ASE 380P 2-ANALYTICAL METHODS II
EM386L MATHEMATICAL METHODS IN APPLIED MECHANICS II
CAM 386L MATHEMATICAL METHODS IN APPLIED ENGINEERING AND SCIENCES

Final Exam. Tuesday, May 15, 2012, 2-5 p.m., ACES 6.304

1. (a) State the elementary Green's formula (integration by parts) in three space dimensions (5 points).

$$\int_D \frac{\partial u}{\partial x_j} v = -\int_D u \frac{\partial v}{\partial x_j} + \int_{\partial D} uvn_i$$

(b) Use the formula to derive integration by parts formula for the curl operator,

$$\int_D (\boldsymbol{\nabla} \times \boldsymbol{E}) \cdot \boldsymbol{F} \, dV = \int_D \boldsymbol{E} \cdot (\boldsymbol{\nabla} \times \boldsymbol{F}) \, dV + \int_{\partial D} (\boldsymbol{n} \times \boldsymbol{E}) \cdot \boldsymbol{F} \, dS \tag{0.1}$$

where $\boldsymbol{n}$ denotes the outward normal unit vector (5 points).

$$\begin{aligned}
\int_D (\boldsymbol{\nabla} \times \boldsymbol{E}) \cdot \boldsymbol{F} &= \int_D \varepsilon_{ijk} \frac{\partial E_k}{\partial x_j} F_i \\
&= -\int_D \varepsilon_{ijk} E_k \frac{\partial F_i}{\partial x_j} + \int_{\partial D} \varepsilon_{ijk} E_k F_i n_j \\
&= \int_D E_k (\varepsilon_{kji} \frac{\partial F_i}{\partial x_j}) + \int_{\partial D} (\varepsilon_{ijk} n_j E_k) F_i \\
&= \int_D \boldsymbol{E} \cdot (\boldsymbol{\nabla} \times \boldsymbol{F}) + \int_{\partial D} (\boldsymbol{n} \times \boldsymbol{E}) \cdot \boldsymbol{F}
\end{aligned}$$

(c) Verify the formula (show that you can compute the involved integrals) for the ball

$$D = \{(x, y, z) \; : \; x^2 + y^2 + z^2 \le a^2\}$$

and,

$$\boldsymbol{E} = (z, x, y), \quad \boldsymbol{F} = (1, 1, 1)$$

Use whatever system of coordinates you want (10 points).

This is very elementary.

$$\boldsymbol{\nabla E} = (1, 1, 1) \quad \boldsymbol{\nabla} \times \boldsymbol{F} = \boldsymbol{0} \quad \boldsymbol{n} \times \boldsymbol{E} = (n_2 y - n_3 x, n_3 z - n_1 y, n_1 x - n_2 z)$$

Formula (0.1) reduces to

$$3 \int_D dV = \int_{\partial D} (n_1(x - y) + n_2(y - z) + n_3(z - x)) \; dS$$

The left-hand side reduces to

$$3\frac{4}{3}\pi a^3 = 4\pi a^3$$

We can take shortcuts in evaluating the surface integral as well,

$$\int_{\partial D} (n_1(x-y) + n_2(y-z) + n_3(z-x))\ dS = \frac{1}{a}\int_{\partial D} (x(x-y) + y(y-z) + z(z-x))\ dS$$
$$= \frac{1}{a}\int_{\partial D} (\underbrace{x^2+y^2+z^2}_{=a^2})\, dS - \underbrace{\frac{1}{a}\int_{\partial D} (xy+yz+zx)\ dS}_{=0 \text{ by symmetry}} = a4\pi a^2 = 4\pi a^3$$

Done.

2. Consider the following initial-value problem.

$$\begin{cases} \ddot{x} - \dot{x} = H(t-1) \\ x(0) = \dot{x}(0) = 0\,, \end{cases}$$

where H denotes the Heaviside function.

(a) Solve the problem using elementary calculus (5 points).

