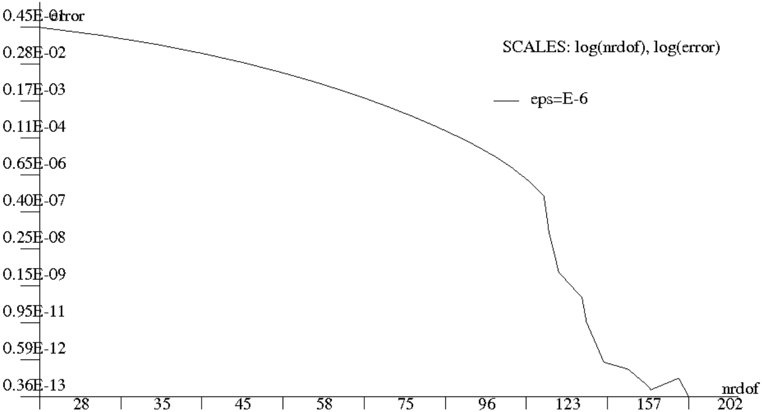
Error representation function ϕ for $\epsilon = 10^{-3}$

$\Delta p = 4$, $p_{max} = 4$



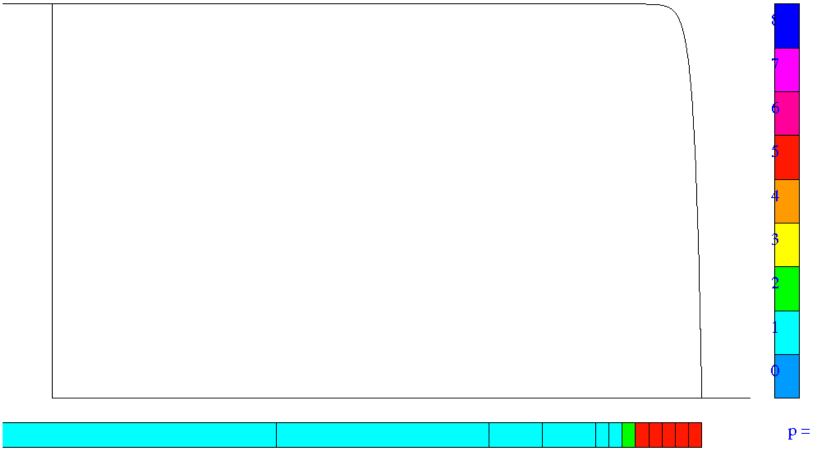
Convergence History for $\epsilon = 10^{-6}$

$\Delta p = 4$, $p_{max} = 4$



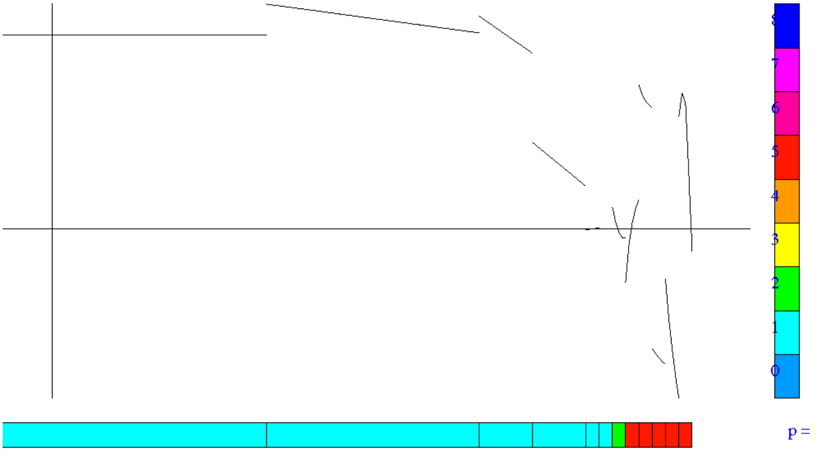
Resolution of boundary layer for $\epsilon = 10^{-6}$

$\Delta p = 4$, $p_{max} = 4$



Error representation function ϕ for $\epsilon = 10^{-6}$

$\Delta p = 4$, $p_{max} = 4$



For $\epsilon = 10^{-7}$ the method falls apart...

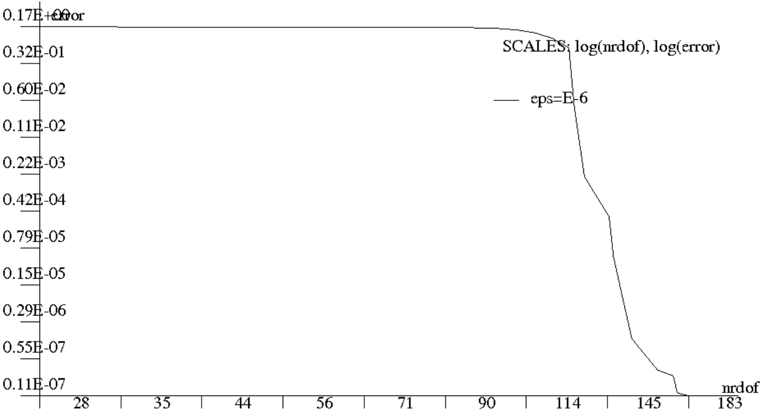
A Remedy

Use a rescaled inner product:

$$(v, \delta v) = \int_{x_{k-1}}^{x_k} (h_k v' \delta v' + v \delta v) w(x) \, dx$$

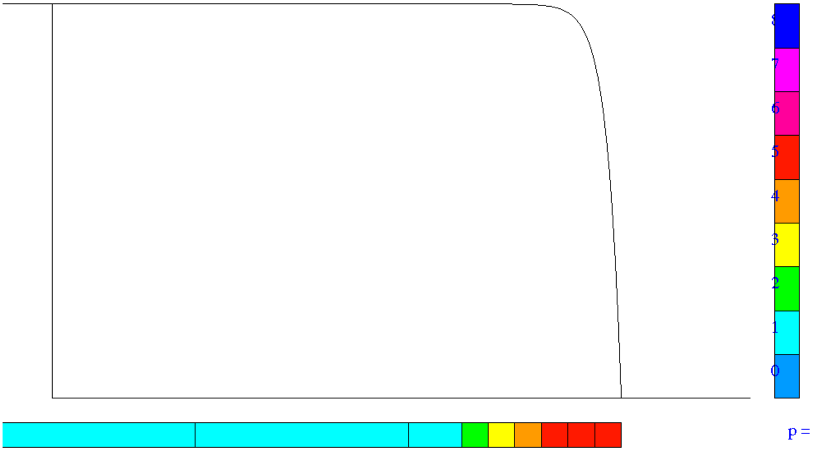
With the rescaled inner product, convergence is no longer guaranteed to be monotone (theory, in practice is...).

Convergence History for $\epsilon = 10^{-6}$ and Rescaled Inner Product



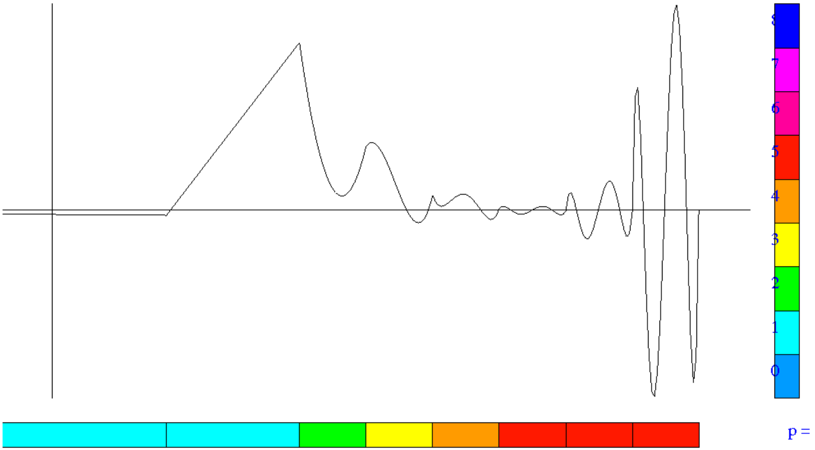
Rescaled Inner Product and $\epsilon = 10^{-6}$

Increment in order to solve local problems $\Delta p = 4$, $p_{max} = 4$



ϕ for $\epsilon = 10^{-6}$ and Rescaled Inner Product

Increment in order to solve local problems $\Delta p = 4$, $p_{max} = 4$



The Limit

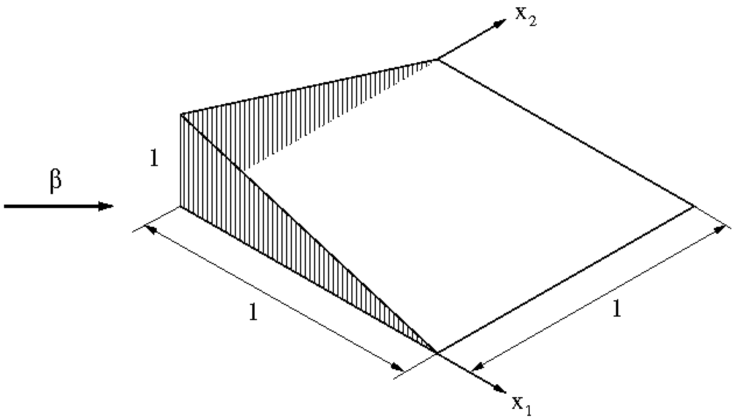
- With the rescaled inner product, we can solve the problem for $\epsilon = 10^{-11}$.
- It is possible to work with $h^\theta, 1 < \theta < 2$ in the rescaled norm but not with $\theta = 2$ (produces wrong refinements).

2D Confusion Problem

2D Convection-Dominated Diffusion

$$\left\{ \begin{aligned} \frac{1}{\epsilon}\boldsymbol{\sigma} - \boldsymbol{\nabla} u &= 0 && \text{in } \Omega \\ -\text{div}\boldsymbol{\sigma} + \text{div}(\boldsymbol{\beta} u) &= f && \text{in } \Omega \\ u &= u_0 && \text{on } \partial\Omega \end{aligned} \right.$$

2D Convection-Dominated Diffusion



Problem definition.

DPG Formulation

$$\begin{cases} \frac{1}{\epsilon}\int_K \boldsymbol{\sigma}\boldsymbol{\tau} & & +\int_K u\,\mathrm{div}\boldsymbol{\tau} & -\int_{\partial K}\hat{u}\tau_n & =0 \\ & & & & \forall\boldsymbol{\tau} \\ \int_K \boldsymbol{\sigma}\boldsymbol{\nabla} v & -\int_{\partial K}\hat{\sigma}_n v & -\int_K u\,\boldsymbol{\beta}\cdot\boldsymbol{\nabla} v & +\int_{\partial K}\hat{u}\beta_n v & =\int_K fv \\ & & & & \forall v \end{cases}$$

Energy setting:

$\tau \in \boldsymbol{H}_w(\mathrm{div}, K),\ v \in H^1_w(K),$

$\boldsymbol{\sigma} \in \boldsymbol{L}^2_{1/w}(K),\ u \in L^2_{1/w}(K),$

$\hat{u} \in \tilde{H}^{1/2}(\Gamma^0_h), \hat{\sigma}_n \in H^{-1/2}(\Gamma_h)$

Flux (Trace) Spaces

$$\Gamma := \bigcup_K \partial K \quad \text{(skeleton)}$$

$$\Gamma_0 := \Gamma - \partial\Omega \quad \text{(internal skeleton)}$$

$$H^{1/2}(\Gamma_0) := \{V|_{\Gamma_0} \,:\, V \in H_0^1(\Omega)\}$$

with the minimum extension norm:

$$\|v\|_{H^{1/2}(\Gamma_0)} := \inf\{\|V\|_{H^1} \,:\, V|_{\Gamma_0} = v\}$$

$$H^{-1/2}(\Gamma) := \{\sigma_n|_\Gamma \,:\, \boldsymbol{\sigma} \in \boldsymbol{H}(\text{div}, \Omega)\}$$

with the minimum extension norm:

$$\|\sigma_n\|_{H^{-1/2}(\Gamma)} := \inf\{\|\boldsymbol{\sigma}\|_{H(\text{div},\Omega)} \,:\, \boldsymbol{\sigma n}|_\Gamma = \sigma_n\}$$

Stability Result

Let $w = 1$ (no weight).
Theorem [D,Gopalakrishan, Sep 2010]
The DPG variational formulation for 2D or 3D confusion problems is well-posed with the inf-sup constant independent of mesh.

Colorollary 1:
There exists $C > 0$:

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_{hp}\|_{L^2(\Omega)} + \|u - u_{hp}\|_{L^2(\Omega)}$$

$$+\|\hat{\sigma}_n - \hat{\sigma}_{n,hp}\|_{H^{-1/2}(\Gamma)} + \|\hat{u} - \hat{u}_{hp}\|_{H^{1/2}(\Gamma_0)}$$

$$\leq C \inf_{\boldsymbol{\sigma}_{hp}, u_{hp}, \hat{\sigma}_{n,hp}, \hat{u}_{hp}} [\ldots]$$

Robustness requires use of weighted norms and appropriate norms for fluxes (in progress...)

Discretization

triangles:

$$\sigma_i, u \in \mathcal{P}^p(K), \quad \hat{\sigma}_n, \hat{u} \in \mathcal{P}^{p_e}(e)$$

quadrilaterals:

$$\sigma_i, u \in \mathcal{Q}^{(p,q)}(K) := \mathcal{P}^p(K) \otimes \mathcal{P}^q(K), \quad \hat{\sigma}_n, \hat{u} \in \mathcal{P}^{p_e}(e)$$

Max rule for determining approximation for fluxes:

triangles: $p_e = \max\{p_1, p_2(, p_3)\} + 1 + \Delta p_e$

quadrilaterals: $p_e = \max\{q_1, q_2(, q_3)\} + \Delta p_e$ (accounting for directionality)

(piecewise polynomials used for 2-1 edges)

Convergence result indicates that we should use

$$\Delta p_e = 1$$

Convergence Result

Assume $w = 1$ and uniform h-refinements.

