EM311M - Dynamics
Exam 1
Monday, Feb 13, 2006

1. A particle is moving in a straight line with a an acceleration $a(t) = a(t - t_0)$, where a is a positive constant and $t_0 > 0$ is a specified instant. At the time t_0, the corresponding position and velocity are x_0 and v_0, respectively. Derive the formulas for position $x(t)$ and velocity $v(t)$ at any time t (5 points)

$$\ddot{x} = a(t - t_0)$$

$$\dot{x} = a\frac{(t-t_0)^2}{2} + c \qquad \dot{x}(t_0) = v_0 \Rightarrow c = v_0$$

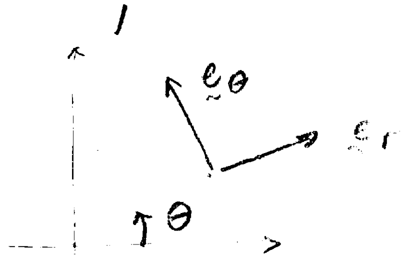
(2) $$\boxed{\dot{x} = a\frac{(t-t_0)^2}{2} + v_0}$$

$$x = a\frac{(t-t_0)^3}{6} + v_0(t-t_0) + d$$

$$x(t_0) = x_0 \Rightarrow d = x_0$$

(3) $$\therefore \boxed{x = a\frac{(t-t_0)^3}{6} + v_0(t-t_0) + x_0}$$

2. At point $r = 1\text{m}$, $\theta = \pi/6$, the polar components of a velocity vector for a particle are $v_r = 0, v_\theta = \cancel{0}$ 1 Calculate the Cartesian components of the velocity vector. (5 points)



$$\underset{\sim}{v} = \overset{0}{v_r}\,\underset{\sim}{e_r} + \overset{1}{v_\theta}\,\underset{\sim}{e_\theta} = \underset{\sim}{e_\theta} = (-\sin\theta, \cos\theta)$$

$$= (-0.5, 0.866)\ \left[\frac{m}{s}\right]$$

(5)

3. Can a particle moving on a curvilinear path have a zero acceleration (vector)? Explain. (5 points)

$$\underset{\sim}{a} = a_t \underset{\sim}{e}_t + a_n \underset{\sim}{e}_n$$

$$a_n = \frac{v^2}{\rho}$$

Particle is moving $\Rightarrow v \neq 0$
Path is curved $\Rightarrow \rho \neq \infty$ $\Big\} \Rightarrow a_n \neq 0$

No!

5

4. Derive the formulas for acceleration vector components a_r and a_θ in the polar system of coordinates (5 points)

$\underset{\sim}{e}_r = (\cos\theta, \sin\theta)$ $\qquad \frac{d\underset{\sim}{e}_r}{d\theta} = (-\sin\theta, \cos\theta) = \underset{\sim}{e}_\theta$

$\underset{\sim}{e}_\theta = (-\sin\theta, \cos\theta)$ $\qquad \frac{d\underset{\sim}{e}_\theta}{d\theta} = (-\cos\theta, -\sin\theta) = -\underset{\sim}{e}_r$

$$\underset{\sim}{r} = r\underset{\sim}{e}_r$$

$$\underset{\sim}{v} = \dot{r}\underset{\sim}{e}_r + r\frac{d\underset{\sim}{e}_r}{d\theta}\dot{\theta} = \underbrace{\dot{r}}_{v_r}\underset{\sim}{e}_r + \underbrace{r\dot{\theta}}_{v_\theta}\underset{\sim}{e}_\theta$$

$$\underset{\sim}{a} = \ddot{r}\underset{\sim}{e}_r + \dot{r}\underbrace{\frac{d\underset{\sim}{e}_r}{d\theta}}_{\underset{\sim}{e}_\theta}\dot{\theta} + \dot{r}\dot{\theta}\underset{\sim}{e}_\theta + r\ddot{\theta}\underset{\sim}{e}_\theta + r\dot{\theta}\underbrace{\frac{d\underset{\sim}{e}_\theta}{d\theta}}_{-\underset{\sim}{e}_r}\dot{\theta}$$

$$= \underbrace{(\ddot{r} - r\dot{\theta}^2)}_{a_r}\underset{\sim}{e}_r + \underbrace{(r\ddot{\theta} + 2\dot{r}\dot{\theta})}_{a_\theta}\underset{\sim}{e}_\theta$$

5

5. A particle moves along a parabola $y = x^2$ with a constant speed v. Determine the Cartesian components of the velocity vector as a function of coordinate x. (5 points)

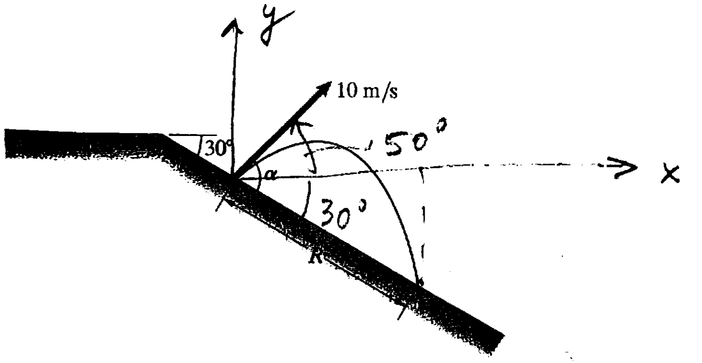
$$y = x^2 \Rightarrow \dot{y} = 2x\dot{x}$$

$$\dot{x}^2 + \dot{y}^2 = v^2 \Rightarrow \dot{x}^2 + (2x\dot{x})^2 = \dot{x}^2(1+4x^2) = v^2$$

$$\therefore \begin{cases} \dot{x} = \pm \dfrac{v}{\sqrt{1+4x^2}} \\ \dot{y} = \pm \dfrac{2xv}{\sqrt{1+4x^2}} \end{cases}$$