This class of problems has been discussed so many times that I will give only the results. The elementary solution is

$$u(t) = \begin{cases} 0 & 0 < t < 1 \\ e^{t-1} - t & t > 1 \end{cases} = H(t-1)(e^{t-1} - t)$$

(b) Compute the Laplace transform of the solution to the initial-value problem (5 points).

$$\hat{u}(s) = \frac{e^{-s}}{s^2(s-1)}$$

(c) Use the Residue Theorem to compute the inverse Laplace transform of the solution in the "Laplace domain" and compare it with the solution obtained using the elementary calculus (10 points).

For $t < 1$ use contour "to the right" to conclude that $u = 0$, for $t > 1$ use contour "to the left". Simple pole at $s = 1$, second order pole at $s = 0$.

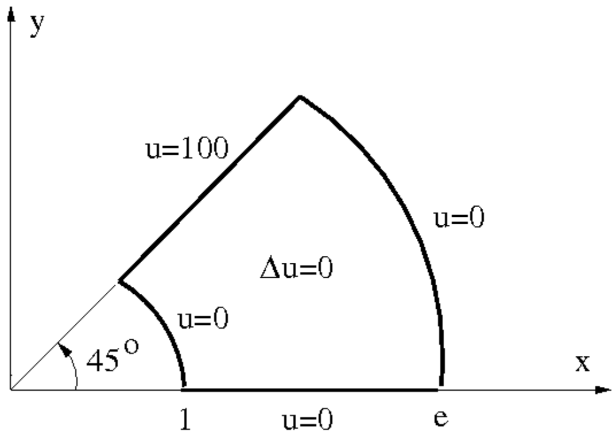

Figure 1: Laplace problem in a conical domain

3. Consider the Laplace problem shown in Fig. 1

   (a) Define a conformal map and explain why the map $w = \ln z$ is conformal. (5 points).

   A map from $\mathbb{R}^2$ into itself is *conformal* if it preserves angles between curves. Any holomorphic map from the complex plane into itself is conformal and function $w = \ln z$ is holomorphic.

   (b) Use the map to transform the Laplace problem in z plane into a corresponding Laplace problem in w plane and solve it. (15 points).

   Transformation

$$w = \ln z = \ln(|z|e^{i\theta}) = \underbrace{\ln|z|}_{\xi} + i\underbrace{\theta}_{\eta}$$

   maps the conical domain into a rectangular domain in plane $w = \xi + i\eta$, shown in Fig. 2. The whole point is in the fact that the composition with holomorphic map sets harmonic functions in x, y into harmonic functions in ξ, η. We can solve the problem in terms of ξ, η using the standard

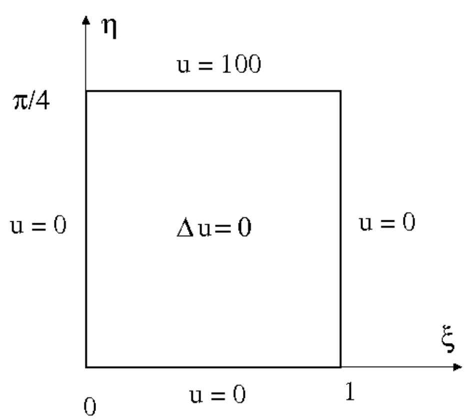

Figure 2: Laplace problem in the transformed domain

separation of variables to obtain

$$u(\xi, \eta) = \sum_{n=1}^{\infty} a_n \sin(n\pi\xi) \sinh(n\pi\eta)$$

where coefficients a_n are determined from the BC at $\eta = \pi/4$,

$$\sum_{n=1}^{\infty} a_n \sin(n\pi\xi) \sinh(\frac{n\pi^2}{4}) = 100$$

Using the fact that functions $\sin(n\pi\xi)$ form an L^2-orthogonal basis in $L^2(0, 1)$, we obtain

$$a_n = \frac{1}{\sinh(\frac{n\pi^2}{4})} \frac{\int_0^1 100 \sin(n\pi\xi)\, d\xi}{\int_0^1 \sin^2(n\pi\xi)\, d\xi} = \begin{cases} \frac{400}{\sinh(\frac{n\pi^2}{4})} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}$$

So,

$$u(\xi, \eta) = \sum_{\text{odd } n=1}^{\infty} \frac{400}{\sinh(\frac{n\pi^2}{4})} \sin(n\pi\xi) \sinh(n\pi\eta)$$

and in terms of x, y,

$$u(x, y) = \sum_{\text{odd } n=1}^{\infty} \frac{400}{\sinh(\frac{n\pi^2}{4})} \sin(n\pi r) \sinh(n\pi\theta)$$

where r, θ are the polar coordinates in x, y plane.