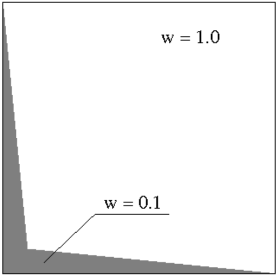
Theorem

For elements of order p and fluxes of order $p+1$,

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_{hp}\|_{L^2(\Omega)} + \|u - u_{hp}\|_{L^2(\Omega)}$$
$$+\|\hat{\sigma}_n - \hat{\sigma}_{n,hp}\|_{H^{-1/2}(\Gamma)} + \|\hat{u} - \hat{u}_{hp}\|_{H^{1/2}(\Gamma_0)}$$
$$\leq Ch^p$$

Norm for Test Functions

$$\|(\boldsymbol{\tau}, v)\|_K^2 = \int_K \left\{ \left|\frac{\partial \tau_1}{\partial x_1} + \frac{\partial \tau_2}{\partial x_2}\right|^2 + |\tau_1|^2 + |\tau_2|^2 + \left|\frac{\partial v}{\partial x_1}\right|^2 + \left|\frac{\partial v}{\partial x_2}\right|^2 + |v|^2 \right\} w(x)\, dx$$



Definition of weight function

Computation of Anisotropy Factor

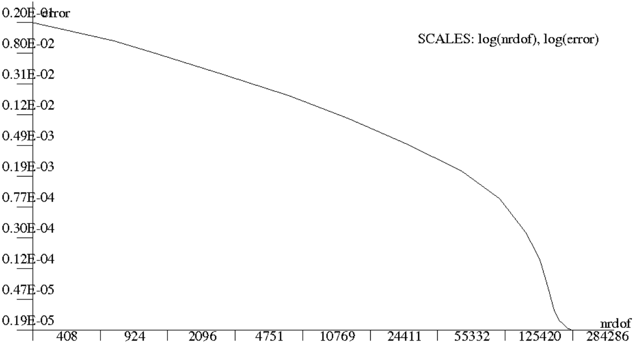
Computation of error function

$$\begin{cases} (\boldsymbol{\tau}, v) \in V_K \\ ((\boldsymbol{\tau}, v), (\delta\boldsymbol{\tau}, \delta v))_K = b_K(U_{hp}, (\delta\boldsymbol{\tau}, \delta v)) - l_K((\delta\boldsymbol{\tau}, \delta v)) \\ \hfill \forall(\delta\boldsymbol{\tau}, \delta v) \in V_K \end{cases}$$

$$c_1 = \int_K (|\tau_1|^2 + |\frac{\partial v}{\partial x_1}|^2) w(x)\, dx \quad c_2 = \int_K (|\tau_2|^2 + |\frac{\partial v}{\partial x_2}|^2) w(x)\, dx$$

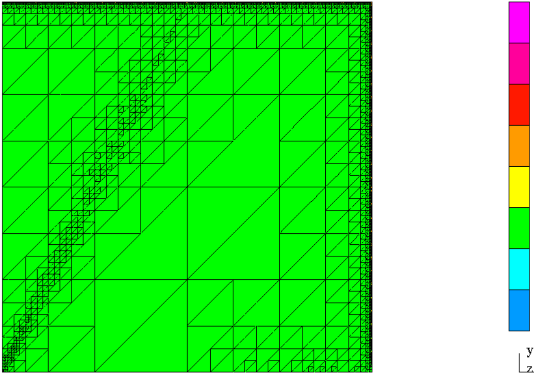
$$\text{Refinement flag} = \begin{cases} 10 & \text{if } c_1 \geq 10 c_2 \\ 01 & \text{if } c_2 \geq 10 c_1 \\ 11 & \text{otherwise} \end{cases}$$

$\epsilon = 10^{-3}$, Triangles



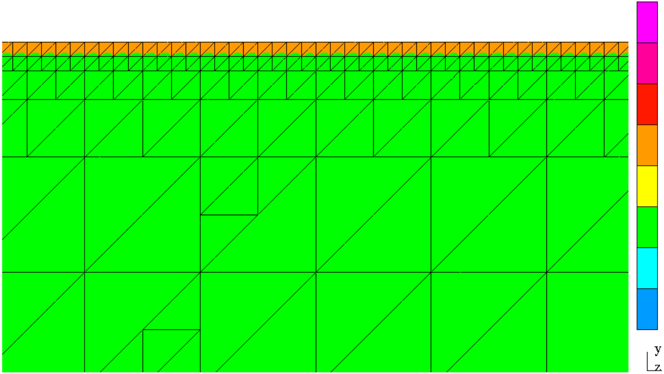
Convergence history

$\epsilon = 10^{-3}$, Triangles



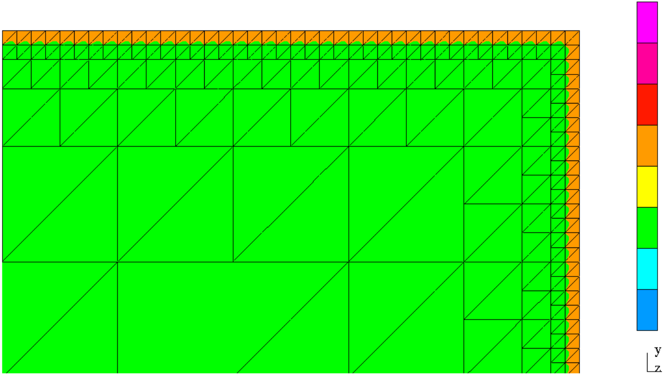
Final mesh after 21 refinements

$\epsilon = 10^{-3}$, Triangles



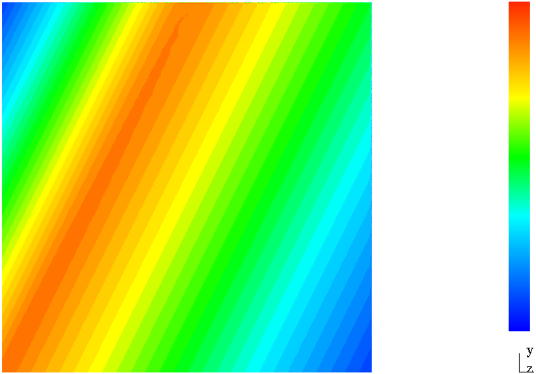
$10^{-2}\times$ zoom on upper boundary

$\epsilon = 10^{-3}$, Triangles



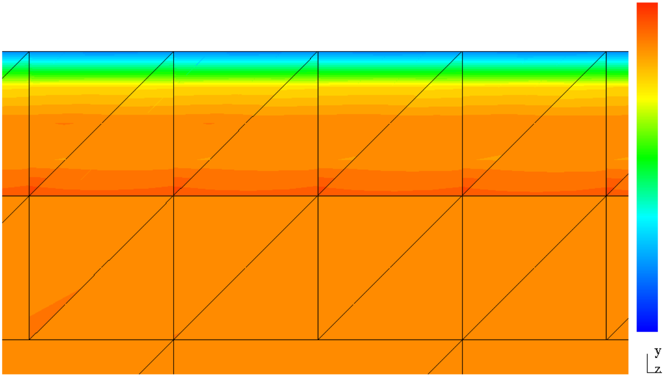
$10^{-2}\times$ zoom on north-east corner

$\epsilon = 10^{-3}$, Triangles



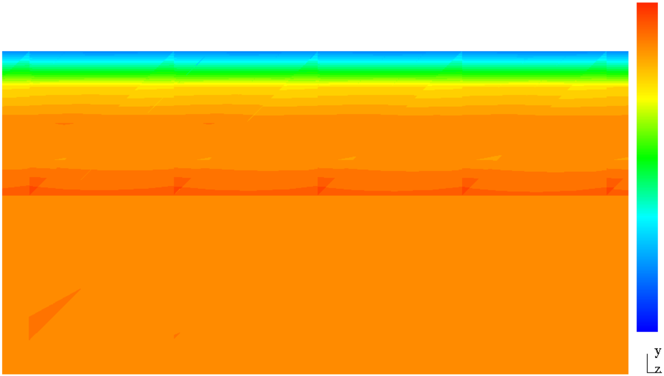
Solution u

$\epsilon = 10^{-3}$, Triangles



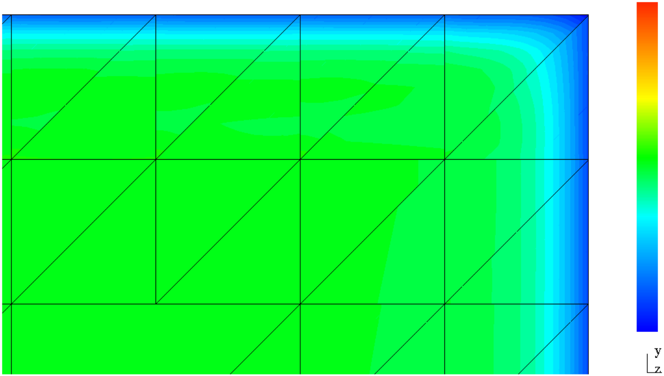
$10^2\times$ zoom on upper boundary. Solution u with the mesh

$\epsilon = 10^{-3}$, Triangles



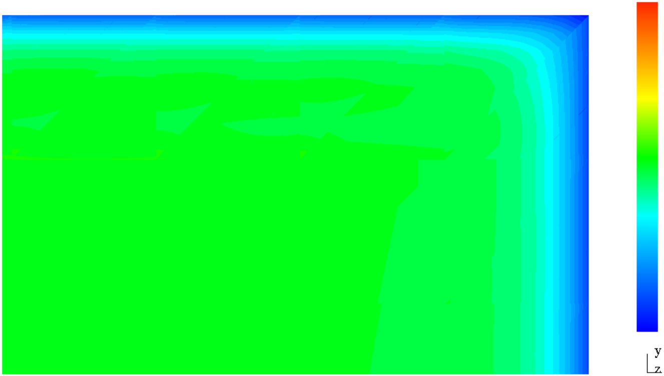
$10^2\times$ zoom on upper boundary. Solution u without the mesh

$\epsilon = 10^{-3}$, Triangles



$10^2\times$ zoom on north-east corner. Solution u with the mesh

$\epsilon = 10^{-3}$, Triangles

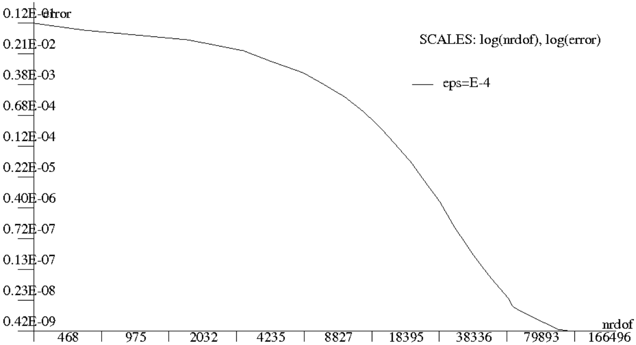


$10^2\times$ zoom on north-east corner. Solution u without the mesh

Limit for Triangles

$\epsilon = 10^{-4}$, almost 1M d.o.f.
(4 years old IBM Think Pad, 1Gb memory, frontal solver for a symmetric problem, no pivoting)

$\epsilon = 10^{-4}$, Quads



Convergence history

Limit for Quads with Standard Inner Product

$\epsilon = 10^{-5}$, almost 0.5M d.o.f.

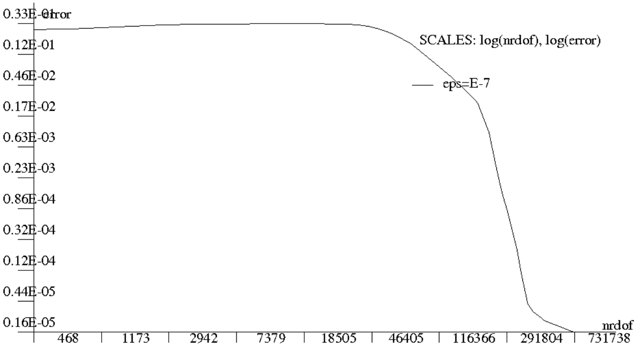
with the following limitations:

- L^2-contribution scaled with factor 10
- aspect ratio $h_1/h_2 \leq 100$
- $p_{max} = 4$

Remedy: Redefined Norm for Test Functions

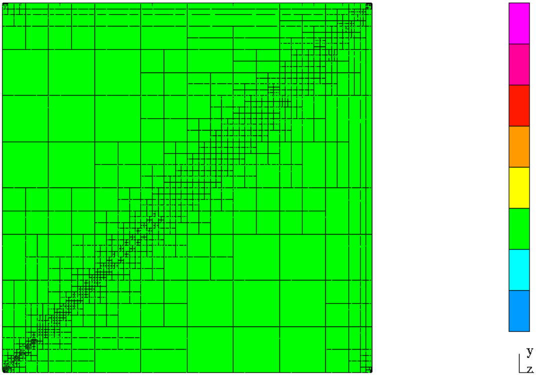
$$
\begin{aligned}
\|(\boldsymbol{\tau}, v)\|_K^2 \quad &= \int_K \Big\{ |\sqrt{h_1}\tfrac{\partial \tau_1}{\partial x_1} + \sqrt{h_2}\tfrac{\partial \tau_2}{\partial x_2}|^2 + |\tau_1|^2 + |\tau_2|^2 \\
&\quad + h_1 |\tfrac{\partial v}{\partial x_1}|^2 + h_2 |\tfrac{\partial v}{\partial x_2}|^2 + |v|^2 \Big\} \, w(x)\, dx
\end{aligned}
$$

$\epsilon = 10^{-7}$



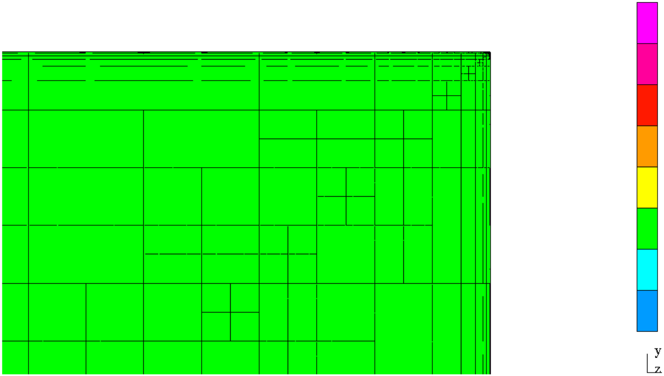
Convergence history for the redefined norm

$\epsilon = 10^{-7}$



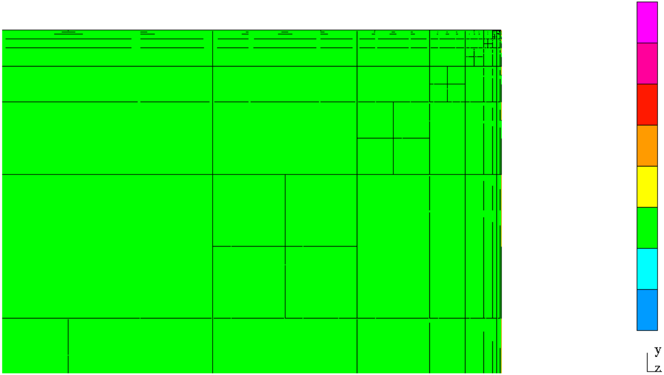
Optimal hp mesh after 45 mesh refinements.

$\epsilon = 10^{-7}$

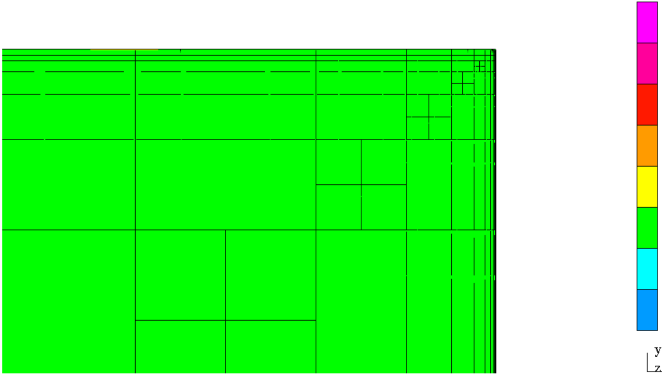


Optimal hp mesh after 45 mesh refinements. Zoom $\times 10$ on the north-east corner.

$\epsilon = 10^{-7}$

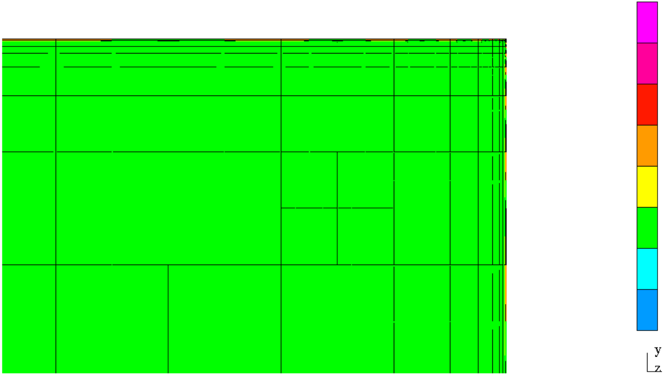


Optimal hp mesh after 45 mesh refinements. Zoom $\times 100$ on the north-east corner.



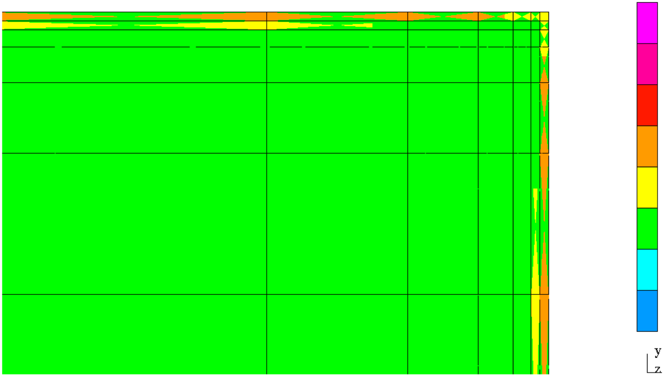
Optimal hp mesh after 45 mesh refinements. Zoom $\times 1000$ on the north-east corner.

$\epsilon = 10^{-7}$



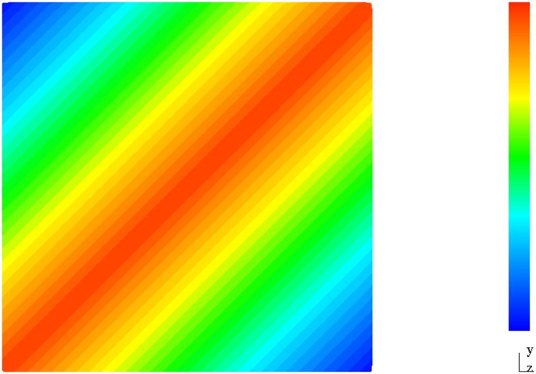
Optimal hp mesh after 45 mesh refinements. Zoom $\times 10000$ on the north-east corner.