5

6. A projectile is launched at 10 m/s from a sloping surface. The angle $\alpha = 80^o$. Determine the range R. (25 points)



$$x(t) = 10 \cos 50^\circ \cdot t = 6.428\, t \quad [m]$$

$$y(t) = -\frac{gt^2}{2} + V_{oy} t$$

$$= -4.905\, t^2 + 10 \sin 50^\circ\, t$$

$$= -4.905 t^2 + 7.66\, t$$

5

At some time t : $\frac{y}{x} = -\tan 30^\circ = -0.5774$

$$-4.905 t^2 + 7.66\, t = -0.5774\,(6.428\, t)$$

$$= -3.711\, t \qquad / : t$$

$$-4.905\, t = -3.711 - 7.66 = -11.37$$

$$t = 2.318\, [s]$$

10

$\therefore$ $x(t) = 14.90\ [m]$

$y(t) = -8.60\ [m]$

5

Check: $\frac{y}{x} = -0.577$ OK ✓

$$R = \sqrt{x^2 + y^2} = 17.20\ [m]$$

5

at $s = 120$ ft, the car is on the first circle, so $\rho = r = 120$ [ft]

$$a_n = \frac{v^2}{\rho} = \frac{5429.2}{120} = 45.24$$

$$a = \sqrt{a_t^2 + a_n^2} = 45.9 \left[\frac{ft}{s^2}\right]$$

10

(b) $v^2 = 59.3^2 + 2 \cdot 7.97 \cdot 160 = 6067 \left[\frac{ft^2}{s^2}\right]$

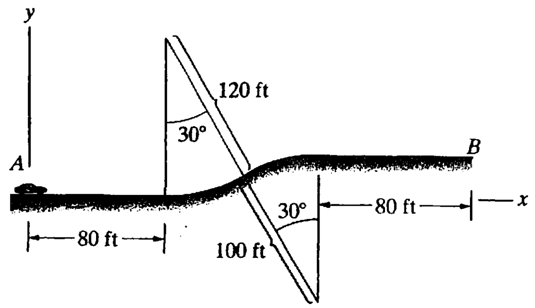
at $s = 160$ ft, the car is on the second circle, so $\rho = r = 100$ [ft]

$$a_n = \frac{v^2}{\rho} = \frac{6067}{100} = 60.7 \left[\frac{ft}{s^2}\right]$$

$$a = \sqrt{a_t^2 + a_n^2} = 61.2 \left[\frac{ft}{s^2}\right]$$

5

7. The car increases its speed at a constant rate from 40 mi/h at A to 60 mi/h at B. Determine the magnitude of its acceleration when it has traveled along the road a distance (a) 120 ft from A, and (b) 160 ft from A. (25 points)



$a_t = \dot{v} = \text{const}$

$a_t = \frac{dv}{ds}\dot{s} = \frac{dv}{ds} v$

$a_t\, ds = v\, dv$

$S_B = 80 + \underbrace{120 \frac{30}{180}\pi}_{62.83} + \underbrace{100 \frac{30}{180}\pi}_{52.36} + 80 = 275.2\ [ft]$

$40\ mi/h = \frac{40 \cdot 5333}{3600} = 59.3\ [\frac{ft}{s}]$

$60\ mi/h = 88.8\ [\frac{ft}{s}]$

$1 mi \approx 1600 m \frac{1 ft}{0.8 m} \approx 5333\ [ft]$

$\int_0^{275.2} a_t\, ds = \int_{59.3}^{88.8} v\, dv$

$a_t \cdot 275.2 = \frac{1}{2}(88.8^2 - 59.3^2)$

$a_t = 7.97\ [\frac{ft}{s^2}]$

10

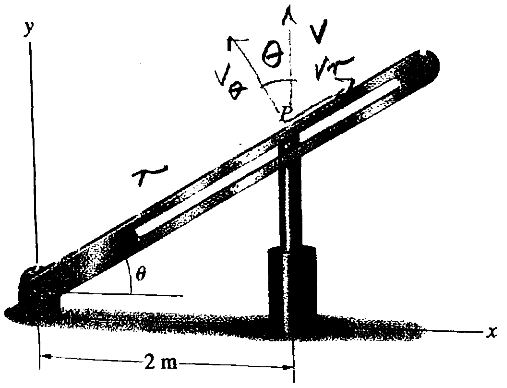
(a) for an arbitrary s $\int_0^s a_t\, ds = \int_{v_A}^{v} v\, dv$

$a_t s = \frac{1}{2}(v^2 - v_A^2)$

so: $v^2 = v_A^2 + 2a_t s$

$v^2 = 59.3^2 + 2 \cdot 7.97 \cdot 120 = 5429.2\ [\frac{ft^2}{s^2}]$

8. The hydraulic actuator moves the pin P upward with constant velocity $v = 2\jmath$ (m/s). Determine the acceleleration of the pin in terms of polar coordinates and the angular acceleration $\ddot{\theta}$ of the slotted bar when $\theta = 35^o$. (25 points)



$V_r = v \sin\theta = 1.147 \left[\frac{m}{s}\right]$

$V_\theta = v \cos\theta = 1.638 \left[\frac{m}{s}\right]$

5

$\underset{\sim}{v} = 2\jmath = \underset{\sim}{const} \Rightarrow \underset{\sim}{a} = \underset{\sim}{0}$!