4. Suppose that we form a ring by taking an insulated heat conductor of length l, bending it into a circle, and joining its ends. Thus we have

$$\alpha^2 u_{xx} = u_t \quad 0 \le x \le l,\ t > 0$$

together with periodicity conditions:

$$u(0,t) = u(l,t) \quad u_x(0,t) = u_x(l,t)$$

and some initial condition:

$$u(x,0) = f(x)$$

(a) Solve by separation of variables (15 points).

Starting with

$$x(x,t) = X(x)T(t)$$

we obtain,

$$-\frac{X''}{X} = -\frac{T'}{\alpha^2 T} = \lambda$$

Since operator $AX = -X''$ with periodic BCs is self-adjoint and positive semi-definite, we can assume $\lambda = k^2, k \ge 0$. Using periodic boundary conditions, we get

$$k = k_n = \frac{2\pi n}{l},\ n = 0, 1, ...$$

with the final solution of the form (Fourier series),

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(\frac{2\pi n x}{l}) + b_n \sin(\frac{2\pi n x}{l}) \right) e^{-\frac{4\pi^2 n^2}{l^2 \alpha^2} t}$$

Matching IC:

$$u(x,0) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(\frac{2\pi n x}{l}) + b_n \sin(\frac{2\pi n x}{l}) \right) = f(x)$$

and using the L^2-orthogonality of sines and cosines, we get,

$$a_0 = \frac{1}{l} \int_0^l f(x)\, dx$$

and

$$a_n = \frac{\int_0^l f(x) \cos(\frac{2\pi n x}{l})\, dx}{\int_0^l \cos^2(\frac{2\pi n x}{l})\, dx}$$

and

$$b_n = \frac{\int_0^l f(x) \sin(\frac{2\pi n x}{l})\, dx}{\int_0^l \sin^2(\frac{2\pi n x}{l})\, dx}$$

(b) Determine the steady-state limit

$$\lim_{t\to\infty} u(x,t)$$

(5 points.)

Only the zero order term survives,

$$\lim_{t\to\infty} u(x,t) = \frac{1}{l}\int_0^l f(x)\,dx$$

5. Consider the perturbed square domain shown in Fig. 3

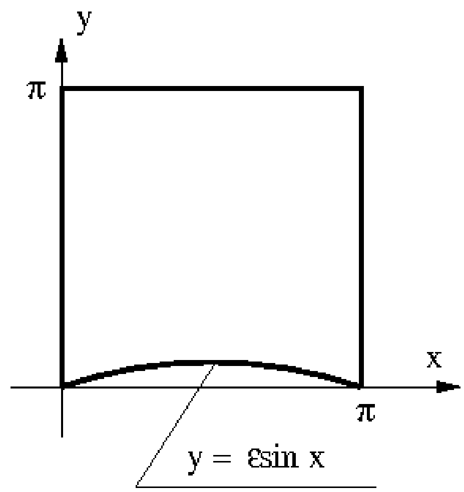

Figure 3: Domain D

(a) Consider the eigenvalue problem defined in the domain D,

$$\begin{cases} -\Delta u &= \lambda u \quad \text{in } D \\ u &= 0 \qquad \text{on } \partial D \end{cases}$$

Argue why all eigenvalues λ are real and positive (5 points).

The operator is self-adjoint and positive definite.

(b) Use perturbation techniques and separation of variables to show that the smallest eigenvalue is of the form

$$\lambda_{min} = 2[1 + \frac{8}{3\pi^2}\epsilon + O(\epsilon^2)]$$

Hint: Recall the way in which we have treated the curved boundary in the airfoil problem discussed in the class (20 points).

