CSE386M/EM386M
FUNCTIONAL ANALYSIS IN THEORETICAL MECHANICS
Fall 2023, Exam 2

1. Define the following notions and provide a non-trivial example (2+2 points each).

    - Quotient vector space.
    - Dual basis.
    - Matrix representation of a linear map in finite dimensional spaces.
    - Adjoint operator.
    - Borel sets.

    See the book.

2. State and prove *three* out of the following four theorems (10 points each).

    - Characterization of lim inf.
    - Rank and Nullity Theorem.
    - Relation between rank of a linear map and the rank of its transpose.
    - Properties of an (abstract) measure (Prop. 3.1.6).

    See the book.

3. This problem enforces understanding of the Weierstrass Theorem. Give examples of:

    (i) a function $f : [0,1] \to \mathbb{R}$ that does not achieve its supremum on $[0,1]$,

    (ii) a continuous function $f : \mathbb{R} \to \mathbb{R}$ that does not achieve its supremum on $\mathbb{R}$.

    (10 points)

    (i) Take

    $$f(x) = \begin{cases} x & x \in [0,1) \\ 0 & x = 1 \end{cases}$$

    (ii) Take $f(x) = x$.

4. An algebra "sanity check". Let a, b be two non-collinear vectors in $\mathbb{R}^3$, and let $(\cdot,\cdot)$ denote the canonical inner product (dot product). Consider the map:

    $$A : \mathbb{R}^3 \ni x \to (x,a)b \in \mathbb{R}^3 .$$

(a) Prove that the map is linear.

(b) Determine range and null space of the operator and, hence, the rank and nullity of the map.

(c) Determine matreix representation of map A in the canonical basis.

(d) Determine the adjoint A^* with respect to the canonical inner product.

(e) Determine the adjoint of A with respect to the weighted inner product:

$$(x, y)_w := 3x_1y_1 + x_2y_2 + 2x_3y_3 \,.$$

(20 points).

Solution:

(a) Linearity follows from the linearlity of the inner product:

$$A(\alpha_1 x_1 + \alpha_2 x_2) = (\alpha_1 x_1 + \alpha_2 x_2, a)b = \alpha_1(x_1, a)b = \alpha_2(x_2, a)b = \alpha_1 Ax_1 + \alpha_2 Ax_2 \,.$$

(b) The null space is the plane consisting of all vectors orthogonal to a,

$$\mathcal{N}(A) := \{x \,:\, Ax = 0\} = \{x \,:\, (x, a) = 0\} \,,$$

and the range is the line spanned by vector b,

$$\mathcal{R}(A) = \{\alpha b \,:\, \alpha \in \mathbb{R}\} =: \mathbb{R}\, a \,.$$

The rank is one and the nullity is two.

(c) Columns of the matrix are components of Ae_i in the canonical basis,

$$A = \begin{pmatrix} a_1b_1 & a_2b_1 & a_3b_1 \\ a_1b_2 & a_2b_2 & a_3b_2 \\ a_1b_3 & a_2b_3 & a_3b_3 \end{pmatrix} \,.$$

(d) We have:

$$(Ax, y) = ((x, a)b, y) = (x, a)(b, y) = (x, \underbrace{(y, b)a}_{=:A^*y}) \,.$$

(e) We have:

$$(Ax, y)_w = ((x, a)b, y)_w = (x, a)(b, y)_w = (x, \hat{a})_w(y, b)_w = (x, \underbrace{(y, b)_w}_{=A^*y})_w$$

where $\hat{a} = (\frac{1}{3}a_1, a_2, \frac{1}{2}a_3)$.

5. Let X, Y be two real finite dimensional vector spaces. Consider two different vector spaces:

   - space $L(X, Y)$ of all linear transformations from X into space Y,
   - space $B(Y^*, X)$ of all bilinear functionals defined on $Y^* \times X$ where Y^* is the dual space of Y.

   (a) Argue why the linear transformations and the bilinear functionals form vector spaces.

   (b) Select bases e_j and g_i for X and Y respectively, and recall (derive) representations for arbitrary linear transformations in $L(X, Y)$ relative to the bases, and bilinear functionals from $B(Y^*, X)$ with respect to bases g_i^*, e_j. Argue why the spaces $L(X, Y)$ and $B(Y^*, X)$ are isomorphic.

   (c) Attempt to construct a *canonical*[1] isomorphism between the two spaces.

   (20 points).

   Solution:

   (a) This follows from the fact that a linear combination of linear maps is a linear map, and a linear combination of bilinear functionals is a bilinear map. In other words, both families are closed with respect to the vector space operations for functions.

   (b) For a linear map $B \in L(X, Y)$, the corresponding matrix representation B_{ij} with respect to bases e_j and g_i is given by the relation:

   $$B_{ij} = \langle g_i^*, Be_j \rangle, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n$$

   where $n = \text{dim}X,\ m = \text{dim}Y$. The space of $m \times n$ matrices, Mat(m, n), is isomorphic (by construction dicussed in the book and class) to the space $L(X, Y)$ of linear transformations.

   Similarly, from the representation formula for bilinear functionals

   $$b(y^*, x) = b(\sum_{i=1}^{m} y_i g_i^*, \sum_{j=1}^{n} x_j e_j) = \sum_{i=1}^{m} \sum_{j=1}^{n} y_i x_j \underbrace{b(g_i^*, e_j)}_{=:b_{ij}}$$

   also implies that the space of bilinear functionals is isomorphic with the space of matrices Mat(m, n).

   Consequently, the two spaces, being isomorphic with the same space, must be isomorphic with each other. In perhaps simpler terms, both spaces are of the same dimension, and all spaces of the same dimension are isomorphic with each other.

[1]Constructed without using any bases

(c) The map can be defined as follows.

$$L(X,Y) \ni A \to \{Y^* \times X \ni (y^*, x) \to \langle y^*, Ax\rangle \in \mathbb{R}\} \in B(Y^*, X)\,.$$

It is well-defined, linear and injective and, therefore, surjective as well.