10

so: $a_r = a_\theta = 0$

at $\theta = 35^\circ$ $\quad r = \frac{2}{\cos\theta} = 2.442 \, [m]$

$V_r = \dot{r} = 1.147 \Rightarrow \dot{r} = 1.147 \left[\frac{m}{s}\right]$

$V_\theta = r\dot{\theta} = 1.638 \Rightarrow \dot{\theta} = 0.67 \left[\frac{rad}{s}\right]$

$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$

$\therefore \ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} = -\frac{2 \cdot 1.147 \cdot 0.67}{2.442}$

$= -0.63 \left[\frac{rad}{s^2}\right]$

10

- Switch to the cartesian components

$$\underset{\sim}{V} = V_r \underset{\sim}{e}_r + V_\theta \underset{\sim}{e}_\theta$$

$$= V_r(\cos\theta\, \underset{\sim}{i} + \sin\theta\, \underset{\sim}{j}) + V_\theta(-\sin\theta\, \underset{\sim}{i} + \cos\theta\, \underset{\sim}{j})$$

$$= \underbrace{(V_r\cos\theta - V_\theta\sin\theta)}_{V_x}\underset{\sim}{i} + \underbrace{(V_r\sin\theta + V_\theta\cos\theta)}_{V_y}\underset{\sim}{j}$$

$$\begin{cases} V_x = -5.76 \;[m/s] \\ V_y = 3.50 \;[m/s] \end{cases}$$

7

Verifications :

- check for r (cosine law)

$$0.4^2 \overset{?}{=} 0.532^2 + 0.2^2 - 2 \cdot 0.532 \cdot 0.2 \cos 40°$$

$$0.16 = 0.16 \quad \text{OK.}$$

- check for speed

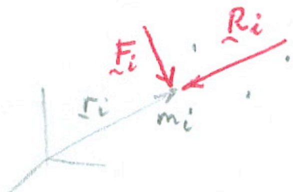
$$V_r^2 + V_\theta^2 \overset{?}{=} V_x^2 + V_y^2$$

$$6.384^2 + 2.16^2 = 5.76^2 + 3.50^2$$

$$45.42 = 45.42 \quad \underline{\text{OK}}$$

EM311M - Dynamics
Exam 2
Monday, Oct 30, 2000

1. Derive the equation of motion for the center of mass of an arbitrary system of particles. (5 points)



$$m_i \ddot{r}_i = F_i + R_i$$

$$\sum_i m_i \ddot{r}_i = \sum_i F_i + \cancel{\sum_i R_i}^{\,0}$$

$$\parallel$$

$$r_c = \frac{\sum_i m_i r_i}{M} \qquad M\ddot{r}_c \qquad \text{so} \qquad \boxed{M\ddot{r}_c = \sum_i F_i}$$

5

2. Define an inertial frame of reference. (5 points)

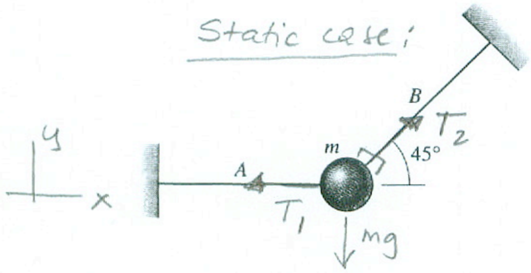
Any system of coordinates in which the Newton laws of motion hold.

5

3. If string A is cut, does the tension in string B remain the same as in the stationary case? Explain. (5 points)

5

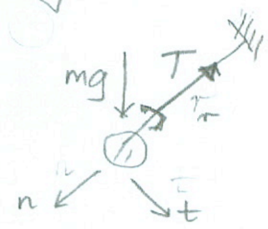
Static case:



$\sum F_y = 0 \quad T_2 \frac{\sqrt{2}}{2} - mg = 0$

$T_2 = \sqrt{2}\, mg$

Dynamic case



$a_n = \frac{v^2}{r} = 0$

$m \cancel{a_n}^{0} = T - mg \cos 45°$

$\therefore T = mg \frac{\sqrt{2}}{2}$

there are different!

4. What does it mean that a force field F is conservative? Give example of a conservative force field and the corresponding potential energy. (5 points)

$\underset{\sim}{F}$ is conservative if there exists a potential $V = V(\underset{\sim}{r})$, such that $\underset{\sim}{F} = -\underset{\sim}{\nabla} V$, e.g. spring force:

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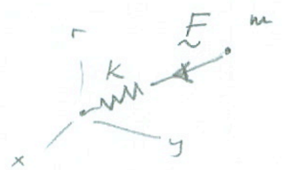
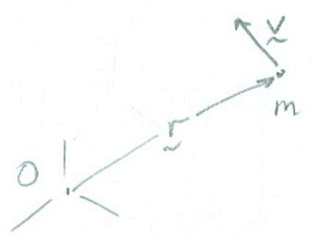


$\underset{\sim}{F} = -k \overbrace{(r - r_0)}^{\Delta r} \frac{\underset{\sim}{r}}{r}$

$V = \frac{1}{2} k \, \Delta r^2$

5. Derive the principle of angular momentum for a single particle. (5 points)

5



$m \ddot{\underset{\sim}{r}} = \underset{\sim}{F} \quad / \; \underset{\sim}{r} \times$

