ASE 380P ANALYTICAL METHODS I
EM386K MATHEMATICAL METHODS IN APPLIED MECHANICS I

Exam 2. Monday, November 28, 2011

1. (a) Formulate the Jordan Canonical Form Theorem for matrices. (8 points)

Let $\boldsymbol{A}$ be an arbitrary matrix, λ an eigenvalue of $\boldsymbol{A}$, and V_λ the corresponding eigenspace. In general,

$$\text{(geometric multiplicity) } m =: \text{dim} V_\lambda \leq \text{ algebraic multiplicity of } \lambda$$

One can select then exactly m different eigenvectors $\boldsymbol{e}_i \in V_\lambda$, each of them starting the so-called Jordan's chain of generalized eigenvectors $\boldsymbol{e}_{i,j}, j = 1, \ldots, N_i$:

$$\boldsymbol{e}_{i1} = \boldsymbol{e}_i, \quad (\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{e}_{i,j} = \boldsymbol{e}_{i,j-1},\ j = 2, \ldots, N_i$$

such that

$$\sum_i^m N_i = \text{ algebraic multiplicity of } \lambda$$

Using the chains of the generalized eigenvectors for columns of the so-called *modal matrix* $\boldsymbol{M}$, the transformation:

$$\boldsymbol{A}' := \boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M}$$

reduces matrix $\boldsymbol{A}$ to its *Jordan canonical form* $\boldsymbol{A}'$ that consists of two diagonals only. Present on the main diagonal are eigenvalues of $\boldsymbol{A}$, with 1's or 0's on the first upper diagonal, forming the so-called Jordan's blocks. Each block corresponds to a single Jordan chain of length N; with the corresponding eigenvalue λ on the main diagonal and $N - 1$ 1's on the first upper diagonal. For instance, a chain of length $N = 3$ produces the Jordan block of the form:

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

In other words, an arbitrary matrix $\boldsymbol{A}$, in general, cannot be reduced to a diagonal matrix but we can come very close to it by reducing it to its Jordan's "almost diagonal" form.

(b) Find the generalized eigenvectors (Jordan's chains) and the corresponding Jordan form for the following matrix

$$\boldsymbol{A} = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 3 \end{pmatrix} \tag{0.1}$$

(12 points)

Clearly, $\lambda = 3$ is a triple eigenvalue. There is only one eigendirection of the form $\boldsymbol{b}_1 = (0, 0, \alpha)^T$. The corresponding generalized eigenvectors are $\boldsymbol{b}_2 = (0, \alpha, \beta)^T$ and $\boldsymbol{b}_3 = (\alpha, \beta, \gamma)^T$. Selecting, for instance, $\alpha = 1, \beta = 0, \gamma = 0$, we get[1]

$$\boldsymbol{M} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \boldsymbol{A}' = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

2. (a) Discuss the connection between the Jordan's chains (generalized eigenvectors) and solution of systems of first order linear ODE's with constant coefficients (5 points).

   Each Jordan's chain of generalized eigenvectors $\boldsymbol{e}_j, j = 1, \ldots, N$ corresponding to eigenvalue λ generates a family of LI solutions of the form:

$$\begin{aligned} \boldsymbol{x}_1 &= \boldsymbol{e}_1 e^{\lambda t} \\ \boldsymbol{x}_2 &= (\boldsymbol{e}_1 t + \boldsymbol{e}_2) e^{\lambda t} \\ &\vdots \\ \boldsymbol{x}_N &= (\boldsymbol{e}_1 t^{N-1} + \boldsymbol{e}_2 t^{N-2} + \ldots + \boldsymbol{e}_N) e^{\lambda t} \end{aligned}$$

   A linear combination of these solutions (with arbitrary coefficients) forms the general solution of the homogeneous system (0.2).

   (b) Find the general solution for the first order system,

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} \tag{0.2}$$

   where matrix $\boldsymbol{A}$ is given by (0.1)(15 points).

$$\boldsymbol{x}(t) = \{C_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + C_2 [\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}] + C_3 [\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} t^2 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}]\} e^{3t}$$

   Comment: This problem was intented to be a "freeby". Once you know the Jordan's chain from Problem 1, you just write down the solution. Many of you have "survived" by utilizing the fact that the system was decoupled, and you can solve it ignoring the Jordan Thm. This worked this time but it may not work next time if I give you a bit more complicated matrix.