$\epsilon = 10^{-7}$



Optimal hp mesh after 45 mesh refinements. Zoom $\times 10^5$ on the north-east corner.

$\epsilon = 10^{-7}$

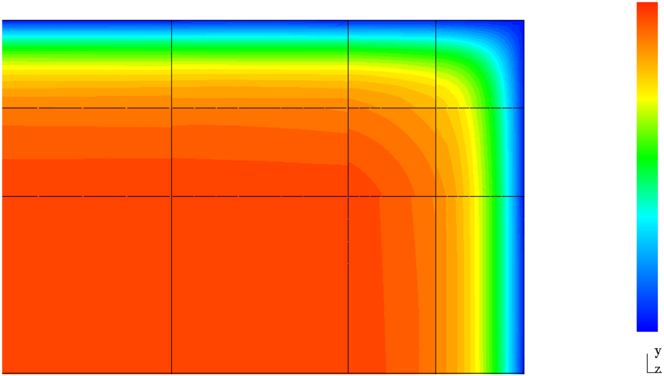


Velocity u.

$\epsilon = 10^{-7}$

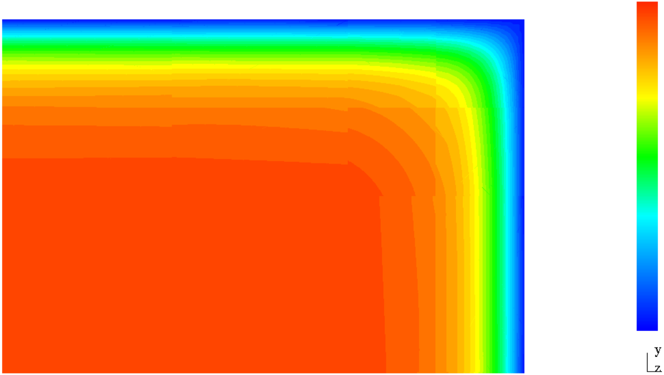
Velocity u. Zoom $\times 10^5$ on the north-east corner.

$\epsilon = 10^{-7}$



Velocity u. Zoom $\times 10^6$ on the north-east corner with the mesh.

$\epsilon = 10^{-7}$



Velocity u. Zoom $\times 10^6$ on the north-east corner w/o the mesh.
OK, is not ideal yet...

Limitations

- aspect ratio $h_1/h_2 \leq 10000$
- $p_{max} = 4$

Lectures

- Petrov-Galerkin Method with Optimal Test Functions.
- Ultraweak variational formulation and the DPG method for convection-dominated diffusion.
- 1D analysis. Adaptivity.
- Wave propagation as an example of a complex-valued problem.
- Systematic choice of test norms. Robustness.
- Convergence proofs.

Ultraweak Variational Formulation and DPG Method for Linear Acoustics

DPG method

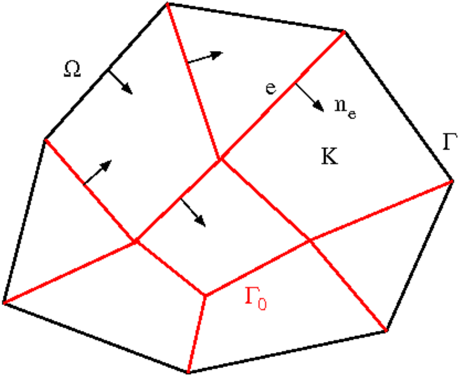
Linear acoustics in frequency domain:

$$\begin{cases} i\omega \boldsymbol{u} + \boldsymbol{\nabla} p & = \boldsymbol{0} \\ i\omega p + \mathrm{div}\boldsymbol{u} & = 0 \end{cases}$$

with, e.g. hard boundary condition:

$$u_n = g$$

DPG method



Elements:K
Edges:e
Skeleton:$\Gamma_h = \bigcup_K \partial K$
Internal skeleton:$\Gamma_h^0 = \Gamma_h - \partial\Omega$

Take an element K. Multiply the equations with test functions $\boldsymbol{v} \in \boldsymbol{H}(\text{div}, K), q \in H^1(K)$:

$$\begin{cases} i\omega\boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{\nabla} p \cdot \boldsymbol{v} & = 0 \\ i\omega p\, q + \text{div}\boldsymbol{u}\, q & = 0 \end{cases}$$

DPG Method

Integrate over the element K:

$$\left\{ \begin{aligned} i\omega \int_K \boldsymbol{u} \cdot \boldsymbol{v} + \int_K \boldsymbol{\nabla} p \cdot \boldsymbol{v} &= 0 \\ i\omega \int_K p\, q + \int_K \mathsf{div} \boldsymbol{u}\, q &= 0 \end{aligned} \right.$$

Integrate by parts (relax) *both* equations:

$$\left\{ \begin{array}{rl} i\omega \int_K \boldsymbol{u} \cdot \boldsymbol{v} - \int_K p \cdot \mathrm{div}\boldsymbol{v} + \int_{\partial K} p v_n & = 0 \\ i\omega \int_K p\, q - \int_K \boldsymbol{u} \cdot \boldsymbol{\nabla} q + \int_{\partial K} u_n\, q\, \mathrm{sgn}(\boldsymbol{n}) & = 0 \end{array} \right.$$

where $u_n = \boldsymbol{u} \cdot \boldsymbol{n}_e$ and

$$\mathrm{sgn}(\boldsymbol{n}) = \left\{ \begin{array}{rl} 1 & \text{if } \boldsymbol{n} = \boldsymbol{n}_e \\ -1 & \text{if } \boldsymbol{n} = -\boldsymbol{n}_e \end{array} \right.$$

DPG Method

Declare traces and fluxes to be independent unknowns:

$$\begin{cases} i\omega \int_K \boldsymbol{u} \cdot \boldsymbol{v} - \int_K p \cdot \mathrm{div}\boldsymbol{v} + \int_{\partial K} \hat{p} v_n & = 0 \\ i\omega \int_K p\, q - \int_K \boldsymbol{u} \cdot \boldsymbol{\nabla} q + \int_{\partial K} \hat{u}_n\, q\, \mathrm{sgn}(\boldsymbol{n}) & = 0 \end{cases}$$

DPG Method

Use BCs to eliminate known fluxes

$$\left\{ \begin{aligned} i\omega \int_K \boldsymbol{u} \cdot \boldsymbol{v} - \int_K p \cdot \mathrm{div}\boldsymbol{v} + \int_{\partial K} \hat{p} v_n &= 0 \\ i\omega \int_K pq - \int_K \boldsymbol{u} \cdot \boldsymbol{\nabla} q + \int_{\partial K - \Gamma} \hat{u}_n q \mathrm{sgn}(\boldsymbol{n}) &= \int_{\partial K \cap \Gamma} g\, q \end{aligned} \right.$$

DPG Method

Sum up over all elements and replace $\boldsymbol{v}, q$ with $\overline{\boldsymbol{v}}, \bar{q}$ to comply with the sesquilinear forms setting,

$$\left\{ \begin{array}{ll} i\omega(\boldsymbol{u},\boldsymbol{v})_\Omega - (u, \mathrm{div}\boldsymbol{v})_{\Omega_h} + < \hat{p}, v_n >_{\Gamma_h} & = 0 \\ i\omega(p,q)_\Omega - (\boldsymbol{u}, \boldsymbol{\nabla} q)_{\Omega_h} + < \hat{u}_n, q >_{\Gamma_h^0} & =< g, q >_\Gamma \end{array} \right.$$

Trace and Flux Spaces

$$
\begin{aligned}
\Gamma_h &:= \bigcup_K \partial K \quad \text{(skeleton)} \\
\Gamma_h^0 &:= \Gamma_h - \partial\Omega \quad \text{(internal skeleton)} \\
H^{1/2}(\Gamma_h) &:= \{q|_{\Gamma_h} \,:\, q \in H^1(\Omega)\} \\
&\quad \text{with the minimum extension norm:} \\
\|q\|_{H^{1/2}(\Gamma_h)} &:= \inf\{\|Q\|_{H^1} \,:\, Q|_{\Gamma_h} = q\} \\
\tilde{H}^{-1/2}(\Gamma_h^0) &:= \{v_n|_{\Gamma_h} \,:\, \boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div}, \Omega)\} \\
&\quad \text{with the minimum extension norm:} \\
\|v_n\|_{\tilde{H}^{-1/2}(\Gamma_h^0)} &:= \inf\{\|\boldsymbol{V}\|_{\boldsymbol{H}_0(\mathrm{div},\Omega)} \,:\, \boldsymbol{V} \cdot \boldsymbol{n}|_{\Gamma_h^0} = \sigma_n\}
\end{aligned}
$$

Functional Setting

Group variables:
Solution $\boldsymbol{U} = (\boldsymbol{u}, p, \hat{u}_n, \hat{p})$:

$$\begin{aligned} & u_1, u_2, p \in L^2(\Omega_h) \\ & \hat{u}_n \in \tilde{H}^{-1/2}(\Gamma_h^0) \\ & \hat{p} \in \tilde{H}^{1/2}(\Gamma_h) \end{aligned}$$

Test function $\boldsymbol{V} = (\boldsymbol{v}, q)$:

$$\begin{aligned} & \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega_h) \\ & q \in H^1(\Omega_h) \end{aligned}$$

Sesquilinear form

$$\begin{aligned} b(\boldsymbol{U}, \boldsymbol{V}) &= -(u, i\omega \boldsymbol{v} + \boldsymbol{\nabla} q)_{\Omega_h} - (p, i\omega q + \mathrm{div}\boldsymbol{v})_{\Omega_h} \\ &+ < \hat{u}_n, q >_{\Gamma_h^0} + < \hat{p}, v_n >_{\Gamma_h} \end{aligned}$$

Punchlines

Local invertibility of Riesz operator

Due to the use of "broken" Sobolev spaces (discontinuous test functions), the Riesz operator is inverted elementwise! Given any (linear) problem, and any trial shape functions, we compute the corresponding optimal test functions on the fly.

Approximate optimal test functions

The locally determined optimal test functions still need to be approximated. This is done using standard Bubnov-Galerkin method and an *enriched space*. If polynomials of order p are used to approximate the unknown velocity and pressure, the approximate optimal test functions are determined using polynomials of order:

$$p + \Delta p$$

Quasi-optimal test norm

Trial norm:

$$\|(\boldsymbol{u}, p, \hat{u}_n, \hat{p})\|_U^2 = \|\boldsymbol{u}\|_{L^2}^2 + \|p\|_{L^2}^2 + \|\hat{u}\|_?^2 + \|\hat{p}\|_?^2$$

Optimal test norm (unfortunately, non-local):

$$\begin{aligned}\|(\boldsymbol{v}, q)\|_{opt}^2 &= \|i\omega\boldsymbol{v} + \boldsymbol{\nabla} q\|_{\Omega_h}^2 + \|i\omega q + \mathsf{div}\boldsymbol{v}\|_{\Omega_h}^2 \\ &+ \mathsf{sup}_{\hat{u}_n,\hat{p}} \tfrac{|<\hat{u}_n, q> + <\hat{p}, v_n>|}{(\|\hat{u}_n\|_?^2 + \|\hat{p}_?^2)^{1/2}}\end{aligned}$$

Quasi-optimal test norm (local):

$$\|(\boldsymbol{v}, q)\|_{opt}^2 = \|i\omega\boldsymbol{v} + \boldsymbol{\nabla} q\|_{\Omega_h}^2 + \|i\omega q + \mathsf{div}\boldsymbol{v}\|_{\Omega_h}^2 + \|\boldsymbol{v}\|^2 + \|q\|^2$$

Robust stability result

Theorem: (Gopalakrishnan, Muga, D, Zitelli, 2011)
Assume: Ω contractable, impedance BC
Use: the quasi-optimal norm to define the minimum energy extension norms for fluxes $\hat{u}_n$ and traces $\hat{p}$.
Then

$$\|(\boldsymbol{v}, q)\|_{opt}^2 \approx \|(\boldsymbol{v}, q)\|_{qopt}^2 \quad \text{(uniformly in } k \text{ and mesh)}$$

Consequently, we get the robust stability in the *desired norm*:

$$\begin{aligned}
&\left(\|\boldsymbol{u} - \boldsymbol{u}_h\|^2 + \|p - p_h\|^2 + \|\hat{u}_n - \hat{u}_{n,h}\| + \|\hat{p} - \hat{p}_h\|^2\right)^{\frac{1}{2}} \\
&\lesssim \|(\boldsymbol{u}, p, \hat{u}_n, \hat{p}) - (\boldsymbol{u}_h, p_h, \hat{u}_{n,h}, \hat{p}_h)\|_E \\
&= \text{BAE of } (\boldsymbol{u}, p, \hat{u}_n, \hat{p}) \text{ in energy norm} \\
&\lesssim \text{BAE of } (\boldsymbol{u}, p, \hat{u}_n, \hat{p}) \text{ in desired norm}
\end{aligned}$$

No pollution in 1D case

In 1D, traces and fluxes and just numbers. Thus, the BAE of fluxes and traces is zero. We get,

$$\left(\|u-u_h\|^2+\|p-p_h\|^2+\|\hat{u}_n-\hat{u}_{n,h}\|+\|\hat{p}-\hat{p}_h\|^2\right)^{\frac{1}{2}}$$

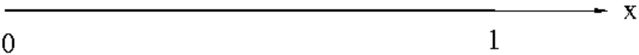
$$\lesssim \inf_{w_h,r_h}\left(\|u-w_h\|^2+\|p-r_h\|^2\right)^{\frac{1}{2}}$$

The BAE of u, p in L^2-error is pollution free.

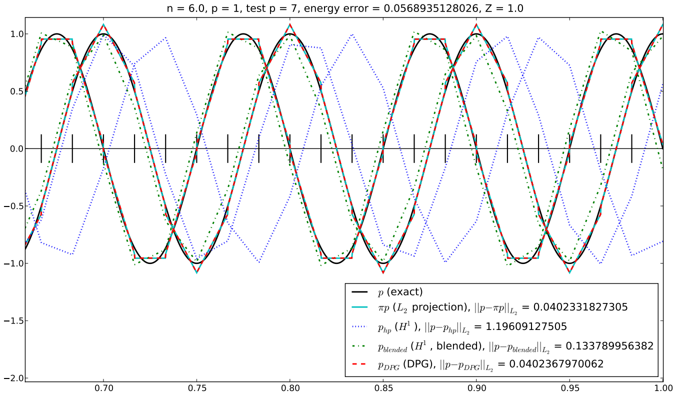
NUMERICAL EXPERIMENTS

1D experiment

Ansatz in time $e^{i\omega t}$,
Exact solution: $u = p = e^{-ikx}$ (going to the right)
BCs:
hard boundary at $x = 0$: $u(0) = 1$
impedance BC at $x = 1$: $u(1) = p(1)$
enriched space: $\Delta p = 6$

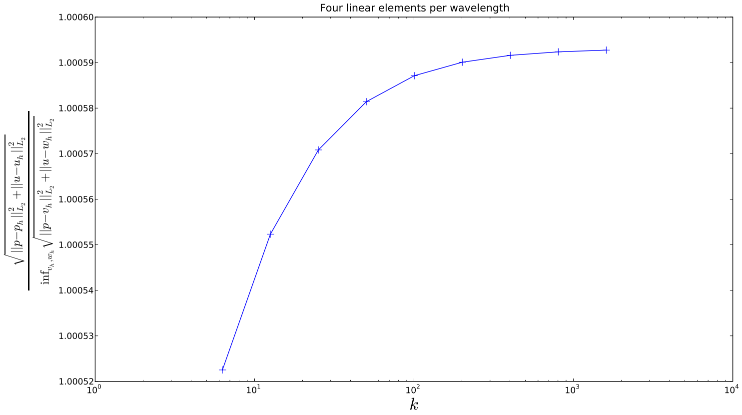


DPG vs. Standard FEs, 6 wavelenghts



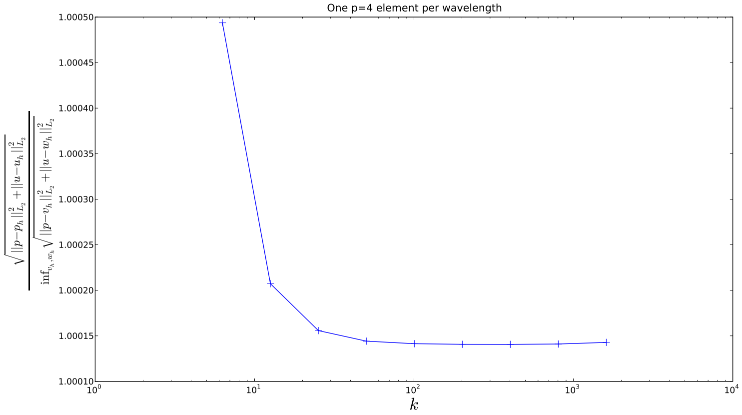
The standard H^1 conforming solution p_{hp} quickly exhibits excessive phase error; it is reduced but still present in p_{blended}

