ASE 380P 2-ANALYTICAL METHODS II
EM386L MATHEMATICAL METHODS IN APPLIED MECHANICS II
CSE 386L MATHEMATICAL METHODS IN APPLIED ENGINEERING AND SCIENCES

Exam 3. Monday, Apr 30, 2012

1. (a) State the Sturm–Liouville theorem (5 points).

   Consider differential operator,

   $$Ly = -(a(x)y')' + c(x)y, \quad x \in (0, l)$$

   accompanied with a combination of any of the boundary conditions:

   - *Dirichlet BC:*
   $$y = 0$$
   - *Neumann BC:*
   $$y' = 0$$
   - *Robin (Cauchy) BC:*
   $$\alpha y + \beta y' = 0$$
   - *Finite energy condition:*
   $$y, y' \text{ finite}$$

   or the *periodic case*:

   $$a(0) = a(l), \quad c(0) = c(l), \quad y(0) = y(l), \quad y'(0) = y'(l)$$

   The operator is then self-adjoint and possesses a sequence of real eigenvalues

   $$\lambda_1 < \lambda_2 < \ldots < \lambda_n \to \infty$$

   with the corresponding eigenvectors y_n providing an L^2-orthogonal basis for space $L^2(0, l)$.

   (b) Consider the problem:

   $$y'' + \lambda y = 0, \quad y(0) = y'(0), \; y(1) = 0$$

   Is this a Sturm–Liouville eigenproblem ? Explain (5 points).

   Yes, it is. $Ly = -y''$, we have Robin BC at $x = 0$ and Dirichlet BC at $x = 1$.

(c) Determine the eigenfunctions for the problem deriving an appropriate transcendental equation for the eigenvalues. (15 points).

The operator $-y''$ with the BCs is positive-definite, so we can assume $\lambda = k^2, k > 0$. This gives

$$y = A \sin kx + B \cos kx$$

Due to the simpler BC at $x = 0$, it is convenient to shift the origin to $x = 1$ and consider the general solution in the form:

$$y = A \sin k(x-1) + B \cos k(x-1)$$

Then BC $y(1) = 0$ implies $B = 0$. BC $y(0) = y'(0)$ implies condition:

$$-\sin k = k \cos k$$

$cosk$ must be different from zero. Indeed, if $\cos k = 0$ then $\sin k \neq 0$[1] and the equation cannot be satisfied. Dividing by $\cos k$, we get a transcendental equation,

$$\tan k = -k$$

with a sequence of roots $k_n \in (n\pi, n\pi + \frac{\pi}{2}), n = 1, \ldots$. The corresponding eigenvectors,

$$y_n = \sin k_n(x-1), \quad n = 1, \ldots$$

form an L^2-orthogonal basis in $L^2(0,1)$, i.e. any function $f \in L^2(0,1)$ can be expanded into the (non-standard) Fourier series:

$$f(x) = \sum_{n=1}^{\infty} f_n \sin k_n(x-1)$$

where

$$f_n = \frac{\displaystyle\int_0^1 f(x) \sin k_n(x-1)\, dx}{[\displaystyle\int_0^1 \sin^2 k_n(x-1)\, dx]^{1/2}}$$

[1] Sine and cosine functions never vanish simultaneously.

2. (a) Consider wave equation in three space dimensions,

$$\Delta u = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}$$

Represent the equation in standard spherical coordinates (5 points.)

A good starting point is the formula for the gradient,

$$\boldsymbol{\nabla} u = \frac{\partial}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial u}{\partial \psi}\boldsymbol{e}_\psi + \frac{1}{r\sin\psi}\frac{\partial u}{\partial \theta}\boldsymbol{e}_\theta$$

Integration by parts (jacobian = $r^2 \sin\psi$) yields then

$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r}) + \frac{1}{r^2\sin\psi}\frac{\partial}{\partial \psi}(\sin\psi\frac{\partial u}{\partial \psi}) + \frac{1}{r^2\sin^2\psi}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}$$

(b) Assume that the solution is point-symmetric (depends only upon radial coordinate r and time t). Provide a classification for second order PDEs and classify the equation (5 points).

Equation

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + \text{ lower order terms } = f$$

is elliptic if $d = AC - B^2 > 0$, parabolic if $d = 0$ and hyperbolic if $d < 0$. In our case $A = 1, C = 1/a^2$, so the equation is hyperbolic.

(c) Demonstrate that the equation may be reexpressed as

$$\frac{\partial^2}{\partial r^2}(ru) = \frac{1}{a^2}\frac{\partial^2}{\partial t^2}(ru)$$

and use the d'Alembert method or any other possible approach to derive the general solution to the problem (15 points).

