A

Curvilinear Systems of Coordinates

A.1 General Formulas

Given a nonlinear transformation between Cartesian coordinates $x_i, i = 1, \ldots, 3$ and general curvilinear coordinates $u_j, j = 1, \ldots, 3$,

$$x_i = x_i(u_j),$$

we introduce the basis vectors as,

$$\boldsymbol{a}_j = \frac{\partial \boldsymbol{r}}{\partial u_j} = \frac{\partial (x_k \boldsymbol{e}_k)}{\partial u_j} = \frac{\partial x_k}{\partial u_j} \boldsymbol{e}_k \, . \tag{A.1}$$

The corresponding cobasis vectors are given by:

$$\boldsymbol{a}^i = \frac{\partial u_i}{\partial x_l} \boldsymbol{e}_l \, . \tag{A.2}$$

Indeed,

$$\begin{aligned} \boldsymbol{a}_j \cdot \boldsymbol{a}^i &= (\frac{\partial x_k}{\partial u_j} \boldsymbol{e}_k) \cdot (\frac{\partial u_i}{\partial x_l} \boldsymbol{e}_l) \\ &= \frac{\partial x_k}{\partial u_j} \frac{\partial u_i}{\partial x_l} \delta_{kl} = \frac{\partial x_k}{\partial u_j} \frac{\partial u_i}{\partial x_k} = \delta_{ij} \, . \end{aligned}$$

The following general formulas can be easily derived and remembered.

$$\begin{aligned} \boldsymbol{\nabla} w &= \frac{\partial w}{\partial u_j} \boldsymbol{a}^j \\ \boldsymbol{\nabla} \cdot \boldsymbol{v} &= \frac{\partial \boldsymbol{v}}{\partial u_j} \cdot \boldsymbol{a}^j \\ \boldsymbol{\nabla} \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{E}}{\partial u_j} \times \boldsymbol{a}^j \\ \boldsymbol{\nabla} \boldsymbol{v} &= \frac{\partial \boldsymbol{v}}{\partial u_j} \otimes \boldsymbol{a}^j \end{aligned} \tag{A.3}$$

A.2 Cylindrical Coordinates

In the cylindrical coordinates (r, θ, z),

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z\,. \end{cases} \tag{A.4}$$

The corresponding basis vectors are :

$$\begin{cases} \boldsymbol{a}_r = \dfrac{\partial \boldsymbol{r}}{\partial r} = \boldsymbol{e}_r \\ \boldsymbol{a}_\theta = \dfrac{\partial \boldsymbol{r}}{\partial \theta} = r\boldsymbol{e}_\theta \\ \boldsymbol{a}_z = \dfrac{\partial \boldsymbol{r}}{\partial z} = \boldsymbol{e}_z \end{cases} \tag{A.5}$$

with the unit vectors $\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_z$ given by:

$$\begin{cases} \boldsymbol{e}_r = (\cos\theta, \sin\theta, 0)^T \\ \boldsymbol{e}_\theta = (-\sin\theta, \cos\theta, 0)^T \\ \boldsymbol{e}_z = (0, 0, 1)^T\,. \end{cases} \tag{A.6}$$

As the system is orthogonal, the calculation of the cobasis vectors reduces to a scaling only,

$$\begin{cases} \boldsymbol{a}^r = \boldsymbol{e}_r \\ \boldsymbol{a}^\theta = \dfrac{1}{r}\boldsymbol{e}_\theta \\ \boldsymbol{a}^z = \boldsymbol{e}_z\,. \end{cases} \tag{A.7}$$

Recording the derivatives of the unit vectors with respect to θ,

$$\frac{\partial \boldsymbol{e}_r}{\partial \theta} = (-\sin\theta, \cos\theta, 0)^T = \boldsymbol{e}_\theta \qquad \frac{\partial \boldsymbol{e}_\theta}{\partial \theta} = (-\cos\theta, -\sin\theta, 0)^T = -\boldsymbol{e}_r\,, \tag{A.8}$$

we specialize easily general formulas A.3 to the cylindrical case.