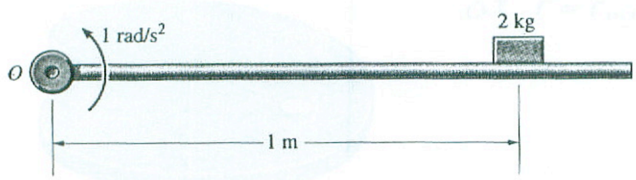
$\underset{\sim}{r} \times m \ddot{\underset{\sim}{r}} = \underbrace{\underset{\sim}{r} \times \underset{\sim}{F}}_{\underset{\sim}{M}_O}$

$\parallel$

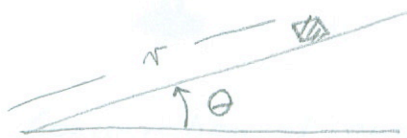
$\underbrace{(\underset{\sim}{r} \times m \dot{\underset{\sim}{r}})^{\cdot}}_{\underset{\sim}{H}_O}$

So $\boxed{\dot{\underset{\sim}{H}}_O = \underset{\sim}{M}_O}$

6. A 2-kg mass rests on a flat horizontal bar. The bar begins rotating *in the vertical plane* about O with a constant angular acceleration of 1 rad/s^2. The mass is observed to slip relative to the bar when the bar is 30^o above the horizontal. What is the static coefficient of friction between the mass and the bar? Does the mass slip toward or away from O? (25 points)



$r = 1m = const \Rightarrow \dot{r} = \ddot{r} = 0$



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$a_r = \ddot{r} - r\omega^2 = -r\omega^2 = -\frac{\pi}{3}\left[\frac{m}{s^2}\right]$

$a_\theta = r\alpha + 2\dot{r}\omega = r\alpha = 1 \quad -''-$

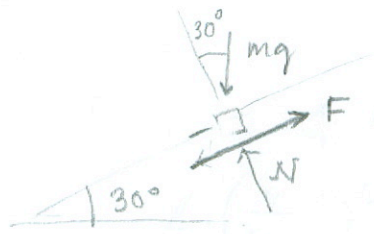
$\left.\begin{aligned}\frac{d\omega}{dt} &= \alpha \\ \frac{d\theta}{dt} &= \omega\end{aligned}\right| \Rightarrow \frac{d\omega}{\alpha} = \frac{d\theta}{\omega} \Rightarrow \omega\, d\omega = \alpha\, d\theta$

5

$\int_0^\omega \omega\, d\omega = \int_0^{\frac{\pi}{6}} \alpha\, d\theta \Rightarrow \frac{\omega^2}{2} = \alpha \cdot \frac{\pi}{6} \Rightarrow \omega^2 = \frac{\pi}{3}$

$\theta: \quad m a_\theta = N - mg\cos 30^\circ \Rightarrow N = m(1 + g\cos 30^\circ)$

$r: \quad m a_r = F - mg\sin 30^\circ$



$\Rightarrow F = m a_r + mg\sin 30^\circ$

15

$= m\left(-\frac{\pi}{3} + g \cdot \frac{1}{2}\right) > 0$

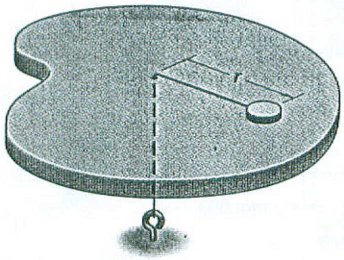
(That indicates that the mass will slide down the bar since F has to oppose the motion!)

$F = \mu \cdot N$

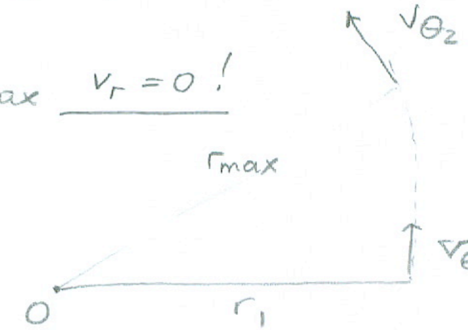
$\mu = \frac{F}{N} = \frac{\cancel{m}(0.5g - \frac{\pi}{3})}{\cancel{m}(1 + g\frac{\sqrt{3}}{2})}$

$= \frac{3.857}{9.495} = 0.406$

7. A 2-kg disk slides on a smooth horizontal table and is connected to an elastic cord whose tension is $T = 6r$ N, where r is the radial position of the disk in meters. The disk is at $r = 1$ m when it is given an initial velocity of 4 m/s in the transverse direction. What is the maximum value of r reached by the disk? (25 points)



at $r = r_{max}$ $\underline{v_r = 0\,!}$



Conservation of angular momentum about O

$$\cancel{m} v_{\theta_1} \cdot r_1 = \cancel{m}\, v_{\theta_2} \cdot r_{max}$$ 10

$$v_{\theta_2} \cdot r_{max} = 4 \cdot 1 = 4 \Rightarrow v_{\theta_2} = \frac{4}{r_{max}}$$