Four linear elements per wavelength



Adhering to a fixed number of elements per wavelength is sufficient to control error

One quartic element per wavelength



Adhering to a fixed number of elements per wavelength is sufficient to control error

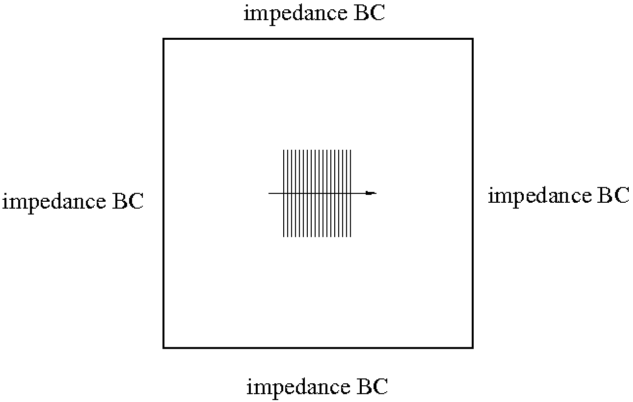
2D experiments

Discretization:

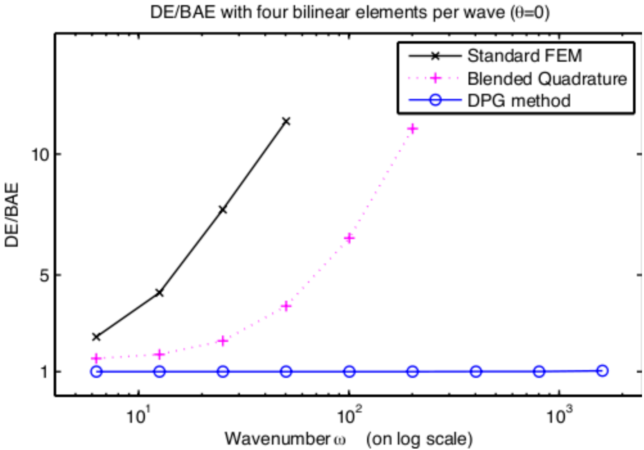
- field variables are discretized using isoparametric L^2-conforming quads of order p,
  $u_1, u_2, p \in \mathcal{P}^p \otimes \mathcal{P}^p$,
- traces are discretized using H^1-conforming elements of order $p+1$,
- fluxes are discretized using L^2-conforming elements of order $p+1$
- optimal test functions are approximated with polynomials of order $p+1+\Delta p$, i.e. $\boldsymbol{v} \in (\mathcal{P}^{p+\Delta p+1} \otimes \mathcal{P}^{p+\Delta p}) \times (\mathcal{P}^{p+\Delta p} \otimes \mathcal{P}^{p+\Delta p+1})$,
  $q \in \mathcal{P}^{p+\Delta p+1} \otimes \mathcal{P}^{p+\Delta p+1}$

2D experiment A

Exact solution: horizontal plane wave
Enriched space: $\Delta p = 2$.



2D experiment A

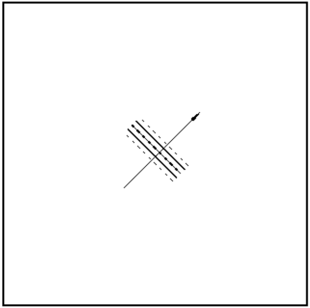


Ratio of L^2 discretization error vs BAE as a function of wave number. DPG vs standard FEs and Ainsworth-Wajid underintegration scheme.

2D experiment B

Exact solution: plane wave along diagonal
Enriched space: $\Delta p = 2$.

impedance BC

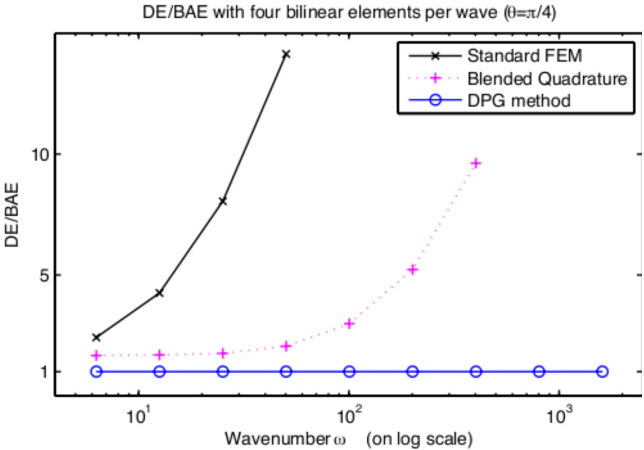


impedance BC

impedance BC

impedance BC

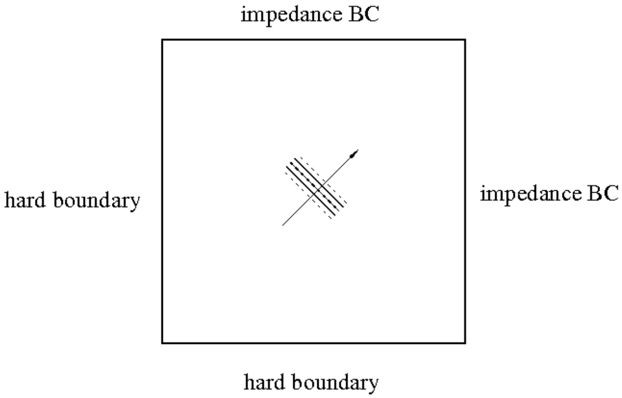
2D experiment B



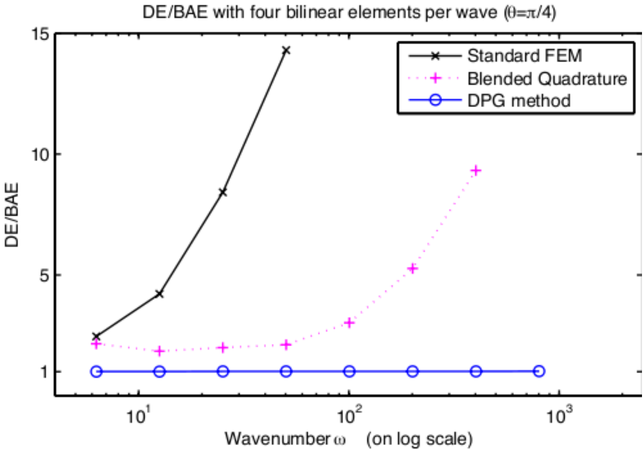
Ratio of L^2 discretization error vs BAE as a function of wave number. DPG vs standard FEs and Ainsworth-Wajid underintegration scheme.

2D experiment C

Exact solution: plane wave along diagonal
Enriched space: $\Delta p = 2$.



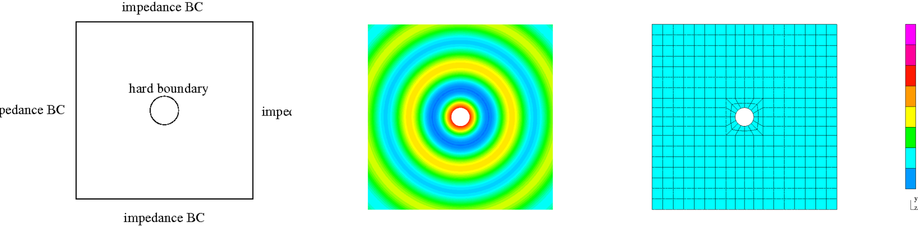
2D experiment C



Ratio of L^2 discretization error vs BAE as a function of wave number. DPG vs standard FEs and Ainsworth-Wajid underintegration scheme.

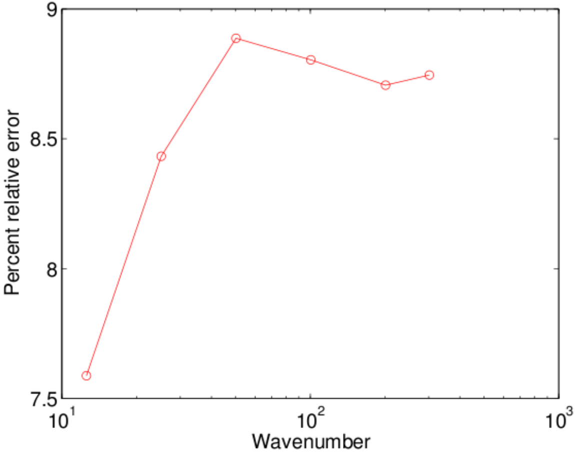
2D experiment D

Exact solution: outgoing cylindrical wave (Hankel functions...)
Enriched space: $\Delta p = 2$.



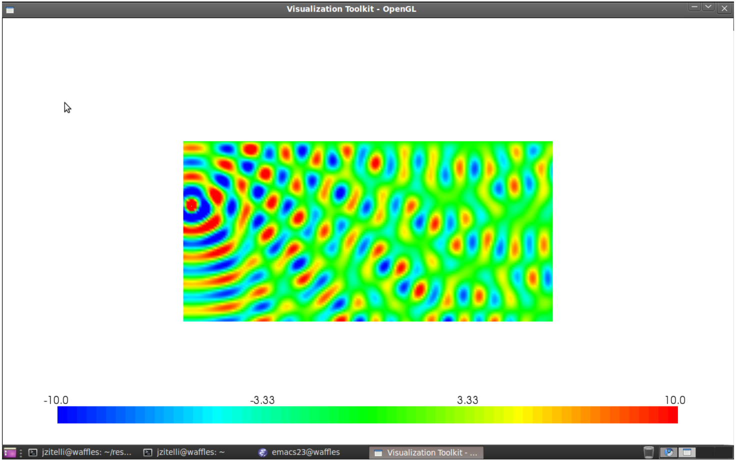
Boundary conditions, real part of pressure, initial mesh for $k = 4\pi$.

2D experiment D



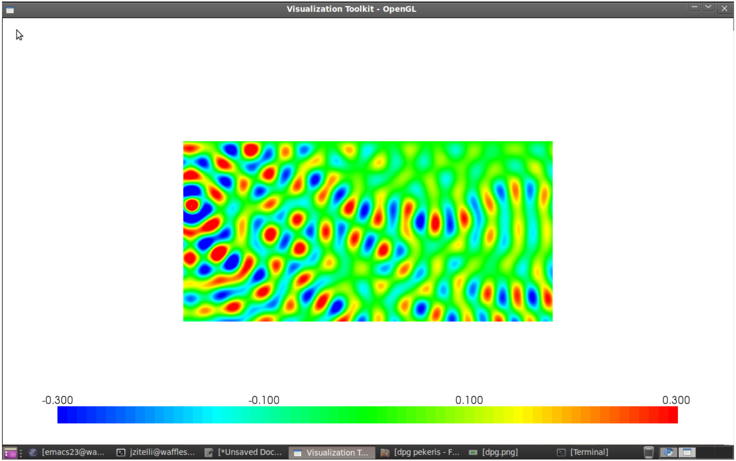
Discretization error as a function of wave number.

Pekeris problem, $k = 50$ (8 wavelengths)



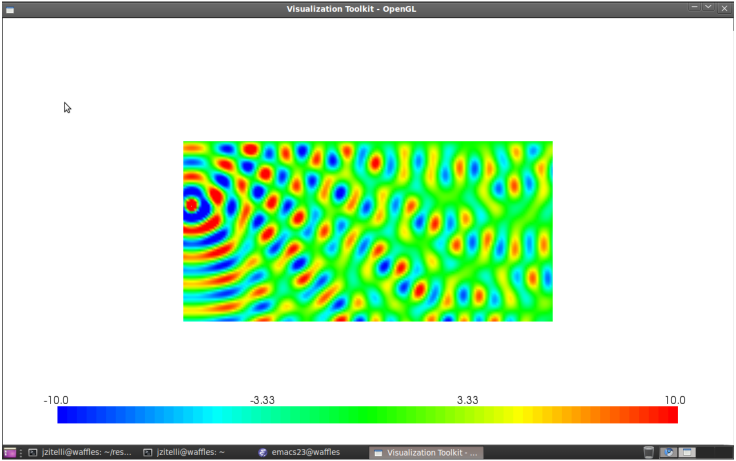
Exact solution (real part of pressure).

Pekeris problem, $k = 50$ (8 wavelengths)



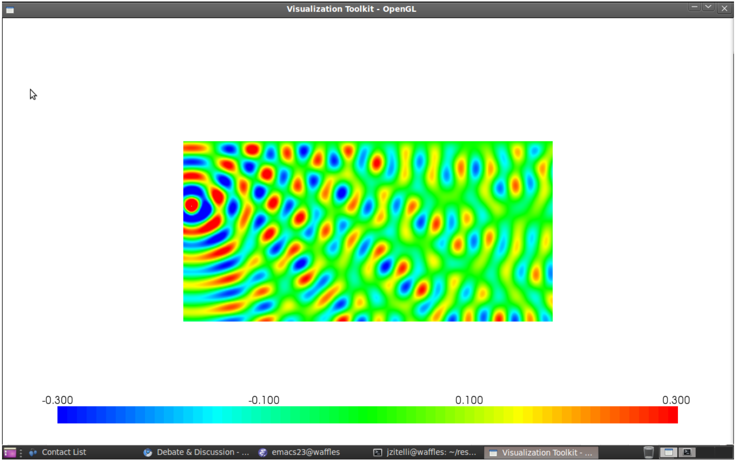
Classical FEs, four biquadratic elements per wavelength.

Pekeris problem, $k = 50$ (8 wavelengths)



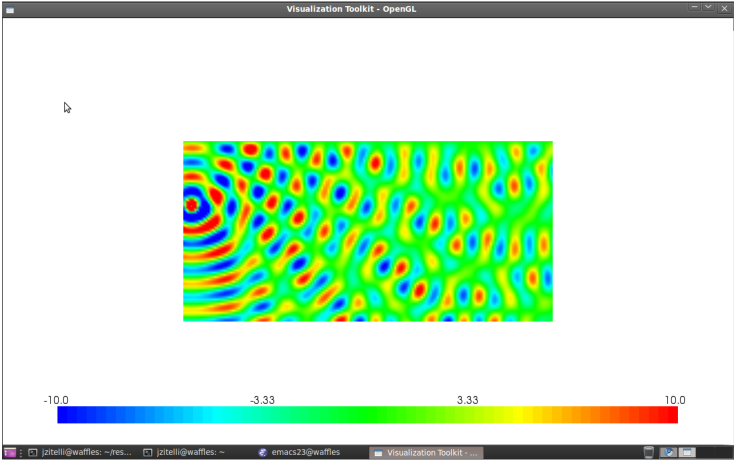
Exact solution (real part of pressure).

Pekeris problem, $k = 50$ (8 wavelengths)



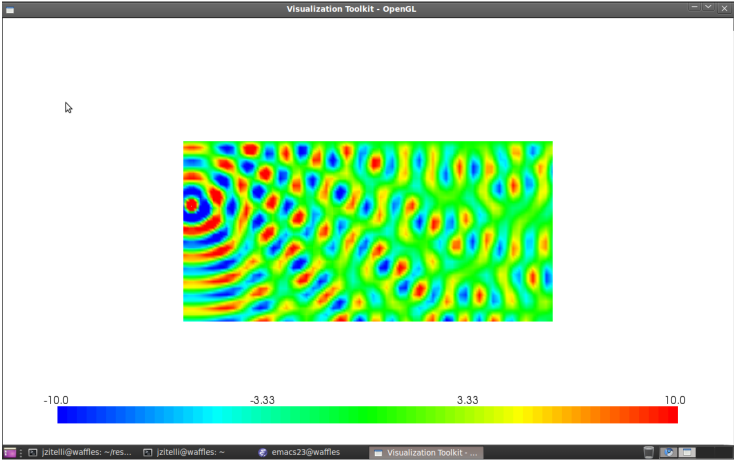
Ainsworth-Wajid quadrature, four biquadratic elements per wavelength.