$$
\begin{aligned}
\boldsymbol{\nabla} w &= \frac{\partial w}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial w}{\partial \theta}\boldsymbol{e}_\theta + \frac{\partial w}{\partial z}\boldsymbol{e}_z \\
\boldsymbol{\nabla}\cdot\boldsymbol{v} &= \frac{\partial}{\partial r}(v_r\boldsymbol{e}_r + v_\theta\boldsymbol{e}_\theta + v_z\boldsymbol{e}_z)\cdot\boldsymbol{e}_r \\
&+\frac{\partial}{\partial \theta}(v_r\boldsymbol{e}_r + v_\theta\boldsymbol{e}_\theta + v_z\boldsymbol{e}_z)\cdot\frac{1}{r}\boldsymbol{e}_\theta \\
&+\frac{\partial}{\partial z}(v_r\boldsymbol{e}_r + v_\theta\boldsymbol{e}_\theta + v_z\boldsymbol{e}_z)\cdot\boldsymbol{e}_z \\
&= \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} \\
\boldsymbol{\nabla}\times\boldsymbol{E} &= -\frac{\partial}{\partial r}(E_r\boldsymbol{e}_r + E_\theta\boldsymbol{e}_\theta + E_z\boldsymbol{e}_z)\times\boldsymbol{e}_r \\
&-\frac{\partial}{\partial \theta}(E_r\boldsymbol{e}_r + E_\theta\boldsymbol{e}_\theta + E_z\boldsymbol{e}_z)\times\frac{1}{r}\boldsymbol{e}_\theta \\
&-\frac{\partial}{\partial z}(E_r\boldsymbol{e}_r + E_\theta\boldsymbol{e}_\theta + E_z\boldsymbol{e}_z)\times\boldsymbol{e}_z \\
&= \left(\frac{1}{r}\frac{\partial E_z}{\partial \theta} - \frac{\partial E_\theta}{\partial z}\right)\boldsymbol{e}_r + \left(\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r}\right)\boldsymbol{e}_\theta + \left(\frac{\partial E_\theta}{\partial r} - \frac{1}{r}\frac{\partial E_r}{\partial \theta} + \frac{E_\theta}{r}\right)\boldsymbol{e}_z \\
\boldsymbol{\nabla}\boldsymbol{u} &= \frac{\partial}{\partial r}(u_r\boldsymbol{e}_r + u_\theta\boldsymbol{e}_\theta + u_z\boldsymbol{e}_z)\otimes\boldsymbol{e}_r \\
&+\frac{\partial}{\partial \theta}(u_r\boldsymbol{e}_r + u_\theta\boldsymbol{e}_\theta + u_z\boldsymbol{e}_z)\otimes\frac{1}{r}\boldsymbol{e}_\theta \\
&+\frac{\partial}{\partial z}(u_r\boldsymbol{e}_r + u_\theta\boldsymbol{e}_\theta + u_z\boldsymbol{e}_z)\otimes\boldsymbol{e}_z \\
&= \left(\frac{\partial u_r}{\partial r}\boldsymbol{e}_r + \frac{\partial u_\theta}{\partial r}\boldsymbol{e}_\theta + \frac{\partial u_z}{\partial r}\boldsymbol{e}_z\right)\otimes\boldsymbol{e}_r \\
&+\left(\frac{\partial u_r}{\partial \theta}\boldsymbol{e}_r + u_r\boldsymbol{e}_\theta + \frac{\partial u_\theta}{\partial \theta}\boldsymbol{e}_\theta - u_\theta\boldsymbol{e}_r + \frac{\partial u_z}{\partial \theta}\boldsymbol{e}_z\right)\otimes\frac{1}{r}\boldsymbol{e}_\theta \\
&+\left(\frac{\partial u_r}{\partial z}\boldsymbol{e}_r + \frac{\partial u_\theta}{\partial z}\boldsymbol{e}_\theta + \frac{\partial u_z}{\partial z}\boldsymbol{e}_z\right)\otimes\boldsymbol{e}_z \\
&= \frac{\partial u_r}{\partial r}\boldsymbol{e}_r\otimes\boldsymbol{e}_r + \frac{\partial u_\theta}{\partial r}\boldsymbol{e}_\theta\otimes\boldsymbol{e}_r + \frac{\partial u_z}{\partial r}\boldsymbol{e}_z\otimes\boldsymbol{e}_r \\
&+\frac{1}{r}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right)\boldsymbol{e}_r\otimes\boldsymbol{e}_\theta + \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right)\boldsymbol{e}_\theta\otimes\boldsymbol{e}_\theta + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\boldsymbol{e}_z\otimes\boldsymbol{e}_\theta \\
&+\frac{\partial u_r}{\partial z}\boldsymbol{e}_r\otimes\boldsymbol{e}_z + \frac{\partial u_\theta}{\partial z}\boldsymbol{e}_\theta\otimes\boldsymbol{e}_z + \frac{\partial u_z}{\partial z}\boldsymbol{e}_z\otimes\boldsymbol{e}_z
\end{aligned}
\tag{A.9}
$$