<u>Principle of work and energy</u>

$$\frac{m v_{\theta_2}^2}{2} = \frac{m v_{\theta_1}^2}{2} + \underbrace{\frac{6}{2}\left(1^2 - r_{max}^2\right)}_{\text{work done by the spring}}$$ 10

$$= 16 + 3 - 3r_{max}^2 = 19 - 3r_{max}^2$$

$$\frac{\cancel{2} \cdot \left(\frac{4}{r_{max}}\right)^2}{\cancel{2}} = 19 - 3r_{max}^2$$

$$\frac{16}{r_{max}^2} = 19 - 3r_{max}^2$$

$r_{max}^2 = x$ 5

$$\frac{16}{x} = 19 - 3x$$

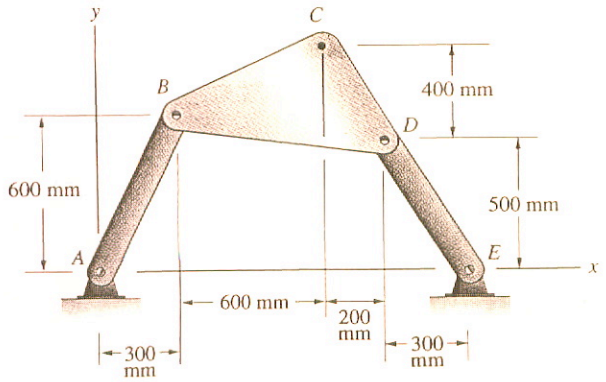
$$16 = 19x - 3x^2 \qquad 3x^2 - 19x + 16 = 0$$

$$x_{1,2} = \frac{+19 \pm 13}{6} = 1 \text{ or } 5.33$$

$x_1 = 1$ corresponds to the initial position, so

$$r_{max} = \sqrt{5.33} = 2.309\ [m]$$

8. Bar AB rotates at 4 rad/s in the counterclockwise direction. Determine the velocity of point C. (25 points)



$$0 = v_E = v_D + \omega_{DE} \times DE$$

$$= v_B + \omega_{BD} \times BD + \omega_{DE} \times DE$$

15

$$= \cancel{v_A}^{\,0} + \omega_{AB} \times AB + \omega_{BD} \times BD + \omega_{DE} \times DE$$

$$= \frac{\begin{matrix}\omega_{AB}(0,0,4)\\ \times\ AB\ (0.3, 0.6, 0)\end{matrix}}{(-2.4, 1.2, 0)} + \frac{\begin{matrix}\omega_{BD}(0,0,\omega_{BD})\\ \times\ BD\ (0.8, -0.1, 0)\end{matrix}}{(0.1\omega_{BD}, 0.8\omega_{BD}, 0)} + \frac{\begin{matrix}\omega_{DE}(0,0,\omega_{DE})\\ \times\ DE\ (0.3, -0.5, 0)\end{matrix}}{(0.5\omega_{DE}; 0.3\omega_{DE}, 0)}$$

so:

$$-2.4 + 0.1\omega_{BD} + 0.5\omega_{DE} = 0$$
$$1.2 + 0.8\omega_{BD} + 0.3\omega_{DE} = 0$$

$$\begin{pmatrix} 0.1 & 0.5 \\ 0.8 & 0.3 \end{pmatrix}\begin{pmatrix}\omega_{BD}\\ \omega_{DE}\end{pmatrix} = \begin{pmatrix}2.4\\ -1.2\end{pmatrix}$$

5

$$\omega_{BD} = \frac{\begin{vmatrix} 2.4 & 0.5 \\ -1.2 & 0.3 \end{vmatrix}}{\begin{vmatrix} 0.1 & 0.5 \\ 0.8 & 0.3 \end{vmatrix}} = \frac{0.72}{-0.37} = -1.946 \left[\frac{rad}{s}\right]$$

$$\omega_{DE} = \frac{\begin{vmatrix} 0.1 & 2.4 \\ 0.8 & -1.2 \end{vmatrix}}{-0.37} = \frac{-2.04}{-0.37} = 5.513 \left[\frac{rad}{s}\right]$$

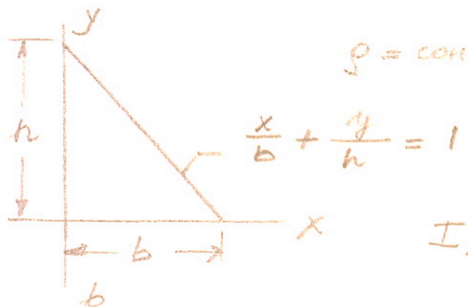
$$v_C = v_B + \omega_{BD} \times BC = (-2.4, 1.2, 0) + \frac{\begin{matrix}\omega_{BD}(0,0,-1.946)\\ \times\ BC\ (0.6, 0.3, 0)\end{matrix}}{(0.584; -1.168; 0)}$$

5

$$= (-1.816; 0.032, 0)$$

EM311M - Dynamics
Exam 3
Monday, Dec 1, 2003

1. Use integration to compute the product of inertia I_{xy} for a triangle below. (5 points)



$\rho = \text{const}$

$$\begin{cases} 0 < x < b \\ 0 < y < h\left(1 - \frac{x}{b}\right) \end{cases}$$

$$I_{xy} = \rho \int_0^b \int_0^{h(1-\frac{x}{b})} xy \, dy \, dx = \rho \int_0^b x \left. \frac{y^2}{2} \right|_0^{h(1-\frac{x}{b})} dx$$

$$= \frac{1}{2}\rho \int_0^b x h^2 \left(1 - \frac{x}{b}\right)^2 dx = \frac{1}{2}\rho h^2 \int_0^b \left(x - \frac{2x^2}{b} + \frac{x^3}{b^2}\right) dx$$

$$= \frac{1}{2}\rho h^2 \left(\frac{x^2}{2} - \frac{2x^3}{3b} + \frac{x^4}{4b^2}\right)\Big|_0^b = \frac{1}{2}\rho b^2 h^2 \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{1}{24}\rho b^2 h^2 = \frac{1}{12} mbh$$