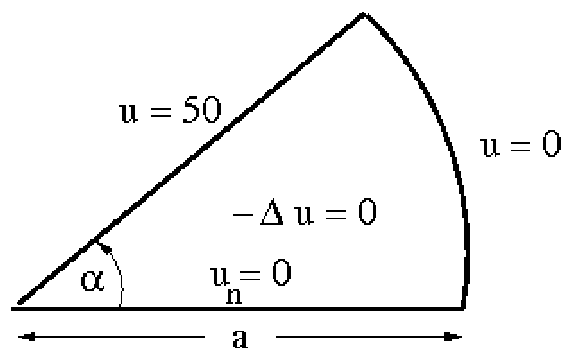
The first part is just simple algebra. Substituting $v = ru$ then, we get the 1D wave equation for which the d'Alembert solution is

$$v(x) = f(x - at) + g(x + at)$$

where f, g are arbitrary functions. This yields

$$u(r) = \frac{1}{r}[f(r - at) + g(r + at)]$$

3. (a) Solve the following 2D boundary-value problem (25 points).



Recalling Laplacian in polar coordinates,

$$-\Delta u = \frac{1}{r}(ru_r)_r - \frac{1}{r^2}u_{\theta\theta}$$

we seek the solution in the form:

$$u = v(r,\theta) + 50$$

trading the non-homogeneous BC in u at $\theta = \alpha$, for a non-homogeneous BC in v at $r = a$,

$$v(a,\theta) = -50$$

Seeking $v = R(r)T(t)$, we get

$$\frac{r(rR')'}{R} = -\frac{\Theta''}{\Theta} = \lambda = k^2 > 0, \quad k > 0$$

Notice that we have obtained a Sturm-Liouville problem in θ and that operator $-\Theta''$ with Dirichlet BCs is positive-definite, so we can assume that λ is real and positive. Solving for Θ, we get,

$$\Theta_n = \cos k\theta, \quad k = k_n = \frac{\pi}{2\alpha} + n\frac{\pi}{\alpha},\ n = 1, 2, \ldots$$

which gives now the Cauchy-Euler equation for $R(r)$,

$$r(rR')' + k^2 R = 0$$

with

$$R(r) = r^{\pm k}$$

Rejecting the singular solution, we get by superposition,

$$v(r,\theta) = \sum_{n=1}^{\infty} A_n r^{k_n} \cos k_n\theta, \quad k_n = \frac{\pi}{2\alpha} + n\frac{\pi}{\alpha},\ n = 1, 2, \ldots$$

Constants A_n are determined from the boundary condition at $r = a$,

$$\sum_{n=1}^{\infty} A_n a^{k_n} \cos k_n\theta = -50$$

$$A_n = a^{-k_n} \frac{\int_0^\alpha (-50)\cos k_n\theta}{[\int_0^\alpha \cos^2 k_n\theta]^{1/2}}$$

4. (a) Define characteristics for a single first order equation,

$$a(x,y,z)u_x + b(x,y,z)u_y + c(x,y,z)u_z = 0$$

and discuss the relation with the concept of prime integrals for a system of ODEs. (5 points).

Characteristics are curves that satisfy the differential equation:

$$\frac{dx}{a(x,y,z)} = \frac{dy}{b(x,y,z)} = \frac{dz}{c(x,y,z)}$$

or, in a parametric form,

$$\frac{dx}{dt} = a, \quad \frac{dy}{dt} = b, \quad \frac{dz}{dt} = c$$

Solution of the original equation is constant along the characteristics, i.e. it is its prime integral.

(b) Determine the general solution of the equation:

$$u_x + u_y + 3u_z = 0$$

(10 points).

The equations for the characteristics yield,

$$\frac{dx}{1} = \frac{dy}{1} \implies y = x + c$$

and

$$\frac{dx}{1} = \frac{dz}{3} \implies z = 3x + d$$

Consequently, $y - x$ and $z - 3x$ are two LI prime integrals, and the general solution of the original equation sis given by,

$$u = f(y - x, z - 3x)$$

where f is an arbitrary function.

(c) Determine the solution of the equation above in the first octant: $x, y, z > 0$ with initial conditions:

$$u(x,y,0) = xy^2;\ u(x,0,z) = zx^2;\ u(0,y,z) = yz^2$$

(10 points).

It is easier to work with the parametric representation of the characteristic

$$x = t + A, \quad y = t + B, \quad z = 3t + C$$

We can scale parameter t to depend upon a point of interest x_0, y_0, z_0 in such a way that,

$$x(0) = x_0, \quad y(0) = y_0, \quad z(0) = z_0$$

This gives,

$$x = t + x_0, \quad y = t + y_0, \quad z = 3t + z_0$$

Depending upon the location of (x_0, y_0, z_0), the characteristics will pass through one of the coordinate planes first, and the IC corresponding to that plane has to be used, For instance, if $x_0 < y_0, z_0/3$ then the characteristics issued at (x_0, y_0, z_0), will intersect plane $x = 0$ first, at the point

$$y = y_0 - x_0, \quad z = z_0 - 3x_0$$

and the value of the solution at $x = 0$ will be equal to the value at (x_0, y_0, z_0),

$$u(x_0, y_0, z_0) = (y_0 - x_0)(z_0 - 3x_0)^2$$

Same reasoning applies to the two remaining cases. Note that the resulting solution is discontinuous.