Utilizing the integration by parts formula,

$$\int \boldsymbol{v}\cdot\boldsymbol{\nabla}\phi\, r dr d\theta dz = -\int(\boldsymbol{\nabla}\cdot\boldsymbol{v})\phi\, r dr d\theta dz\ ,$$

we can derive the formula for the divergence of a vector field in the divergence form,

$$\boldsymbol{\nabla}\cdot\boldsymbol{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}\ . \tag{A.10}$$

By the same token, we can utilize the corresponding identity for tensors,

$$\int \boldsymbol{\sigma} : \boldsymbol{\nabla}\boldsymbol{v}\, r dr d\theta dz = -\int(\mathbf{div}\boldsymbol{\sigma})\cdot\boldsymbol{v}\, r dr d\theta dz\ ,$$

to derive the formula for the divergence of the tensor field $\boldsymbol{\sigma}$,

$$
\begin{aligned}
\mathbf{div}\boldsymbol{\sigma} = &\left(\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rr}) + \frac{1}{r}\left(\frac{\partial \sigma_{r\theta}}{\partial \theta} - \sigma_{\theta\theta}\right) + \frac{\partial \sigma_{rz}}{\partial z}\right)\boldsymbol{e}_r \\
&+ \left(\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{\theta r}) + \frac{1}{r}\left(\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \sigma_{r\theta}\right) + \frac{\partial \sigma_{\theta z}}{\partial z}\right)\boldsymbol{e}_\theta \\
&+ \left(\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{zr}) + \frac{1}{r}\frac{\partial \sigma_{z\theta}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z}\right)\boldsymbol{e}_z \, .
\end{aligned} \tag{A.11}
$$

Finally, a similar exercise stemming from the formula,

$$
\int \boldsymbol{E} \cdot (\boldsymbol{\nabla} \times \boldsymbol{F})\, r dr d\theta dz = \int (\boldsymbol{\nabla} \times \boldsymbol{E}) \cdot \boldsymbol{F}\, r dr d\theta dz \, ,
$$

yields an equivalent formula for the curl in a slightly different form,

$$
\boldsymbol{\nabla} \times \boldsymbol{E} = \left(\frac{1}{r}\frac{\partial E_z}{\partial \theta} - \frac{\partial E_\theta}{\partial z}\right)\boldsymbol{e}_r + \left(\frac{\partial E_r}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}(rE_z) + \frac{E_z}{r}\right)\boldsymbol{e}_\theta + \left(\frac{1}{r}\frac{\partial}{\partial r}(rE_\theta) - \frac{1}{r}\frac{\partial E_r}{\partial \theta}\right)\boldsymbol{e}_z \, . \tag{A.12}
$$

A.3 Spherical Coordinates

In the spherical coordinates (r, ψ, θ),

$$
\begin{cases}
x = r\sin\psi\cos\theta \\
y = r\sin\psi\sin\theta \\
z = r\cos\psi \, .
\end{cases} \tag{A.13}
$$

The corresponding basis vectors are :

$$
\begin{cases}
\boldsymbol{a}_r = \dfrac{\partial \boldsymbol{r}}{\partial r} = \boldsymbol{e}_r \\
\boldsymbol{a}_\psi = \dfrac{\partial \boldsymbol{r}}{\partial \psi} = r\boldsymbol{e}_\psi \\
\boldsymbol{a}_\theta = \dfrac{\partial \boldsymbol{r}}{\partial \theta} = r\sin\psi\boldsymbol{e}_\theta
\end{cases} \tag{A.14}
$$

with the unit vectors $\boldsymbol{e}_r, \boldsymbol{e}_\psi, \boldsymbol{e}_\theta$ given by:

$$
\begin{cases}
\boldsymbol{e}_r = (\sin\psi\cos\theta, \sin\psi\sin\theta, \cos\psi)^T \\
\boldsymbol{e}_\psi = (\cos\psi\cos\theta, \cos\psi\sin\theta, -\sin\psi)^T \\
\boldsymbol{e}_\theta = (-\sin\theta, \cos\theta, 0)^T \, .
\end{cases} \tag{A.15}
$$