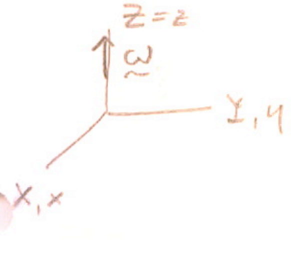
5

2. System of coordinates $Oxyz$ is rotating with respect to a fixed system of coordinates $OXYZ$ around the $Z = z$ axis with a constant angular velocity ω. Particle P is moving with a constant velocity vector $\boldsymbol{v} = (v_X, 0, 0)$ in the fixed system of coordinates. Compute the Coriolis acceleration of the particle in frame $Oxyz$, at the instant when the two systems coincide with each other. (5 points)

$$\underset{\sim}{v}_P = \underset{\sim}{v}_O + \underset{\sim}{v}_{P/rel} + \underset{\sim}{\omega} \times \underset{\sim}{OP}$$

$$\underset{\sim}{a}_P = \underset{\sim}{a}_O + \underset{\sim}{a}_{P/rel} + \underset{\sim}{\omega} \times \underset{\sim}{v}_{P/rel} + \underset{\sim}{\alpha} \times \underset{\sim}{OP} + \underset{\sim}{\omega} \times \left(\underset{\sim}{v}_{P/rel} + \underset{\sim}{\omega} \times \underset{\sim}{OP}\right)$$

$$= \underset{\sim}{a}_O + \underset{\sim}{\alpha} \times \underset{\sim}{OP} + \underset{\sim}{\omega} \times \left(\underset{\sim}{\omega} \times \underset{\sim}{OP}\right) + \underset{\sim}{a}_{P/rel} + \underbrace{2\underset{\sim}{\omega} \times \underset{\sim}{v}_{P/rel}}_{\text{Coriolis acceleration } \underset{\sim}{a}_c}$$



$$\underset{\sim}{v}_{P/rel} = \underset{\sim}{v}_P - \cancel{\underset{\sim}{v}_O}^{\,0} - \underset{\sim}{\omega} \times \underset{\sim}{OP}$$

$$= (v_X, 0, 0) - \begin{array}{r} (0, 0, \omega) \\ \times\ (x, 0, 0) \\ \hline (0, \omega x, 0) \end{array} = (v_X, -\omega x, 0)$$

$$\underset{\sim}{a}_c = 2 \begin{array}{r} \underset{\sim}{\omega}\ (0, 0, \omega) \\ \times\ \underset{\sim}{v}_{P/rel}\ (v_X, -\omega x, 0) \\ \hline (-\omega^2 x, \omega v, 0) \end{array} = \underline{(-2\omega^2 x, 2\omega v, 0)}$$

5

3. Derive the principle of angular impulse and momentum for an arbitrary system of particles. (5 points)

Consider case: O fixed.

$$(\underset{\sim}{r}_i \times m_i \underset{\sim}{v}_i)^\cdot = \cancel{\dot{\underset{\sim}{r}}_i \times m_i \underset{\sim}{v}_i}^{\,\underset{\sim}{0}} + \underset{\sim}{r}_i \times m_i \dot{\underset{\sim}{v}}_i$$

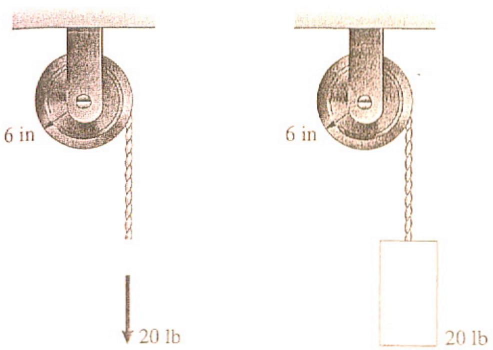
$$\underbrace{\Big(\sum_i \underset{\sim}{r}_i \times m_i \underset{\sim}{v}_i\Big)^\cdot}_{\substack{\text{angular momentum} \\ \underset{\sim}{H}_o}} = \sum_i \underset{\sim}{r}_i \times m_i \dot{\underset{\sim}{v}}_i$$

$$= \sum_i \underset{\sim}{r}_i \times (\underset{\sim}{F}_i + \underset{\sim}{R}_i)$$

$$= \underbrace{\sum_i \underset{\sim}{r}_i \times \underset{\sim}{F}_i}_{\text{moment of active forces } \underset{\sim}{M}_o} + \cancel{\sum_i \underset{\sim}{r}_i \times \underset{\sim}{R}_i}^{\,\underset{\sim}{0}}$$

5

4. Analyze the two scenarios depicted below. In which case do you expect a smaller initial acceleration of the system ? Explain, why. (5 points)

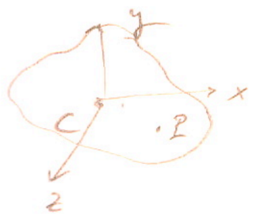


(a) (b)

In the second. Forces are identical but the inertia is bigger.