Expanding in ϵ,

$$\begin{aligned} u(x,y) &= u_0(x,y) + \epsilon u_1(x,y) + \epsilon^2 u_2(x,y) + \dots \\ \lambda &= \lambda_0 + \epsilon\lambda_1 + \dots \end{aligned}$$

Plugging into the BC on the lower edge, we get

$$\begin{aligned} u(x,\epsilon\sin x) &= u_0(x,\epsilon\sin x) + \epsilon u_1(x,\epsilon\sin x) + \dots \\ &= u_0(x,0) + \tfrac{\partial u_0}{\partial y}(x,0)\epsilon\sin x + \epsilon[u_1(x,0) + \tfrac{\partial u_1}{\partial y}(x,0)\epsilon\sin x] + \dots \end{aligned}$$

Collecting terms corresponding to ϵ^0 and ϵ^1 in PDE and BCs, we get the standard eigenvalue problem for u_0 *in the square domain*,

$$\begin{cases} -\Delta u_0 &= \lambda u_0 \qquad \text{in } \Omega := (0,\pi)^2 \\ u_0 &= 0 \qquad \text{on } \partial\Omega \end{cases}$$

and an additional BVP for u_1 and λ_1 in Ω,

$$\begin{cases} -\Delta u_1 - \lambda_0 u_1 &= \lambda_1 u_0 & \text{in } \Omega \\ u_1 &= 0 & \text{on the upper edge and the vertical edges} \\ u_1(x,0) &= -\frac{\partial u_0}{\partial y}(x,0)\sin x & \text{on the lower edge} \end{cases}$$

Standard separation of variables yields[1]:

$$\lambda_0 = 2, \quad u_0(x,y) = C \sin x \sin y$$

This gives the following BVP for u_1,

$$\begin{cases} -\Delta u_1 - 2u_1 &= \lambda_1 \sin x \sin y & \text{in } \Omega \\ u_1 &= 0 & \text{on the upper edge and the vertical edges} \\ u_1(x,0) &= -\sin^2 x & \text{on the lower edge} \end{cases}$$

There are two questions here: a/ how do we solve the problem ? b/ what is the defining condition for constant λ_1. The first question is relatively easy to answer. We will use the ansatz:

$$u_1 = v + w$$

where v is a function satisfying the non-homogeneous BC and w satisfies the homogeneous BC. There are many "lifts" to the BC data. The simplest, probably, is:

$$v(x,y) = -\sin^2 x \cos(y/2)$$

If we plug this into the PDE, we get the following equation for unknown function w,

$$-\Delta w - 2w = \lambda_1 \sin x \sin y + (\Delta v + 2v) = \lambda_1 \sin x \sin y + (2 - \frac{23}{4}\sin^2 x)\cos(y/2) \quad (0.2)$$

along with homogeneous BCs.

The crucial observation here is that this is a problem with resonance. Recall discussion in class. If we expand w and right-hand side f into eigenvectors ϕ_i of the Laplace operator[2],

$$w(x,y) = \sum w_i \phi_i(x,y), \quad f(x,y) = \sum f_i \phi_i(x,y)$$

we obtain the following formulas for the coefficients w_i,

$$w_i = \frac{f_i}{\lambda_i - 2}$$

where λ_i are the eigenvalues of the Laplacian. Now, for the first eigenvalue $\lambda = 2$, the denominator is zero (resonance) and we do not have a solution unless the numerator vanishes as well

[1] We are looking for the *smallest* eigenvalue.

[2] Recall that eigenvectors of *any* self-adjoint operator provide an L^2-orthogonal basis.

(in this case we have infinitely many solutions). This is exactly the defining condition for the unknown λ_1. The first spectral component of the right-hand side (0.2) must be L^2-orthogonal to the first eigenvector of the Laplacian in the square domain,

$$\int_0^\pi \int_0^\pi \left(\lambda_1 \sin x \sin y + (2 - \frac{23}{4} \sin^2 x) \cos(y/2) \right) \sin x \sin y \, dx dy = 0$$

Computing the integrals, we obtain

$$\lambda_1 = \frac{16}{3\pi^2}$$