As the system is orthogonal, the calculation of the cobasis vectors reduces to a scaling only,

$$
\begin{cases}
\boldsymbol{a}^r = \boldsymbol{e}_r \\
\boldsymbol{a}^\psi = \dfrac{1}{r}\boldsymbol{e}_\psi \\
\boldsymbol{a}^\theta = \dfrac{1}{r\sin\psi}\boldsymbol{e}_\theta \, .
\end{cases} \tag{A.16}
$$

Recording the derivatives of the unit vectors with respect to (r, ψ, θ),

$$\begin{aligned} &\frac{\partial \boldsymbol{e}_r}{\partial r} = \boldsymbol{0} \quad \frac{\partial \boldsymbol{e}_r}{\partial \psi} = \boldsymbol{e}_\psi \quad \frac{\partial \boldsymbol{e}_r}{\partial \theta} = \sin\psi\, \boldsymbol{e}_\theta \\ &\frac{\partial \boldsymbol{e}_\psi}{\partial r} = \boldsymbol{0} \quad \frac{\partial \boldsymbol{e}_\psi}{\partial \psi} = -\boldsymbol{e}_r \quad \frac{\partial \boldsymbol{e}_\psi}{\partial \theta} = \cos\psi\, \boldsymbol{e}_\theta \\ &\frac{\partial \boldsymbol{e}_\theta}{\partial r} = \boldsymbol{0} \quad \frac{\partial \boldsymbol{e}_\theta}{\partial \psi} = \boldsymbol{0} \quad \frac{\partial \boldsymbol{e}_\theta}{\partial \theta} = -\sin\psi\, \boldsymbol{e}_r - \cos\psi\, \boldsymbol{e}_\psi\ , \end{aligned} \tag{A.17}$$

we specialize easily general formulas A.3 to the spherical case.