5

5. Derive the formula for the kinetic energy of a rigid body undergoing an arbitrary planar motion. (5 points)



$$\underset{\sim}{v}_P = \underset{\sim}{v}_C + \underset{\sim}{\omega} \times \underset{\sim}{CP} = (v_{Cx}, v_{Cy}, 0) + (-\omega y, \omega x, 0)$$

$$\underset{\sim}{\omega}\,(0,0,\omega) \times \underset{\sim}{CP}\,(x, y, 0) = (-\omega y, \omega x, 0)$$

$$\underset{\sim}{v}_P^2 = (v_{Cx} - \omega y,\ v_{Cy} + \omega x,\ 0)^2$$

$$T = \frac{1}{2}\int_V \rho\, \underset{\sim}{v}_P^2\, dV = \frac{1}{2}\int_V \rho\{(v_{Cx} - \omega y)^2 + (v_{Cy} + \omega x)^2\}\, dV$$

$$= \frac{1}{2}\int_V \rho(v_{Cx}^2 + v_{Cy}^2)\, dV - v_{Cx}\omega \cancel{\int_V \rho y\, dV}^{\,0} + v_{Cy}\omega \cancel{\int_V \rho x\, dV}^{\,0} + \frac{1}{2}\Big(\int_V \rho(x^2+y^2)\, dV\Big)\omega^2$$

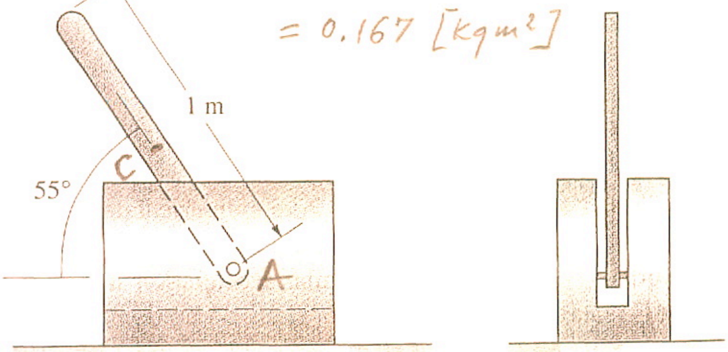
$$= \underline{\frac{1}{2} m \underset{\sim}{v}_C^2 + \frac{1}{2} I_z \omega^2}$$

5

6. The 2-kg slender bar and ~~K~~ 5-kg block are released from rest in the position shown. If friction is negligible, what is the block's acceleration at that instant? (25 points)

$I_c = \frac{1}{12} 2 \cdot 1^2$

$= 0.167 \, [\mathrm{kgm^2}]$



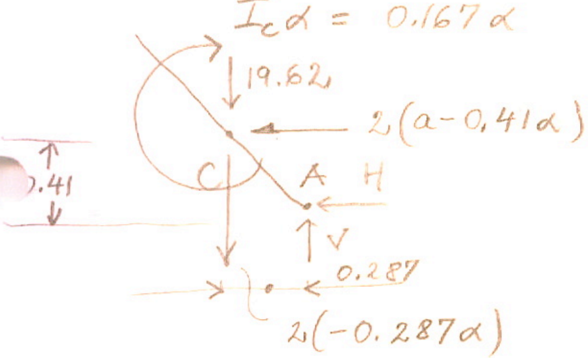
Kinematics

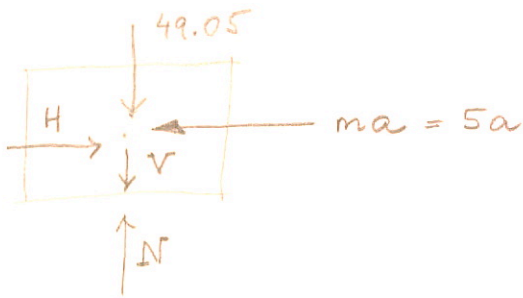
$$\underset{\sim}{a}_C = \underset{\sim}{a}_A + \underset{\sim}{\alpha} \times \underset{\sim}{AC} - \cancel{\omega^2}^{\,0} \underset{\sim}{AC}$$

$$= (a, 0, 0) + \frac{\underset{\sim}{\alpha}\,(0, 0, \alpha) \; \times \; \underset{\sim}{AC}\,(-0.287,\ 0.410,\ 0)}{(-0.41\alpha, -0.287\alpha, 0)}$$

$$= (a - 0.41\alpha, -0.287\alpha, 0)$$ (5)

d'Alembert diagrams





(5)

Bar: $\Sigma M_A = 0$ $\quad 19.62 \cdot 0.287 + 2(-0.287\alpha)\,0.287$

$+ 2(a - 0.41\alpha)\,0.41 - 0.167\alpha = 0$ (5)

$\Rightarrow \quad 0.82a = -5.631 + 0.165\alpha + 0.336\alpha + 0.167\alpha$

$= -5.631 + 0.668\alpha$

(1)

Bar + block: $\Sigma F_x = 0$ (5)

$-2(a - 0.41\alpha) - 5a = 0$

$7a = 0.82\alpha \Rightarrow a = 0.117\alpha$ (2)

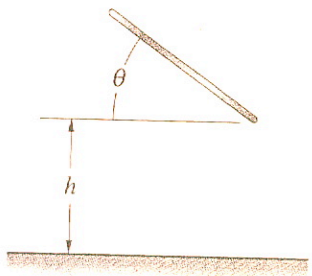
Substituting into (1) $\Rightarrow$ $0.096\alpha = -5.631 + 0.668\alpha$

$\therefore \alpha = 9.844 \left[\frac{\mathrm{rad}}{\mathrm{s^2}}\right]$

$\Rightarrow a = 1.151 \left[\frac{\mathrm{m}}{\mathrm{s^2}}\right]$ (5)

$$I_c = \frac{1}{12} 2\, 1^2 = 0.167 \text{ [kg m}^2\text{]}$$

7. The slender bar is released from rest with $\theta = 45°$ and falls a distance $h = 1$m onto the smooth floor. The length of the bar is 1 m and its mass is 2 kg. If the coefficient of restitution of the impact is $e = 0.4$, what is the angular velocity of the bar just after it hits the floor? (25 points)