Pekeris problem, $k = 50$ (8 wavelengths)



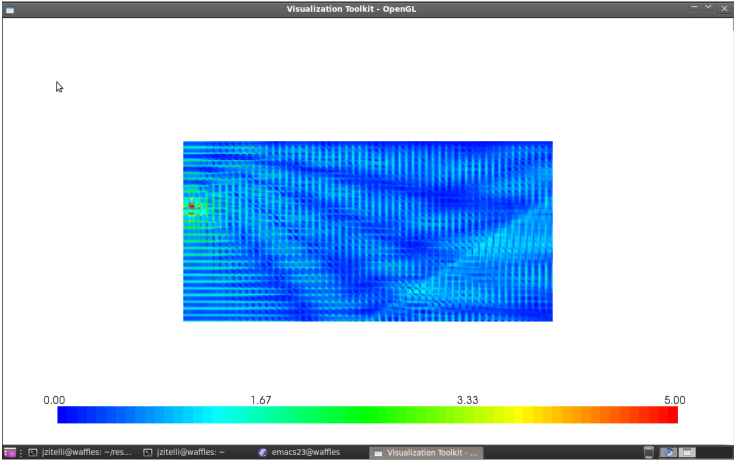
Exact solution (real part of pressure).

Pekeris problem, $k = 50$ (8 wavelengths)



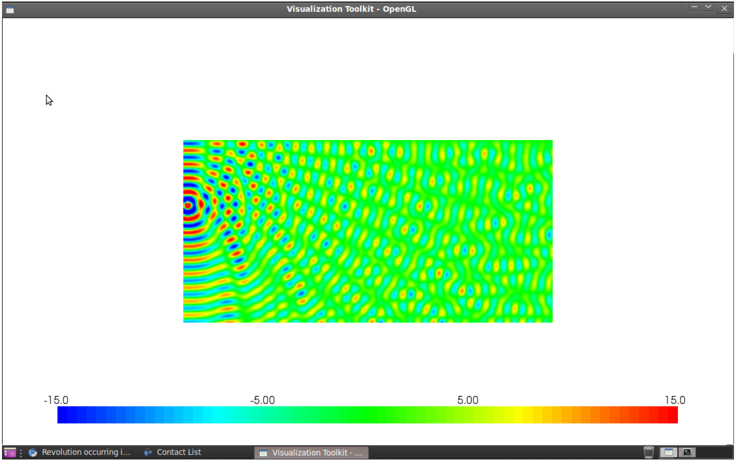
DPG method, four bilinear elements per wavelength.

Pekeris problem, $k = 50$ (8 wavelengths)



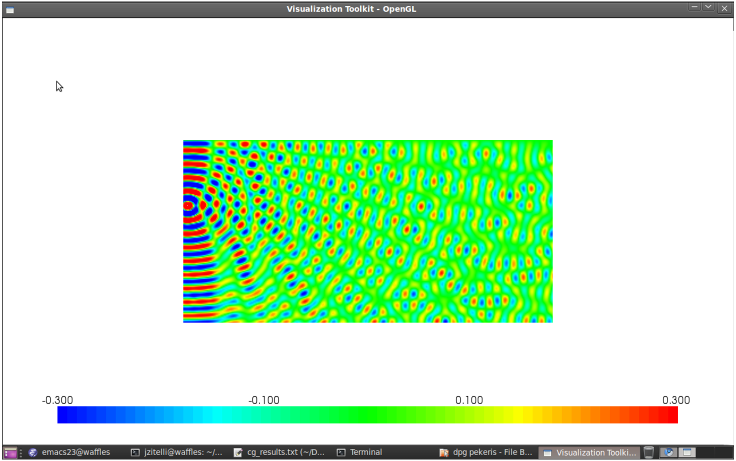
Error for the DPG method.

Pekeris problem, $k = 100$ (16 wavelenghts)



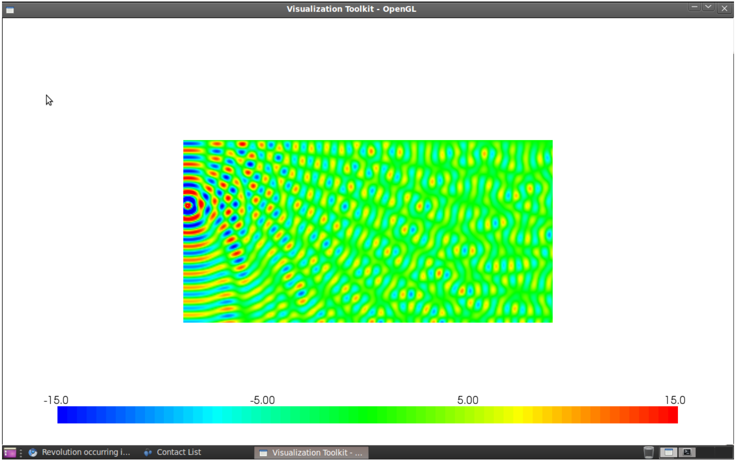
Exact solution (real part of pressure).

Pekeris problem, $k = 100$ (16 wavelenghts)



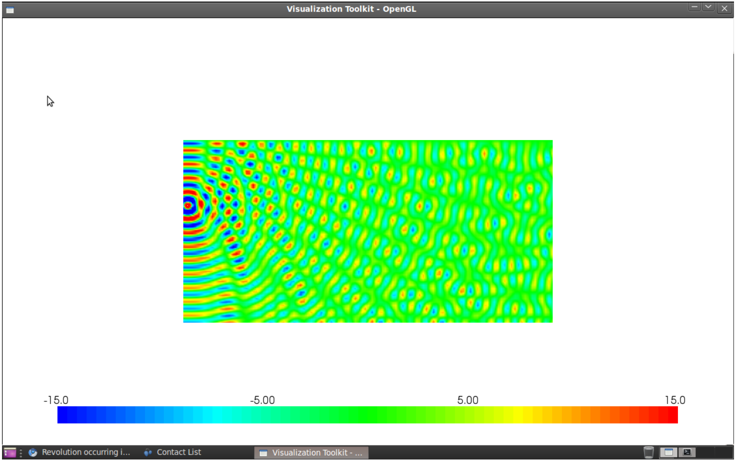
Ainsworth-Wajid quadrature, four biquadratic elements per wavelength.

Pekeris problem, $k = 100$ (16 wavelenghts)



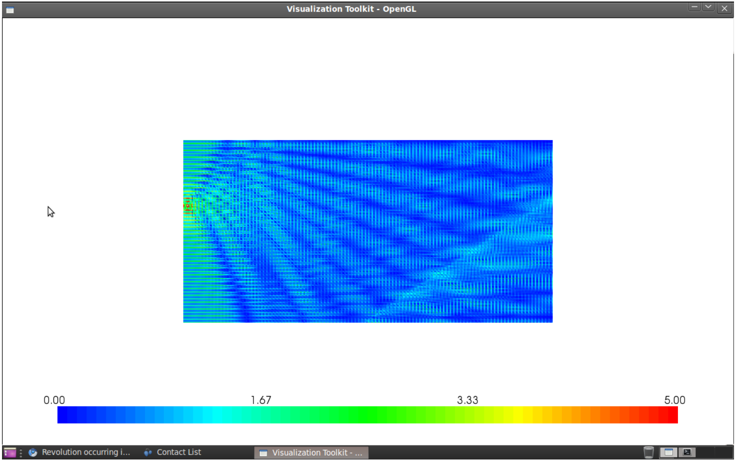
Exact solution (real part of pressure).

Pekeris problem, $k = 100$ (16 wavelenghts)



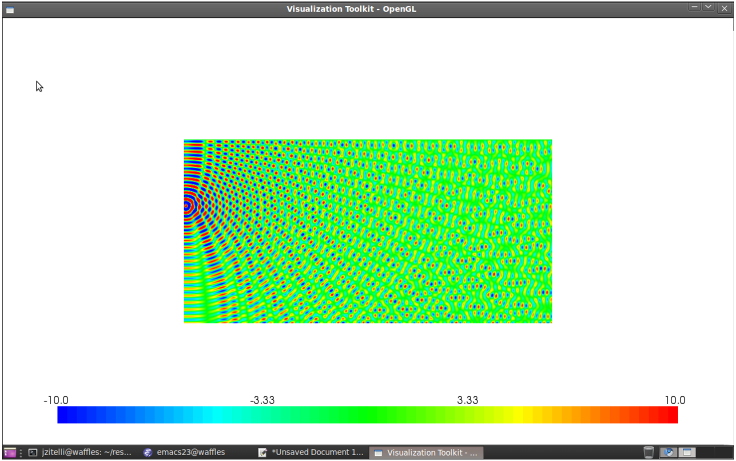
DPG method, four bilinear elements per wavelength.

Pekeris problem, $k = 100$ (16 wavelenghts)



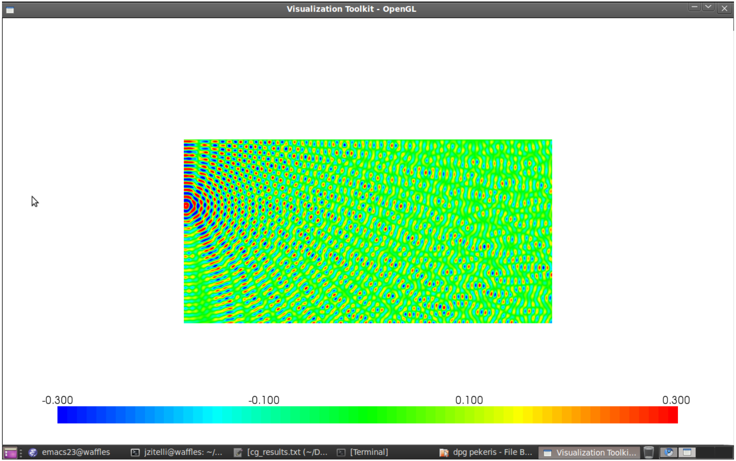
Error for the DPG method.

Pekeris problem, $k = 200$ (32 wavelenghts)



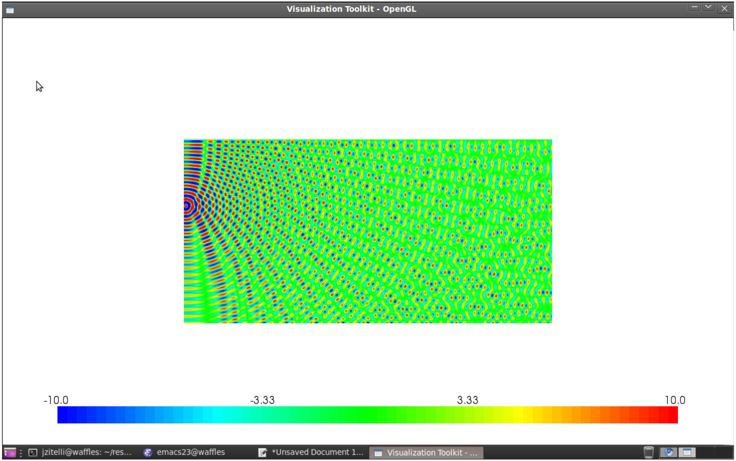
Exact solution (real part of pressure).

Pekeris problem, $k = 200$ (32 wavelenghts)



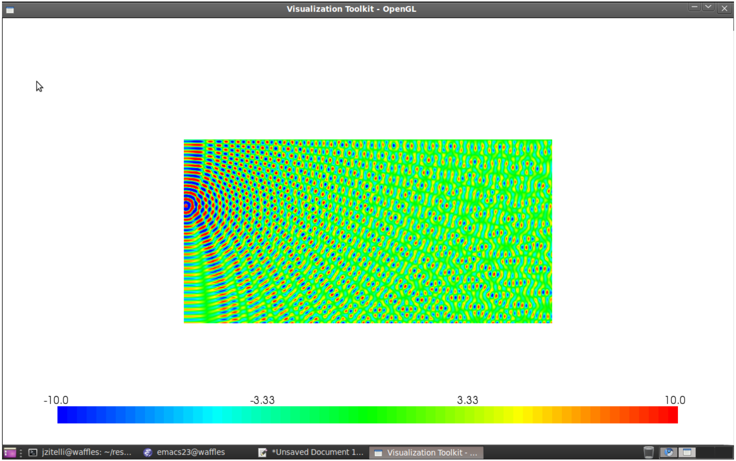
Ainsworth-Wajid quadrature, four biquadratic elements per wavelength.

Pekeris problem, $k = 200$ (32 wavelenghts)



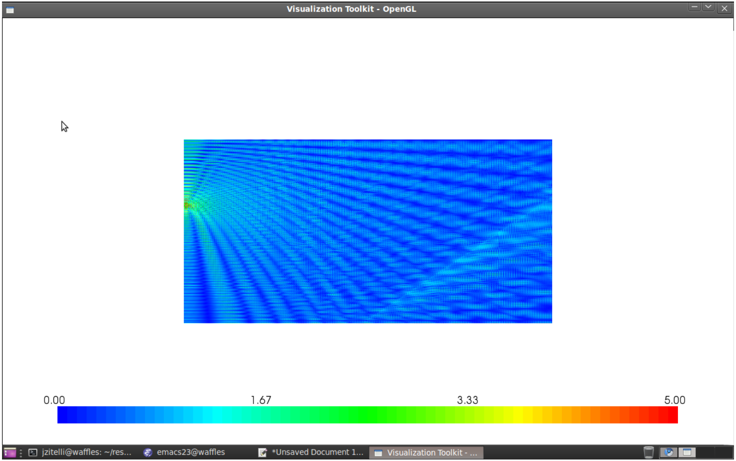
Exact solution (real part of pressure).

Pekeris problem, $k = 200$ (32 wavelenghts)



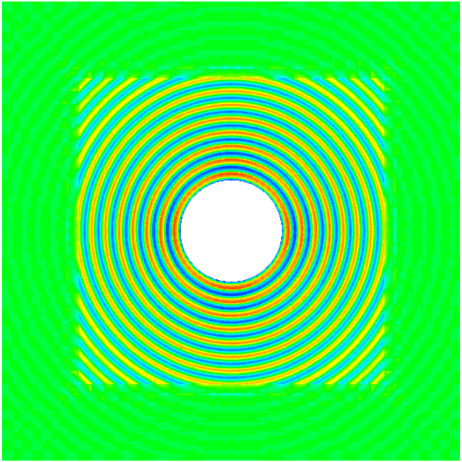
DPG method, four bilinear elements per wavelength.

Pekeris problem, $k = 200$ (32 wavelenghts)



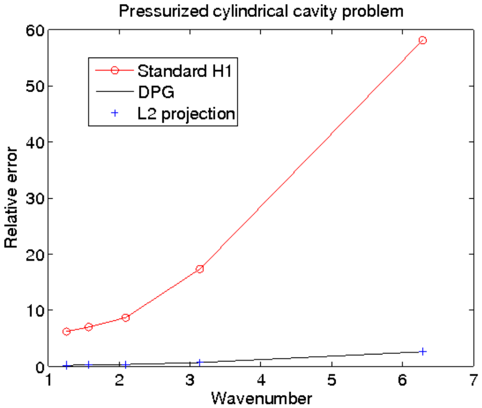
Error for the DPG method.

2D elastodynamics



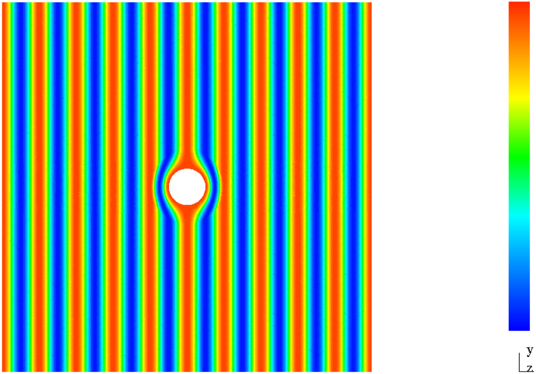
Pressurized cylindrical cavity problem with PML layer. Radial component of velocity.