$$\begin{aligned} \boldsymbol{\nabla} w &= \frac{\partial w}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial w}{\partial \psi}\boldsymbol{e}_\psi + \frac{1}{r\sin\psi}\frac{\partial w}{\partial \theta}\boldsymbol{e}_\theta \\ \boldsymbol{\nabla}\cdot\boldsymbol{v} &= \frac{\partial}{\partial r}(v_r\boldsymbol{e}_r + v_\psi\boldsymbol{e}_\psi + v_\theta\boldsymbol{e}_\theta)\cdot\boldsymbol{e}_r \\ &+\frac{\partial}{\partial \psi}(v_r\boldsymbol{e}_r + v_\psi\boldsymbol{e}_\psi + v_\theta\boldsymbol{e}_\theta)\cdot\frac{1}{r}\boldsymbol{e}_\psi \\ &+\frac{\partial}{\partial \theta}(v_r\boldsymbol{e}_r + v_\psi\boldsymbol{e}_\psi + v_\theta\boldsymbol{e}_\theta)\cdot\frac{1}{r\sin\psi}\boldsymbol{e}_\theta \\ &= \frac{\partial v_r}{\partial r} + 2\frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\psi}{\partial \psi} + \frac{v_\psi}{r\tan\psi} + \frac{1}{r\sin\psi}\frac{\partial v_\theta}{\partial \theta} \\ \boldsymbol{\nabla}\times\boldsymbol{E} &= -\frac{\partial}{\partial r}(E_r\boldsymbol{e}_r + E_\psi\boldsymbol{e}_\psi + E_\theta\boldsymbol{e}_\theta)\times\boldsymbol{e}_r \\ &-\frac{\partial}{\partial \psi}(E_r\boldsymbol{e}_r + E_\psi\boldsymbol{e}_\psi + E_\theta\boldsymbol{e}_\theta)\times\frac{1}{r}\boldsymbol{e}_\psi \\ &-\frac{\partial}{\partial \theta}(E_r\boldsymbol{e}_r + E_\psi\boldsymbol{e}_\psi + E_\theta\boldsymbol{e}_\theta)\times\frac{1}{r\sin\psi}\boldsymbol{e}_\theta \\ &= \left(\frac{1}{r}\frac{\partial E_\theta}{\partial \psi} - \frac{1}{r\sin\psi}\frac{\partial E_\psi}{\partial \theta} + \frac{E_\theta}{r\tan\psi}\right)\boldsymbol{e}_r + \left(\frac{1}{r\sin\psi}\frac{\partial E_r}{\partial \theta} - \frac{\partial E_\theta}{\partial r} - \frac{E_\theta}{r}\right)\boldsymbol{e}_\psi + \left(\frac{\partial E_\psi}{\partial r} - \frac{1}{r}\frac{\partial E_r}{\partial \psi} + \frac{E_\psi}{r}\right)\boldsymbol{e}_\theta \\ \boldsymbol{\nabla}\boldsymbol{u} &= \frac{\partial}{\partial r}(u_r\boldsymbol{e}_r + u_\psi\boldsymbol{e}_\psi + u_\theta\boldsymbol{e}_\theta)\otimes\boldsymbol{e}_r \\ &+\frac{\partial}{\partial \psi}(u_r\boldsymbol{e}_r + u_\psi\boldsymbol{e}_\psi + u_\theta\boldsymbol{e}_\theta)\otimes\frac{1}{r}\boldsymbol{e}_\psi \\ &+\frac{\partial}{\partial \theta}(u_r\boldsymbol{e}_r + u_\psi\boldsymbol{e}_\psi + u_\theta\boldsymbol{e}_\theta)\otimes\frac{1}{r\sin\psi}\boldsymbol{e}_\theta \\ &= \left(\frac{\partial u_r}{\partial r}\boldsymbol{e}_r + \frac{\partial u_\psi}{\partial r}\boldsymbol{e}_\psi + \frac{\partial u_\theta}{\partial r}\boldsymbol{e}_\theta\right)\otimes\boldsymbol{e}_r \\ &+\left(\frac{\partial u_r}{\partial \psi}\boldsymbol{e}_r + u_r\boldsymbol{e}_\psi + \frac{\partial u_\psi}{\partial \psi}\boldsymbol{e}_\psi - u_\psi\boldsymbol{e}_r + \frac{\partial u_\theta}{\partial \psi}\boldsymbol{e}_\theta\right)\otimes\frac{1}{r}\boldsymbol{e}_\psi \\ &+\left(\frac{\partial u_r}{\partial \theta}\boldsymbol{e}_r + u_r\sin\psi\boldsymbol{e}_\theta + \frac{\partial u_\psi}{\partial \theta}\boldsymbol{e}_\psi + u_\psi\cos\psi\boldsymbol{e}_\theta + \frac{\partial u_\theta}{\partial \theta}\boldsymbol{e}_\theta + u_\theta(-\sin\psi\boldsymbol{e}_r - \cos\psi\boldsymbol{e}_\psi)\right)\otimes\frac{1}{r\sin\psi}\boldsymbol{e}_\theta \\ &= \frac{\partial u_r}{\partial r}\boldsymbol{e}_r\otimes\boldsymbol{e}_r + \frac{\partial u_\psi}{\partial r}\boldsymbol{e}_\psi\otimes\boldsymbol{e}_r + \frac{\partial u_\theta}{\partial r}\boldsymbol{e}_\theta\otimes\boldsymbol{e}_r \\ &+\frac{1}{r}\left(\frac{\partial u_r}{\partial \psi} - u_\psi\right)\boldsymbol{e}_r\otimes\boldsymbol{e}_\psi + \frac{1}{r}\left(\frac{\partial u_\psi}{\partial \psi} + u_r\right)\boldsymbol{e}_\psi\otimes\boldsymbol{e}_\psi + \frac{1}{r}\frac{\partial u_\theta}{\partial \psi}\boldsymbol{e}_\theta\otimes\boldsymbol{e}_\psi \\ &+\frac{1}{r\sin\psi}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\sin\psi\right)\boldsymbol{e}_r\otimes\boldsymbol{e}_\theta + \frac{1}{r\sin\psi}\left(\frac{\partial u_\psi}{\partial \theta} - u_\theta\cos\psi\right)\boldsymbol{e}_\psi\otimes\boldsymbol{e}_\theta \\ &+\frac{1}{r\sin\psi}\left(u_r\sin\psi + u_\psi\cos\psi + \frac{\partial u_\theta}{\partial \theta}\right)\boldsymbol{e}_\theta\otimes\boldsymbol{e}_\theta \end{aligned} \tag{A.18}$$