3. (a) Define: irregular distributions, Dirac's delta and a δ-sequence (8 points).

   A distribution L is said to be *regular* if there exists an L^2_{loc} function g that generates it, i.e.

$$< L, \phi >= \int_\Omega g\phi$$

   If such a function does not exist, we say that the distribution is *irregular*. Dirac's delta:

$$\delta \; : \; \mathcal{D}(I\!\!R) \ni \phi \rightarrow \phi(0) \in I\!\!R$$

[1]Note that the Jordan's chain is not unique, you can select other values for α, β and γ. For this choice of the generalized eigenevectors though, the corresponding modal matrix happens to be orthonormal. This makes the computation of $\boldsymbol{M}^{-1}$ easy as $\boldsymbol{M}^{-1} = \boldsymbol{M}^T$ (I hope you remember how to check that a matrix is orthonormal *without* computing its inverse). All of this is just for a sanity check. We *do know* that, after the transformation, we must obtain the Jordan's block.

is an example of an irregular distribution. Any sequence of functions $w_k(x)$ that generates a corresponding sequence of regular distributions converging to Dirac's delta, is called a δ-sequence,

$$\int w_k(x)\phi(x)\,dx \to \phi(0), \quad k \to \infty$$

If $w \geq 0$ and $\int_{-\infty}^{\infty} w(x)\,dx = 1$ then $w_k(x) := kw(kx)$ is a δ-sequence (Thm 4.3 in the book).

(b) Verify if the following sequence is a δ-sequence,

$$y_k(x) = \frac{k}{\sqrt{\pi}} e^{-k^2x^2}$$

(12 points).

We only need to check that

$$\int_{-\infty}^{\infty} e^{x^2}\,dx = \sqrt{\pi}$$

(see page 58 for the use of double integral to compute it).

4. (a) Define the Laplace transform. Specify precisely the conditions which have to be satisfied by a function that can be Laplace transformed (5 points).

$$\bar{f}(s) := \int_0^{\infty} f(t)e^{-st}\,dt$$

Function f must be piecewise smooth and it may grow at most exponentially, i.e.

$$|f(t)| \leq Ce^{\gamma t}, \quad \text{for some } \gamma \in \mathbb{R}$$

The new function is a complex-valued function of complex argument s. For comparison, Fourier transform also produces a complex-valued function but of a real argument.

(b) Laplace transform the following initial-value problem and solve it in the Laplace domain (no need to transform it back...).

$$\begin{cases} t\ddot{x} + \dot{x} + tx = 0 \\ x(0) = 1, \quad \dot{x}(0) = 0 \end{cases}$$

(15 points).

See Example 6.5, page 109.

5. (a) Formulate the Frobenius Theorem (5 points).

See Thm. 22.6, page 425.

(b) Use the method of Frobenius to find the general solution of the following equation.

$$4xy'' + 2y' + y = 0$$

(15 points).

$x = 0$ is a regular singular point. Look for the solution in the form of the series:

$$\sum_0^{\infty} c_n x^{n+r}$$

Substituting into the equation, we get,

$$\sum_{0}^{\infty} 4(n+r)(n+r-1)c_n x^{n+r-1} + \sum_{0}^{\infty} 2(n+r)c_n x^{n+r-1} + \sum_{0}^{\infty} c_n x^{n+r} = 0$$

Shiftting the index in the first two series, we get,

$$\sum_{-1}^{\infty} 4(n+1+r)(n+r)c_{n+1} x^{n+r} + \sum_{-1}^{\infty} 2(n+1+r)c_{n+1} x^{n+r} + \sum_{0}^{\infty} c_n x^{n+r} = 0$$

Equating to zero the coefficent corresponding to $n = -1$, we get,

$$2c_0 r(2r-1) = 0$$

and to $n \geq 0$,

$$2(n+1+r)[2(n+r)+1]c_{n+1} + c_n = 0$$

The first (indicial) equation gives two values for r, $r = 0, \frac{1}{2}$, the second one generates a recursion formula for the coefficients,

$$c_{n+1} = -\frac{c_n}{2(n+1+r)[2(n+r)+1]}, \quad n \geq 0$$

We get two LI solutions, a combination of which defines the general solution to the equation.