2D elastodynamics



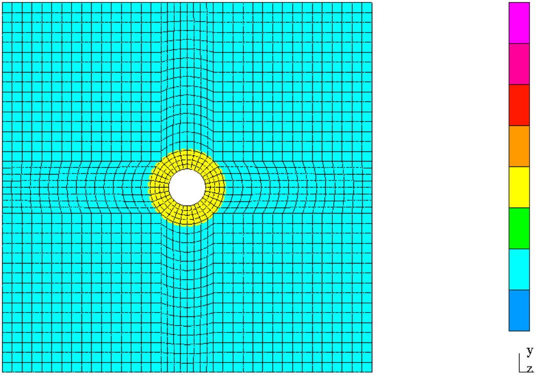
Pressurized cylindrical cavity problem with PML layer. Comparison of relative L^2 error for standard FEs and DPG with the BAE for increasing wave numbers.

2D acoustics (electromagnetics) cloaking problem



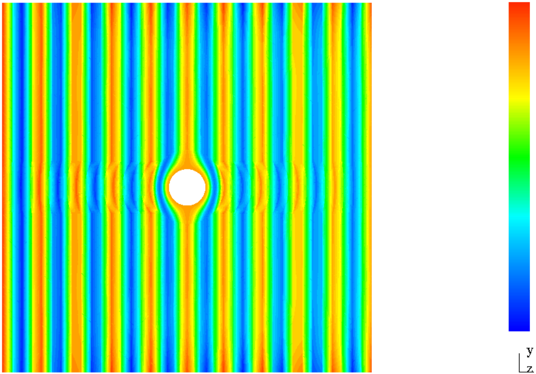
Exact solution (pressure or magnetic field)

2D acoustics (electromagnetics) cloaking problem



An hp mesh (4 bilinear elements per wavelength)

2D acoustics (electromagnetics) cloaking problem



Numerical solution (pressure or magnetic field)

Lectures

- Petrov-Galerkin Method with Optimal Test Functions.
- Ultraweak variational formulation and the DPG method for convection-dominated diffusion.
- 1D analysis. Adaptivity.
- Wave propagation as an example of a complex-valued problem.
- Systematic choice of test norms. Robustness.
- Convergence proofs.

A Recipe:

How to Construct a Robust DPG Method

for the Confusion Problem

(and Any Other Linear Problem as Well)

Step 1: Decide what you want

We want the L^2 robustness in u:

$$\|u\| \lesssim \|(u, \boldsymbol{\sigma}, \hat{u}, \hat{q})\|_E$$

($a \lesssim b$ means that there exists a constant C, independent of ϵ such that $a \leq Cb$). This implies

$$\begin{aligned} \|u - u_h\| &\lesssim \|(u - u_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \hat{u} - \hat{u}_h, \hat{q} - \hat{q}_h)\|_E \\ &= \underbrace{\inf_{(u_h, \boldsymbol{\sigma}_h, \hat{u}_h, \hat{q}_h)} \|(u - u_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \hat{u} - \hat{u}_h, \hat{q} - \hat{q}_h)\|_E}_{\text{Best Approximation Error (BAE)}} \\ &\leq C(\epsilon) h^p \end{aligned}$$

Step 2: Select a special test function...

$$
\begin{aligned}
b((u,\boldsymbol{\sigma},\hat{u},\hat{q}),(v,\boldsymbol{\tau})) &= (\boldsymbol{\sigma}, \tfrac{1}{\epsilon}\boldsymbol{\tau} + \boldsymbol{\nabla} v)_{\Omega_h} + (u, \mathrm{div}\boldsymbol{\tau} - \boldsymbol{\beta}\cdot\boldsymbol{\nabla} v)_{\Omega_h} \\
&\quad - < \hat{u}, \tau_n >_{\Gamma_h^0} - < \hat{q}, v >_{\Gamma_h}
\end{aligned}
$$

Choose a test function $(v, \boldsymbol{\tau})$ such that

$$
\left\{
\begin{array}{rl}
v \in H_0^1(\Omega),\ \boldsymbol{\tau} \in \boldsymbol{H}(\mathrm{div},\Omega) & \\
\frac{1}{\epsilon}\boldsymbol{\tau} + \boldsymbol{\nabla} v & = 0 \\
\mathrm{div}\boldsymbol{\tau} - \boldsymbol{\beta}\cdot\boldsymbol{\nabla} v & = u
\end{array}
\right.
$$

Then

$$
\begin{aligned}
\|u\|^2 &= b((u,\boldsymbol{\sigma},\hat{u},\hat{q}),(v,\boldsymbol{\tau})) = \frac{b((u,\boldsymbol{\sigma},\hat{u},\hat{q}),(v,\boldsymbol{\tau}))}{\|(v,\boldsymbol{\tau})\|_V}\|(v,\boldsymbol{\tau})\|_V \\
&\le \sup_{(v,\boldsymbol{\tau})} \frac{b((u,\boldsymbol{\sigma},\hat{u},\hat{q}),(v,\boldsymbol{\tau}))}{\|(v,\boldsymbol{\tau})\|_V}\|(v,\boldsymbol{\tau})\|_V = \|(u,\boldsymbol{\sigma},\hat{u},\hat{q})\|_E\, \|(v,\boldsymbol{\tau})\|_V
\end{aligned}
$$

… and request stability of the adjoint problem

Consequently, we need to select the test norm in such a way that

$$\|(v, \boldsymbol{\tau})\|_V \lesssim \|u\|$$

This gives,

$$\|u\|^2 \lesssim \|(u, \boldsymbol{\sigma}, \hat{u}, \hat{q})\|_E \, \|u\|$$

Dividing by $\|u\|$, we get what we wanted.
The point: Construction of a robust DPG reduces to the classical stability analysis for the adjoint equation!

Step 3: Study the stability of the adjoint equation

Theorem (Generalization of Erickson-Johnson Theorem) (Heuer,D., 2011)

$$\left.\begin{array}{r} \|v\| \\ \|\boldsymbol{\beta}\cdot\boldsymbol{\nabla} v\|_w, \sqrt{\epsilon}\|\boldsymbol{\nabla} v\| \\ \|\mathrm{div}\boldsymbol{\tau}\|_{w+\epsilon}, \frac{1}{\epsilon}\|\boldsymbol{\beta}\cdot\boldsymbol{\tau}\|_w, \frac{1}{\sqrt{\epsilon}}\|\boldsymbol{\tau}\| \end{array}\right\} \lesssim \|u\|$$

where $w = O(1)$ is a weight vanishing on the inflow boundary that satisfies some "mild" assumptions.

The terms on the left-hand side are our "Lego" blocks with which we can build different test norms.

Step 4: Construct test norm(s)

Quasi-optimal test norm:

$$\|(v, \boldsymbol{\tau})\|_1^2 := \|v\|^2 + \|\frac{1}{\epsilon}\boldsymbol{\tau} + \boldsymbol{\nabla} v\|^2 + \|\mathrm{div}\boldsymbol{\tau} - \boldsymbol{\beta} \cdot \boldsymbol{\nabla} v\|^2$$

Weighted norm:

$$\|(v, \boldsymbol{\tau})\|_2^2 := \epsilon\|v\|^2 + \|\boldsymbol{\beta} \cdot \boldsymbol{\nabla} v\|_w^2 + \epsilon\|\boldsymbol{\nabla} v\|^2 + \|\boldsymbol{\tau}\|_{w+\epsilon}^2 + \|\mathrm{div}\boldsymbol{\tau}\|_{w+\epsilon}^2$$

Remark: Both choices imply also L^2-robustness in $\boldsymbol{\sigma}$, as well as in traces and fluxes measured in special energy norms.

Estimates for $\boldsymbol{\sigma}, \hat{u}, \hat{q}$

Same methodology can be used to design a test norm that will imply,

$$\|\boldsymbol{\sigma}\| \lesssim \|(\boldsymbol{\sigma}, u, \hat{u}, \hat{q})\|_E$$

In fact both quasioptimal and weighted norms imply the robust estimate for $\boldsymbol{\sigma}$. They also imply a robust estimate for traces and fluxes measured in a minimum extension norm implied by the problem,

$$(*) \qquad \|(\hat{u}, \hat{q})\|^2 := \|\frac{1}{\epsilon}\Sigma - \boldsymbol{\nabla} U\|^2 + \| - \mathrm{div}\Sigma + \boldsymbol{\beta} \cdot \boldsymbol{\nabla} U\|^2$$

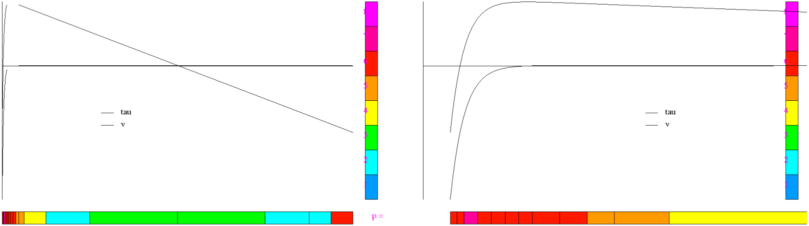
where Σ, U are extensions of $\hat{u}, \hat{q}$ from mesh skeleton to the whole domain,

$$U = \hat{u} \text{ on } \Gamma_h^0, \quad (\Sigma - \boldsymbol{\beta} U) \cdot \boldsymbol{n}_e = \hat{q} \text{ on } \Gamma_h$$

that minimize the right hand side of $(*)$.

Pros and cons for both test norms

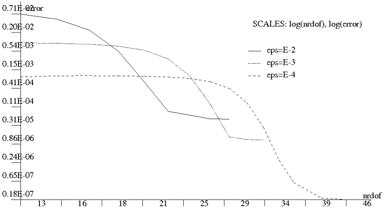
- The quasi-optimal test norm produces strong boundary layers that need to be resolved, also in 1D,

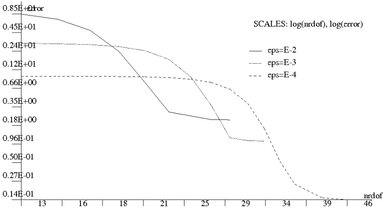


Left: τ and v components of the optimal test function corresponding to trial function $u = 1$ and element size $h = 0.25$, along with the optimal hp subelement mesh. Right: 10 $\times$ zoom on the left end of the element.
Determining optimal test functions is expensive.

- The weighted test norm produces no boundary layers. Solving for the optimal test functions is inexpensive.
- Quasi-optimal test norm yields better estimates for the best approximation error measured in the corresponding energy norm.

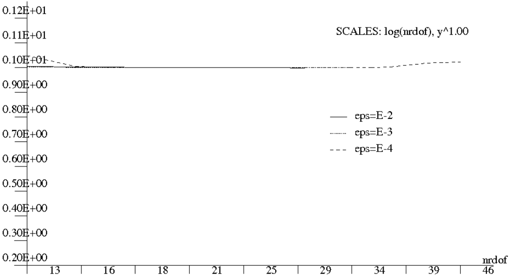
1D: Quasi-Optimal Test Norm, $\epsilon = 10^{-2}, 10^{-3}, 10^{-4}$





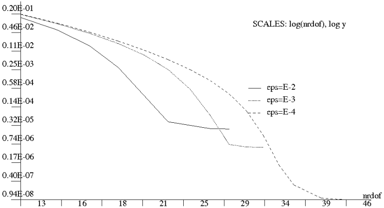
Left: convergence in energy error. Right: convergence in relative L^2-error for the field variables (in percent of their L^2-norm).

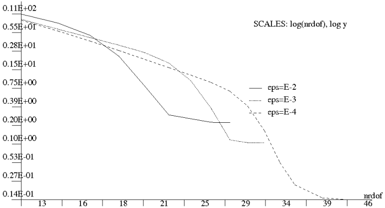
1D: Quasi-Optimal Test Norm, $\epsilon = 10^{-2}, 10^{-3}, 10^{-4}$



Ratio of L^2 and energy norms.

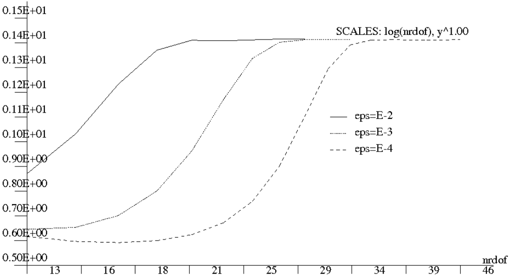
1D: Weighted Test Norm, $\epsilon = 10^{-2}, 10^{-3}, 10^{-4}$





Left: convergence in energy error. Right: convergence in relative L^2-error for the field variables (in percent of their L^2-norm).

1D: Weighted Test Norm, $\epsilon = 10^{-2}, 10^{-3}, 10^{-4}$

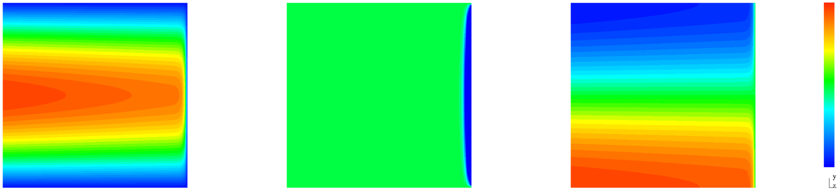


Ratio of L^2 and energy norms.

2D: Model problem of Erickson and Johnson

$$\Omega = (0,1)^2, \quad \boldsymbol{\beta} = (1,0), f = 0, \qquad u_0 = \begin{cases} \sin \pi y & \text{on } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

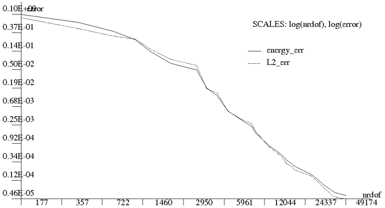
The problem can be solved analytically using separation of variables.



Velocity u and "stresses" σ_x, σ_y (using scale for σ_y) for $\epsilon = 0.01$.

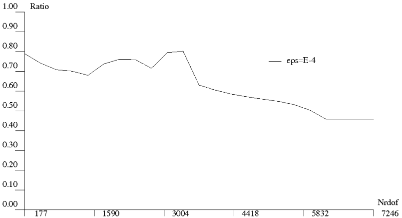
2D: Weighted norm, $\epsilon = 10^{-2}$

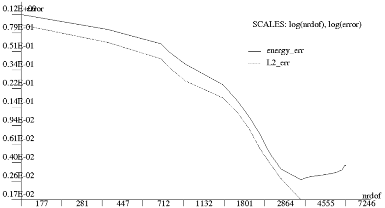
Weight: $w = x$.



Ratio of energy and L^2 errors (left), energy vs L^2 error (right) for 29 hp-adaptive meshes. Relative $L2$-error range 12.6 - 0.00068 % .

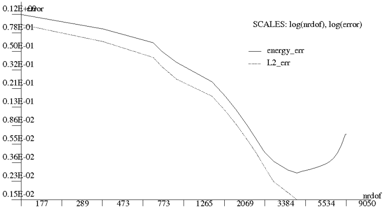
2D: Weighted norm, $\epsilon = 10^{-4}$





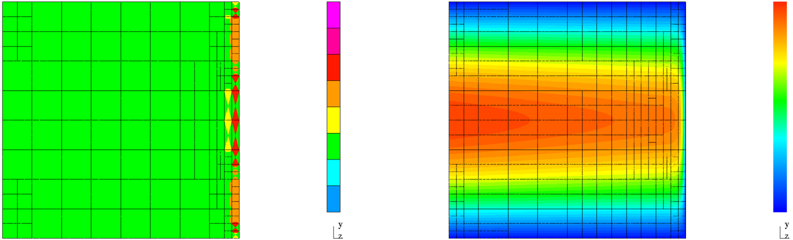
Ratio of energy and L^2 errors (left), energy vs L^2 error (right) for 23 hp-adaptive meshes. Relative $L2$-error range 13.5 - 0.24 % .

2D: Weighted norm, $\epsilon = 10^{-5}$



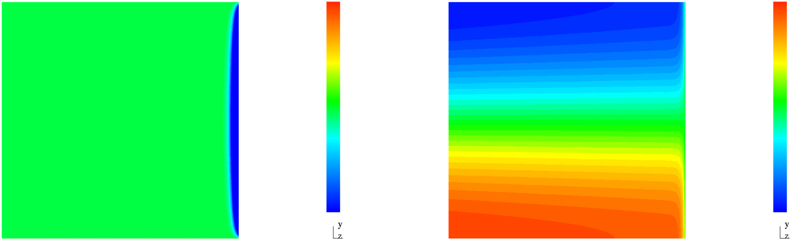
Ratio of energy and L^2 errors (left), energy vs L^2 error (right) for 27 hp-adaptive meshes. Relative $L2$-error range 13.5 - 0.21 % .