Utilizing the integration by parts formula,

$$\int_\Omega \boldsymbol{v}\cdot\boldsymbol{\nabla}\eta\, r^2\sin\psi dr d\psi d\theta = -\int_\Omega (\boldsymbol{\nabla}\cdot\boldsymbol{v})\eta\, r^2\sin\psi dr d\psi d\theta\ ,$$

we can derive the formula for the divergence of a vector field in the divergence form,

$$\boldsymbol{\nabla}\cdot\boldsymbol{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\psi}\frac{\partial}{\partial\psi}(v_\psi\sin\psi) + \frac{1}{r\sin\psi}\frac{\partial v_\theta}{\partial\theta}\ . \tag{A.19}$$

By the same token, we can take advantage of the integration by parts formula for tensors,

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\nabla}\boldsymbol{v}\, r^2\sin\psi dr d\psi d\theta = -\int_\Omega (\mathbf{div}\boldsymbol{\sigma})\cdot\boldsymbol{v}\, r^2\sin\psi dr d\psi d\theta\ ,$$

to derive the formula for the divergence of the tensor field $\boldsymbol{\sigma}$,

$$\begin{aligned}\mathbf{div}\boldsymbol{\sigma} = &\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\sigma_{rr}) + \frac{1}{r}\left(\frac{1}{\sin\psi}\frac{\partial}{\partial\psi}(\sigma_{r\psi}\sin\psi) - \sigma_{\psi\psi}\right) + \frac{1}{r}\left(\frac{1}{\sin\psi}\frac{\partial\sigma_{r\theta}}{\partial\theta} - \sigma_{\theta\theta}\right)\right)\boldsymbol{e}_r\\ &+\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\sigma_{\psi r}) + \frac{1}{r}\left(\frac{1}{\sin\psi}\frac{\partial}{\partial\psi}(\sigma_{\psi\psi}\sin\psi) + \sigma_{r\psi}\right) + \frac{1}{r}\left(\frac{1}{\sin\psi}\frac{\partial\sigma_{\psi\theta}}{\partial\theta} - \frac{\sigma_{\theta\theta}}{\tan\psi}\right)\right)\boldsymbol{e}_\psi\\ &+\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\sigma_{\theta r}) + \frac{1}{r}\left(\frac{1}{\sin\psi}\frac{\partial}{\partial\psi}(\sigma_{\theta\psi}\sin\psi) + \sigma_{r\theta}\right) + \frac{1}{r}\left(\frac{1}{\sin\psi}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{\sigma_{\psi\theta}}{\tan\psi}\right)\right)\boldsymbol{e}_\theta\ .\end{aligned} \tag{A.20}$$

Finally, a similar exercise stemming from the formula,

$$\int_\Omega \boldsymbol{E}\cdot(\boldsymbol{\nabla}\times\boldsymbol{F})\, r^2\sin\psi dr d\psi d\theta = \int_\Omega (\boldsymbol{\nabla}\times\boldsymbol{E})\cdot\boldsymbol{F}\, r^2\sin\psi dr d\psi d\theta\ ,$$

yields an equivalent formula for the curl in a slightly different form,

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{E} = &\frac{1}{r\sin\psi}\left(\frac{\partial}{\partial\psi}(\sin\psi E_\theta) - \frac{\partial E_\psi}{\partial\theta}\right)\boldsymbol{e}_r + \frac{1}{r}\left(\frac{1}{\sin\psi}\frac{\partial E_r}{\partial\theta} - \frac{1}{r}\frac{\partial}{\partial r}(r^2E_\theta) + E_\theta\right)\boldsymbol{e}_\psi\\ &+\frac{1}{r}\left(-\frac{1}{\sin\psi}\frac{\partial(\sin\psi E_r)}{\partial\psi} + \frac{E_r}{\tan\psi} + \frac{1}{r}\frac{\partial}{\partial r}(r^2E_\psi) - E_\psi\right)\boldsymbol{e}_\theta\\ = &\frac{1}{r\sin\psi}\left(\frac{\partial}{\partial\psi}(\sin\psi E_\theta) - \frac{\partial E_\psi}{\partial\theta}\right)\boldsymbol{e}_r + \frac{1}{r}\left(\frac{1}{\sin\psi}\frac{\partial E_r}{\partial\theta} - \frac{\partial(rE_\theta)}{\partial r}\right)\boldsymbol{e}_\psi + \frac{1}{r}\left(\frac{\partial(rE_\psi)}{\partial r} - \frac{\partial E_r}{\partial\psi}\right)\boldsymbol{e}_\theta\ .\end{aligned} \tag{A.21}$$