2D: Weighted norm, $\epsilon = 10^{-2}$



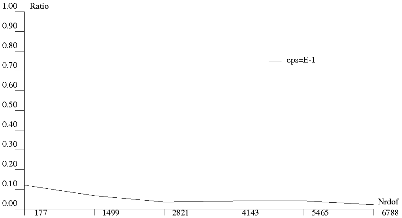
Optimal hp mesh corresponding to 0.006 % L^2 error and the corresponding u component of the solution.

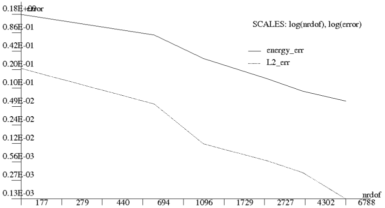
2D: Weighted norm, $\epsilon = 10^{-2}$



σ_x and σ_y components of the solution.

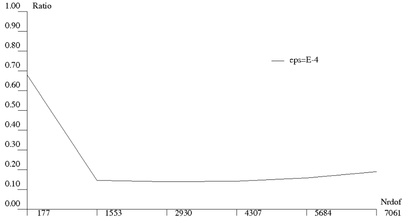
2D: Quasi-optimal norm, $\epsilon = 10^{-1}$

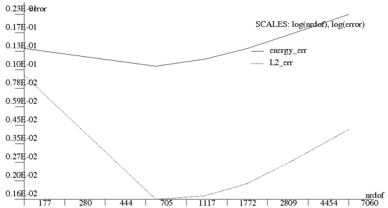




Ratio of energy and L^2 errors (left), energy vs L^2 error (right) for 5 h-adaptive meshes. Relative $L2$-error range 4.3 - 0.0267 % . Optimal test functions obtained with $\Delta p = 6$.

2D: Quasi-optimal norm, $\epsilon = 10^{-4}$

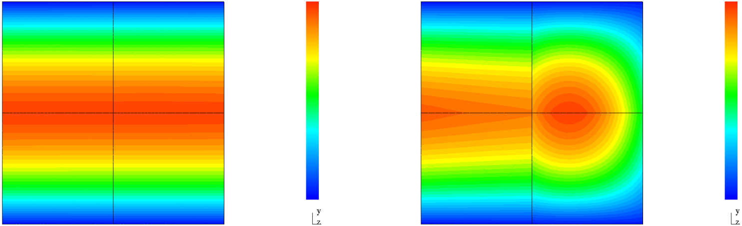




Ratio of energy and L^2 errors (left), energy vs L^2 error (right) for 6 h-adaptive meshes.

Relative $L2$-error range 1.3 - 0.6 % . Optimal test functions obtained with Shishkin meshes and $\Delta p = 2$. The non-monotone behavior of the energy error indicates a significant error in the resolution of optimal test functions.

2D: Eye-ball norm comparison for $\epsilon = 10^{-4}$



Velocity u on the initial mesh of four quadratic elements for quasi-optimal (left) and weighted (right) norms.

Lectures

- Petrov-Galerkin Method with Optimal Test Functions.
- Ultraweak variational formulation and the DPG method for convection-dominated diffusion.
- 1D analysis. Adaptivity.
- Wave propagation as an example of a complex-valued problem.
- Systematic choice of test norms. Robustness.
- Convergence proofs.

Convergence Analysis in Multidimensions

Poisson Problem

$$\begin{cases} u &= u_0 \quad \text{on } \partial\Omega \\ -\boldsymbol{\nabla}\cdot(\boldsymbol{\alpha}\boldsymbol{\nabla} u) + \boldsymbol{\beta}\cdot\boldsymbol{\nabla} u &= f \quad \text{in } \Omega \end{cases}$$

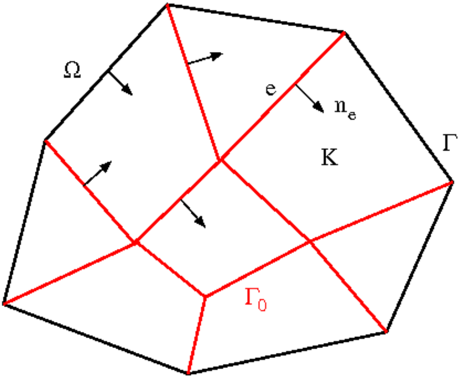
For a moment $\boldsymbol{\beta} = \mathbf{0}$.

Poisson Problem

First order system:

$$\begin{cases} \boldsymbol{\alpha}^{-1}\boldsymbol{\sigma} - \boldsymbol{\nabla} u &= \mathbf{0} \quad \text{in } \Omega \\ \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} &= f \quad \text{in } \Omega \\ u &= u_0 \quad \text{on } \partial\Omega \end{cases}$$

DPG Method



Elements:K
Edges:e
Skeleton:$\Gamma_h = \bigcup_K \partial K$
Internal skeleton:$\Gamma_h^0 = \Gamma_h - \partial\Omega$

Take an element K. Multiply the equations with test functions $\boldsymbol{\tau} \in \boldsymbol{H}(\text{div}, K), v \in H^1(K)$:

$$\begin{cases} (\boldsymbol{\alpha}^{-1}\boldsymbol{\sigma}) \cdot \boldsymbol{\tau} - (\boldsymbol{\nabla} u) \cdot \boldsymbol{\tau} &= 0 \\ (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}) v &= f v \end{cases}$$

DPG Method

Integrate over the element K:

$$\left\{ \begin{aligned} \int_K (\boldsymbol{\alpha}^{-1}\boldsymbol{\sigma}) \cdot \boldsymbol{\tau} - \int_K (\boldsymbol{\nabla} u) \cdot \boldsymbol{\tau} &= 0 \\ \int_K (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}) v &= \int_K f v \end{aligned} \right.$$

DPG Method

Integrate by parts (relax) *both* equations:

$$\left\{ \begin{array}{rl} \int_K (\boldsymbol{\alpha}^{-1}\boldsymbol{\sigma}) \cdot \boldsymbol{\tau} + \int_K u \,\mathrm{div}\boldsymbol{\tau} - \int_{\partial K} u \,\tau_n & = 0 \\ - \int_K \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} v + \int_{\partial K} q \,\mathrm{sgn}(\boldsymbol{n}) v & = \int_K f v \end{array} \right.$$

where $q = \boldsymbol{\sigma}\boldsymbol{n}_e$ and

$$\mathrm{sgn}(\boldsymbol{n}) = \left\{ \begin{array}{rl} 1 & \text{if } \boldsymbol{n} = \boldsymbol{n}_e \\ -1 & \text{if } \boldsymbol{n} = -\boldsymbol{n}_e \end{array} \right.$$

Declare fluxes to be independent unknowns:

$$\begin{cases} \int_K (\boldsymbol{\alpha}^{-1}\boldsymbol{\sigma}) \cdot \boldsymbol{\tau} + \int_K u \, \mathrm{div}\boldsymbol{\tau} - \int_{\partial K} \hat{u} \, \tau_n & = 0 \\ - \int_K \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} v + \int_{\partial K} \hat{q} \, \mathrm{sgn}(\boldsymbol{n}) v & = \int_K f v \end{cases}$$

where $q = \boldsymbol{\sigma}\boldsymbol{n}_e$ and

$$\mathrm{sgn}(\boldsymbol{n}) = \begin{cases} 1 & \text{if } \boldsymbol{n} = \boldsymbol{n}_e \\ -1 & \text{if } \boldsymbol{n} = -\boldsymbol{n}_e \end{cases}$$

Use BCs to eliminate known fluxes

$$\left\{ \begin{array}{rl} \int_K (\boldsymbol{\alpha}^{-1}\boldsymbol{\sigma}) \cdot \boldsymbol{\tau} + \int_K u \, \mathsf{div}\boldsymbol{\tau} - \int_{\partial K - \partial\Omega} \hat{u} \, \tau_n & = + \int_{\partial K \cap \partial\Omega} u_0 \, \tau_n \\ - \int_K \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} v + \int_{\partial K} \hat{q} \, \mathsf{sgn}(\boldsymbol{n}) v & = \int_K f v \end{array} \right.$$

Trace and Flux Spaces

$$
\begin{aligned}
\Gamma_h &:= \bigcup_K \partial K \quad \text{(skeleton)} \\
\Gamma_h^0 &:= \Gamma_h - \partial\Omega \quad \text{(internal skeleton)} \\
\tilde{H}^{1/2}(\Gamma_h^0) &:= \{V|_{\Gamma_h^0} \;:\; V \in H_0^1(\Omega)\} \\
& \quad \text{with the minimum extension norm:} \\
\|v\|_{\tilde{H}^{1/2}(\Gamma_h^0)} &:= \inf\{\|V\|_{H^1} \;:\; V|_{\Gamma_h^0} = v\} \\
H^{-1/2}(\Gamma_h) &:= \{\sigma_n|_{\Gamma_h} \;:\; \boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div},\Omega)\} \\
& \quad \text{with the minimum extension norm:} \\
\|\sigma_n\|_{H^{-1/2}(\Gamma_h)} &:= \inf\{\|\boldsymbol{\sigma}\|_{H(\mathrm{div},\Omega)} \;:\; \boldsymbol{\sigma n}|_{\Gamma_h} = \sigma_n\}
\end{aligned}
$$

Functional Setting

Group variables:
Solution $\boldsymbol{U} = (u, \boldsymbol{\sigma}, \hat{u}, \hat{q})$:

$$u, \sigma_1, \sigma_2 \in L^2(\Omega_h)$$
$$\hat{u} \in \tilde{H}^{1/2}(\Gamma_h^0)$$
$$\hat{q} \in H^{-1/2}(\Gamma_h)$$

Test function $\boldsymbol{V} = (\boldsymbol{\tau}, v)$:

$$\boldsymbol{\tau} \in \boldsymbol{H}(\mathrm{div}, \Omega_h)$$
$$v \in H^1(\Omega_h)$$

Variational problem:

$$b(\boldsymbol{U}, \boldsymbol{V}) = l(\boldsymbol{V}), \quad \forall \boldsymbol{V}$$

Simple facts

- Form b is continuous
- $b(U,V) = 0, \forall V$ implies $U = 0$.

In operator terms,

$$b(U,V) =< BU, V >=< U, B^*V >$$

B is injective, B, B^* are well-defined and continuous.

Well-Posedness

Theorem 1
The DPG variational formulation is well-posed with a mesh-independent inf-sup constant.

Theorem 2
There exists a mesh-independent $C > 0$:

$$\|u - u_{hp}\|_{L^2(\Omega)} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_{hp}\|_{L^2(\Omega)}$$
$$+\|\hat{u} - \hat{u}_{hp}\|_{\tilde{H}^{1/2}(\Gamma_h^0)} + \|\hat{q} - \hat{q}_{hp}\|_{H^{-1/2}(\Gamma_h)}$$
$$\leq C \inf_{\boldsymbol{\sigma}_{hp}, u_{hp}, \hat{q}_{hp}, \hat{u}_{hp}} [...]$$

where $u_{hp}, \boldsymbol{\sigma}_{hp}, \hat{u}_{hp}, \hat{q}_{hp}$ is the DPG FE solution.

Optimal Test Norm

Define:

$$\|\boldsymbol{V}\|_o = \|B^*V\| = \sup_{\boldsymbol{U}} \frac{|b(\boldsymbol{U},\boldsymbol{V})|}{\|\boldsymbol{U}\|_U}$$

$$= \sup_{u,\sigma,\hat{u},\hat{q}} \frac{(u, -\mathrm{div}\boldsymbol{\tau})_\Omega + (\boldsymbol{\sigma}, \boldsymbol{\alpha}^{-1}\boldsymbol{\tau} - \boldsymbol{\nabla} v)_\Omega + < \hat{u}, \tau_n >_{\Gamma_h^0} + < v, \hat{q} >_{\Gamma_h}}{(\|u\|^2 + \|\boldsymbol{\sigma}\|^2 + \|\hat{u}\|^2 + \|\hat{q}\|^2)^{1/2}}$$

$$= \left(\|\mathrm{div}\boldsymbol{\tau}\|^2 + \|\boldsymbol{\alpha}^{-1}\boldsymbol{\tau} - \boldsymbol{\nabla} v\|^2 + \|[v]\|^2_{\Gamma_h^0} + \|[\tau_n]\|^2_{\Gamma_h} \right)^{1/2}$$

where

$$\|[v]\|_{\Gamma_h^0} = \sup_{w \in H(\mathrm{div},\Omega)} \frac{< v, w_n >_{\Gamma_h}}{\|\boldsymbol{w}\|_{H(\mathrm{div},\Omega)}}$$

$$\|[\tau_n]\|_{\Gamma_h} = \sup_{w \in H_0^1(\Omega)} \frac{< w, \tau_n >_{\Gamma_h^0}}{\|w\|_{H^1(\Omega)}}$$

Equivalence of Norms

We will show that the standard and optimal norms are equivalent, i.e.

$$\|\boldsymbol{V}\| \leq C\|\boldsymbol{V}\|_o \quad \text{and} \quad \|\boldsymbol{V}\|_o \leq C\|\boldsymbol{V}\|$$

The second inequality is straightforward, we will focus on the first one.
Conclusions:

- B^* is injective,
- b satisfies the inf-sup condition (B is bounded below).

Consequently, Nečas - Babuška (Generalized Lax-Milgram, Lions, Banach Closed Range) Theorem implies that the variational problem is well-posed. Theorem 2 follows.

Proof

Take $\boldsymbol{\tau} \in \boldsymbol{H}(\text{div}, \Omega_h), v \in H^1(\Omega_h)$. Denote

$$\begin{aligned} \boldsymbol{\alpha}^{-1}\boldsymbol{\tau} - \boldsymbol{\nabla} v =: &\quad \boldsymbol{f} \\ \text{div}\boldsymbol{\tau} =: &\quad g \end{aligned}$$

Need to show the bounds:

$$\|\boldsymbol{\tau}\|_{H(\text{div},\Omega_h)}, \|v\|_{H^1(\Omega_h)} \le C(\|\boldsymbol{f}\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)} + \|[v]\|_{\Gamma_h^0} + \|[\tau_n]\|_{\Gamma_h})$$

Step 1: $\boldsymbol{f} = \boldsymbol{0}, g = 0$.
Consider the weighted Helmholtz decomposition:

$$\boldsymbol{\tau} = \boldsymbol{\alpha}\boldsymbol{\nabla}\psi + \boldsymbol{\nabla} \times \boldsymbol{z}, \quad \psi \in H_0^1(\Omega), \boldsymbol{z} \in \boldsymbol{H}(\textbf{curl}, \Omega)$$

Potentials $\psi, \boldsymbol{\tau}$ are unique, orthogonal in the weighted $(\boldsymbol{\alpha}^{-1}\cdot, \cdot)$ L^2-product, and depend continuously upon $\boldsymbol{\tau}$.

Step 1: $f, g = 0$

$$\|\boldsymbol{\tau}\|^2_{\alpha^{-1}} = (\boldsymbol{\alpha}^{-1}\boldsymbol{\tau}, \boldsymbol{\tau}) = (\boldsymbol{\alpha}^{-1}\boldsymbol{\tau}, \boldsymbol{\alpha}\boldsymbol{\nabla}\psi + \boldsymbol{\nabla} \times \boldsymbol{z})_{\Omega_h}$$

Step 1: $\boldsymbol{f}, g = 0$

$$
\begin{aligned}
\|\boldsymbol{\tau}\|^2_{\alpha^{-1}} &= (\boldsymbol{\alpha}^{-1}\boldsymbol{\tau}, \boldsymbol{\tau}) = (\boldsymbol{\alpha}^{-1}\boldsymbol{\tau}, \boldsymbol{\alpha}\boldsymbol{\nabla}\psi + \boldsymbol{\nabla} \times \boldsymbol{z})_{\Omega_h} \\
&= (\boldsymbol{\tau}, \boldsymbol{\nabla}\psi)_{\Omega_h} + (\boldsymbol{\nabla} v, \boldsymbol{\nabla} \times \boldsymbol{z})_{\Omega_h}
\end{aligned}
$$

Step 1: $f, g = 0$

$$
\begin{aligned}
\|\boldsymbol{\tau}\|^2_{\alpha^{-1}} &= (\boldsymbol{\alpha}^{-1}\boldsymbol{\tau}, \boldsymbol{\tau}) = (\boldsymbol{\alpha}^{-1}\boldsymbol{\tau}, \boldsymbol{\alpha}\boldsymbol{\nabla}\psi + \boldsymbol{\nabla} \times \boldsymbol{z})_{\Omega_h} \\
&= (\boldsymbol{\tau}, \boldsymbol{\nabla}\psi)_{\Omega_h} + (\boldsymbol{\nabla} v, \boldsymbol{\nabla} \times \boldsymbol{z})_{\Omega_h} \\
&= -(\mathrm{div}\boldsymbol{\tau}, \psi)_{\Omega_h} + < \psi, \tau_n >_{\Gamma_h} + < v, (\boldsymbol{\nabla} \times \boldsymbol{z}) \cdot \boldsymbol{n} >_{\Gamma_h^0}
\end{aligned}
$$

Step 1: $f, g = 0$

$$
\begin{aligned}
\|\boldsymbol{\tau}\|^2_{\alpha^{-1}} &= (\boldsymbol{\alpha}^{-1}\boldsymbol{\tau}, \boldsymbol{\tau}) = (\boldsymbol{\alpha}^{-1}\boldsymbol{\tau}, \boldsymbol{\alpha}\boldsymbol{\nabla}\psi + \boldsymbol{\nabla} \times \boldsymbol{z})_{\Omega_h} \\
&= (\boldsymbol{\tau}, \boldsymbol{\nabla}\psi)_{\Omega_h} + (\boldsymbol{\nabla} v, \boldsymbol{\nabla} \times \boldsymbol{z})_{\Omega_h} \\
&= -(\mathrm{div}\boldsymbol{\tau}, \psi)_{\Omega_h} + < \psi, \tau_n >_{\Gamma_h} + < v, (\boldsymbol{\nabla} \times \boldsymbol{z}) \cdot \boldsymbol{n} >_{\Gamma_h^0} \\
&= \frac{< \psi, \tau_n >_{\Gamma_h}}{\|\psi\|_{H^1(\Omega)}} \|\psi\|_{H^1(\Omega)} + \frac{< v, (\boldsymbol{\nabla} \times \boldsymbol{z}) \cdot \boldsymbol{n} >_{\Gamma_h^0}}{\|\boldsymbol{\nabla} \times \boldsymbol{z}\|_{H(\mathrm{div},\Omega)}} \|\boldsymbol{\nabla} \times \boldsymbol{z}\|_{L^2(\Omega)}
\end{aligned}
$$

Step 1: $f, g = 0$

$$
\begin{aligned}
\|\boldsymbol{\tau}\|^2_{\alpha^{-1}} &= (\boldsymbol{\alpha}^{-1}\boldsymbol{\tau}, \boldsymbol{\tau}) = (\boldsymbol{\alpha}^{-1}\boldsymbol{\tau}, \boldsymbol{\alpha}\boldsymbol{\nabla}\psi + \boldsymbol{\nabla} \times \boldsymbol{z})_{\Omega_h} \\
&= (\boldsymbol{\tau}, \boldsymbol{\nabla}\psi)_{\Omega_h} + (\boldsymbol{\nabla} v, \boldsymbol{\nabla} \times \boldsymbol{z})_{\Omega_h} \\
&= -(\mathrm{div}\boldsymbol{\tau}, \psi)_{\Omega_h} + < \psi, \tau_n >_{\Gamma_h} + < v, (\boldsymbol{\nabla} \times \boldsymbol{z}) \cdot \boldsymbol{n} >_{\Gamma_h^0} \\
&= \frac{< \psi, \tau_n >_{\Gamma_h}}{\|\psi\|_{H^1(\Omega)}} \|\psi\|_{H^1(\Omega)} + \frac{< v, (\boldsymbol{\nabla} \times \boldsymbol{z}) \cdot \boldsymbol{n} >_{\Gamma_h^0}}{\|\boldsymbol{\nabla} \times \boldsymbol{z}\|_{H(\mathrm{div},\Omega)}} \|\boldsymbol{\nabla} \times \boldsymbol{z}\|_{L^2(\Omega)} \\
&\leq \sup_{w \in H_0^1(\Omega)} \frac{< w, \tau_n >_{\Gamma_h}}{\|w\|_{H^1(\Omega)}} \|\psi\|_{H^1(\Omega)} + \sup_{w \in H(\mathrm{div},\Omega)} \frac{< v, w_n >_{\Gamma_h^0}}{\|\boldsymbol{w}\|_{H(\mathrm{div},\Omega)}} \|\boldsymbol{\nabla} \times \boldsymbol{z}\|_{L^2(\Omega)}
\end{aligned}
$$

Step 1: $f, g = 0$

$$
\begin{aligned}
\|\boldsymbol{\tau}\|^2_{\alpha^{-1}} &= (\boldsymbol{\alpha}^{-1}\boldsymbol{\tau}, \boldsymbol{\tau}) = (\boldsymbol{\alpha}^{-1}\boldsymbol{\tau}, \boldsymbol{\alpha}\boldsymbol{\nabla}\psi + \boldsymbol{\nabla} \times \boldsymbol{z})_{\Omega_h} \\
&= (\boldsymbol{\tau}, \boldsymbol{\nabla}\psi)_{\Omega_h} + (\boldsymbol{\nabla} v, \boldsymbol{\nabla} \times \boldsymbol{z})_{\Omega_h} \\
&= -(\mathsf{div}\boldsymbol{\tau}, \psi)_{\Omega_h} + < \psi, \tau_n >_{\Gamma_h} + < v, (\boldsymbol{\nabla} \times \boldsymbol{z}) \cdot \boldsymbol{n} >_{\Gamma_h^0} \\
&= \frac{< \psi, \tau_n >_{\Gamma_h}}{\|\psi\|_{H^1(\Omega)}} \|\psi\|_{H^1(\Omega)} + \frac{< v, (\boldsymbol{\nabla} \times \boldsymbol{z}) \cdot \boldsymbol{n} >_{\Gamma_h^0}}{\|\boldsymbol{\nabla} \times \boldsymbol{z}\|_{H(\mathsf{div},\Omega)}} \|\boldsymbol{\nabla} \times \boldsymbol{z}\|_{L^2(\Omega)} \\
&\leq \sup_{w \in H_0^1(\Omega)} \frac{< w, \tau_n >_{\Gamma_h}}{\|w\|_{H^1(\Omega)}} \|\psi\|_{H^1(\Omega)} + \sup_{w \in H(\mathsf{div},\Omega)} \frac{< v, w_n >_{\Gamma_h^0}}{\|\boldsymbol{w}\|_{H(\mathsf{div},\Omega)}} \|\boldsymbol{\nabla} \times \boldsymbol{z}\|_{L^2(\Omega)} \\
&\leq C \left(\|[v]\|_{\Gamma_h^0} + \|[\tau_n]\|_{\Gamma_h} \right) \|\boldsymbol{\tau}\|_{\alpha^{-1}}
\end{aligned}
$$

Step 1: $f, g = 0$

Consequently,

$$\|\nabla v\|_{L^2(\Omega_h)} \leq C \left(\|[v]\|_{\Gamma_h^0} + \|[\tau_n]\|_{\Gamma_h} \right)$$

as well.

Discrete Poincaré Inequality:

$$\|v\|_{\Omega_h} \leq C \left(\|\nabla v\|_{\Omega_h} + \|[v]\|_{\Gamma_h^0} \right)$$

gives

$$\|v\|_{H^1(\Omega_h)} \leq C \left(\|[v]\|_{\Gamma_h^0} + \|[\tau_n]\|_{\Gamma_h} \right)$$

Step 2: $\boldsymbol{f}, g \neq 0$

Let $\boldsymbol{\tau}_1 \in \boldsymbol{H}(\mathsf{div}, \Omega), v_1 \in H_0^1(\Omega)$ such that

$$\begin{cases} \boldsymbol{\alpha}^{-1}\boldsymbol{\tau}_1 - \boldsymbol{\nabla} v_1 & = \boldsymbol{f} \\ \mathsf{div}\boldsymbol{\tau}_1 & = g \end{cases}$$

Brezzi's Theory implies

$$\|\boldsymbol{\tau}_1\|_{H(\mathsf{div},\Omega)}, |v_1\|_{H^1(\Omega)} \leq C(\|\boldsymbol{f}\| + \|g\|)$$

Final step: replace $\boldsymbol{\tau}, v$ with $\boldsymbol{\tau} - \boldsymbol{\tau}_1, v - v_1$ and use Step 1 result. Note that jump terms for $\boldsymbol{\tau} - \boldsymbol{\tau}_1, v - v_1$ are controlled by the original jump terms and norms of $\boldsymbol{\tau}_1, v_1$.

Generalization to Convection-Dominated Diffusion

In Step 1, use the decomposition:

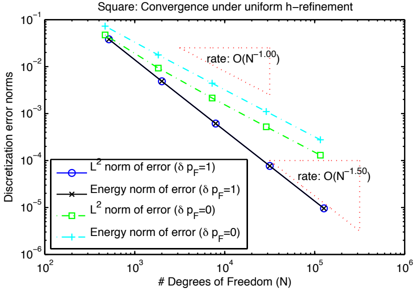
$$\boldsymbol{\tau} = (\boldsymbol{\alpha}\boldsymbol{\nabla}\psi + \boldsymbol{\beta}\psi) + \boldsymbol{\nabla} \times \boldsymbol{z}, \quad \psi \in H_0^1(\Omega), \boldsymbol{z} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$$

Numerical Experiments

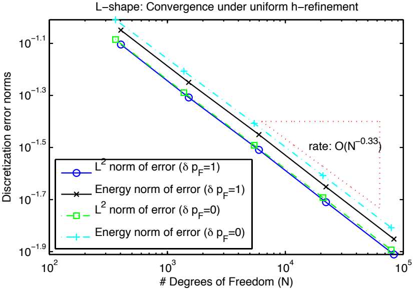
Test problems:

- Square domain with $u(x, y) = \sin(\pi x)\sin(\pi y)$,
- L-shape domain with $u(r, \theta) = r^{2/3} \sin\left(\frac{2}{3}(\theta + \frac{\pi}{2})\right)$

Uniform h-convergence rates



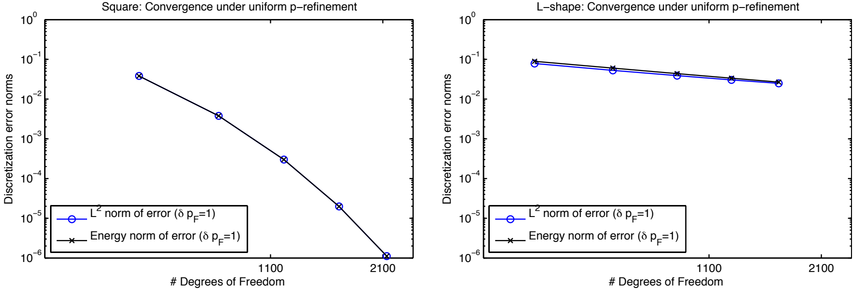
(a) The square case



(b) The case of the L-shaped domain

Figure: h-convergence rates for the two examples

Uniform p-convergence rates

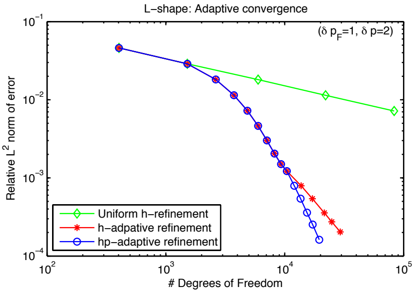


(a) Results from the square domain

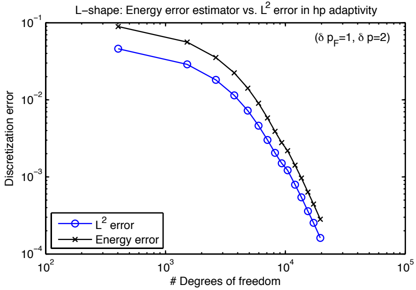
(b) Results from the L-shaped domain

Figure: p-convergence rates for the two examples

Adaptivity



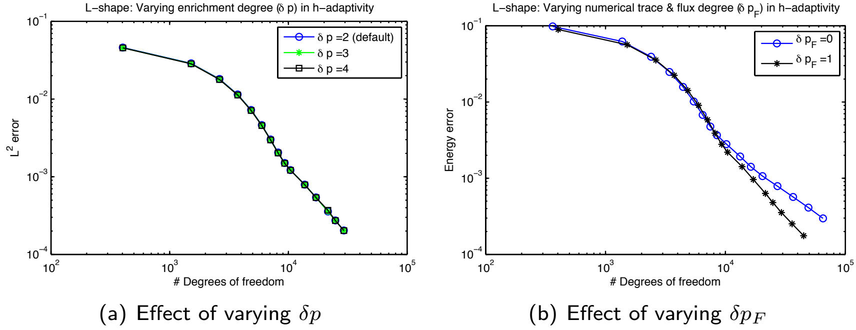
(a) Comparison of convergence of adaptive schemes



(b) Energy error estimator vs. L^2-error

Figure: Convergence curves from adaptive schemes

Adaptivity - cont.

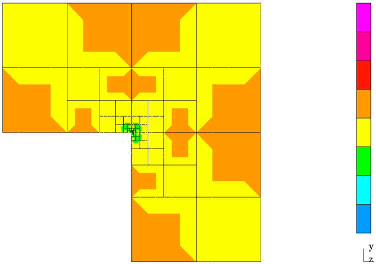


(a) Effect of varying δp

(b) Effect of varying δp_F

Figure: Convergence curves from adaptive schemes

Some Color to Finish



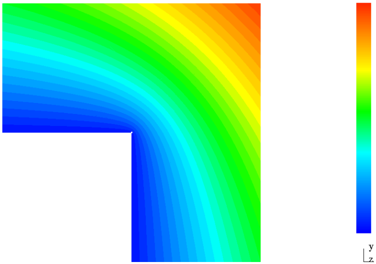


Figure: Left: The hp mesh found by the hp-adaptive algorithm after 15 refinements. (Color scale represents polynomial degrees.) Right: The corresponding solution u. (Color scale represent solution values.)

Thank You !

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DPG Code

A General Variational Problem

$$
\left\{
\begin{array}{l}
u_1, \dots, u_N \in L^2(\Omega), \quad f_1, \dots, f_L \in H^{1/2}(\Gamma_h), \quad g_1, \dots, g_M \in H^{-1/2}(\Gamma_h) \\
\int_K (\sum_{j=1}^N a_{ij} u_j) \, \mathrm{div} \boldsymbol{q}_i + \int_{\partial K} f_i \, q_{in} + \int_K (\sum_{j=1}^N \boldsymbol{b}_{ij} u_j) \cdot \boldsymbol{q}_i \\
= \int_K A_i \, \mathrm{div} \boldsymbol{q}_i + \int_{\partial K} F_i \, q_{in} + \int_K \boldsymbol{B}_i \cdot \boldsymbol{q}_i \\
\hfill \boldsymbol{q}_i \in \boldsymbol{H}(\mathrm{div}, K), \; i = 1, \dots, L \\
\int_K (\sum_{j=1}^N \boldsymbol{c}_{ij} u_j) \, \boldsymbol{\nabla} v_i + \int_{\partial K} g_i \, v_i + \int_K (\sum_{j=1}^N d_{ij} u_j) \, v_i \\
= \int_K \boldsymbol{C}_i \, \boldsymbol{\nabla} v_i + \int_{\partial K} G_i \, v_i + \int_K d_i \, v_i \\
\hfill v_i \in H^1(K), \; i = 1, \dots, M
\end{array}
\right.
$$

Number of (field) unknowns equals number of (scalar) equations,

$$N = 2L + M$$

General Test Norm

$$\|(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_L;\ v_1, \ldots, v_M)\|^2$$
$$= \sum_{j=1}^{N} \int_K |\sum_{i=1}^{L} a_{ij}\, \text{div}\boldsymbol{q}_i + \sum_{i=1}^{L} \boldsymbol{b}_{ij} \cdot \boldsymbol{q}_i + \sum_{i=1}^{M} \boldsymbol{c}_{ij} \cdot \boldsymbol{\nabla} v_i + \sum_{i=1}^{M} d_{ij} v_i|^2$$
$$+ \sum_{l=1}^{L} \int_K e_l |\boldsymbol{q}_l|^2 + \sum_{m=1}^{M} \int_K f_m |v_m|^2$$