Stage 1 : Free fall, use the work and energy principle

$$\cancel{T_1}^{\,0} + U_{12} = T_2$$

$$mgh = \frac{1}{2} m v^2$$

$$v = \sqrt{2gh} = 4.43 \text{ [m/s]} \downarrow$$

3

Stage 2 : Impact

- Conservation of linear momentum in x direction $\Rightarrow V_{C_x}^{after} = 0$ 5

- Conservation of angular momentum wrt fixed (on the floor) point A 5

$$H_A^{before} = H_A^{after}$$

$$\underset{\sim}{H}_A^{before} = \underset{\sim}{AC} \times m\underset{\sim}{V}_C^{before} + \cancel{\underset{\sim}{H}_C^{before}}^{\,0} = \begin{array}{r} \underset{\sim}{AC}\ (-0.353,\ 0.353,\ 0) \\ \times\ m\underset{\sim}{V}_C^{before}\ (0,\ -8.86,\ 0) \\ \hline (0,\ 0,\ 3.132) \end{array} \quad \text{[m kg m/s]}$$

$$\underset{\sim}{H}_A^{after} = \begin{array}{r} \underset{\sim}{AC}\ (-0.353,\ 0.353,\ 0) \\ \times\ m\underset{\sim}{V}_C^{after}\ (0,\ mV_{C_y}^{after},\ 0) \\ \hline (0,\ 0,\ -0.706V_{C_y}^{after}) \end{array} + (0,\ 0,\ 0.167\,\omega^{after})$$

$$\boxed{3.132 = -0.706\, V_{C_y}^{after} + 0.167\,\omega^{after}} \quad (*)$$

- $0.4 = -\dfrac{V_{A_y}^{after}}{V_{A_y}^{before}} \Rightarrow V_{A_y}^{after} = -0.4 \cdot (-4.43) = 1.772$ [m/s] ↑

5

7 continued

- Kinematics

$$\underset{\sim}{V}_C = \underset{\sim}{V}_A + \underset{\sim}{\omega} \times \underset{\sim}{AC} = (V_{Ax}, V_{Ay}, 0) + \underset{(-0.353\omega, -0.353\omega, 0)}{\underline{\begin{matrix} \times\ \underset{\sim}{\omega}\ (0, 0, \omega) \\ \underset{\sim}{AC}\ (-0.353, 0.353, 0) \end{matrix}}}$$

$$V_{cy}^{after} = V_{Ay}^{after} - 0.353\omega^{after} = 1.772 - 0.353\,\omega^{after} \qquad (5)$$

- Substituting into (*)

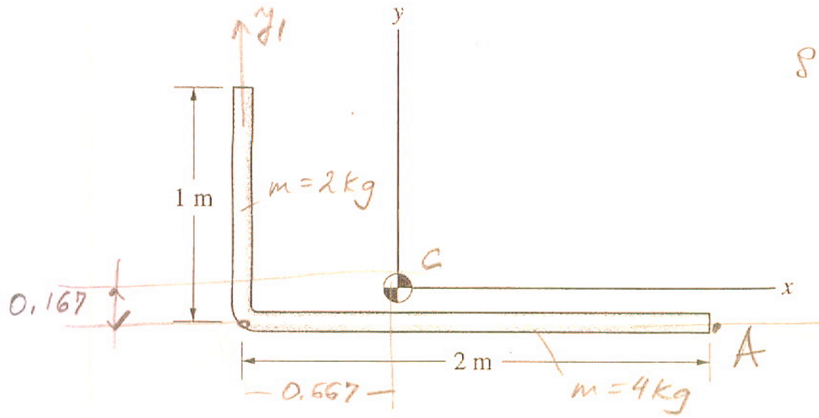
$$3.132 = -0.706\,(1.772 - 0.353\omega^{after}) + 0.167\,\omega^{after}$$

$$3.132 = -1.251 + (0.249 + 0.167)\,\omega^{after}$$

$$\omega^{after} = 10.54 \left[\frac{rad}{s^2}\right] \qquad (2)$$

8. The bar shown below is stationary relative to an inertial reference frame when the force $\boldsymbol{F} = 12\boldsymbol{k}$ (N) is applied at the right end of the bar. No other forces or couples act on the bar. Determine the bar's angular acceleration relative to the inertial reference frame. (25 points)

total mass $m = 6$ kg



ρ = const $\Rightarrow$ center of mass coincides with the centroid

$$x_{1C} = \frac{2 \cdot 1}{3} = 0.667\ [m]$$

$$y_{1C} = \frac{1 \cdot 0.5}{3} = 0.167\ [m]$$

Step 1: Compute tensor of inertia at C

$$I_x = \underbrace{0 + 4 \cdot 0.167^2}_{I_x^-} + \underbrace{\tfrac{1}{12} 2 \cdot 1^2 + 2\,(0.5 - 0.167)^2}_{I_x'} = 0.5\ [kg\,m^2]$$

$$I_y = \underbrace{\tfrac{1}{12} 4\, 2^2 + 4\,(1 - 0.667)^2}_{I_y^-} + \underbrace{0 + 2 \cdot 0.667^2}_{I_y'} = 2.67\ [kg\,m^2]$$

$$I_z = I_x + I_y = 3.17\ [kg\,m^2]$$

$$I_{xy} = \underbrace{0 - 4\,(1 - 0.667)\,0.167}_{I_{xy}^-} + \underbrace{0 - 2 \cdot 0.667\,(0.5 - 0.167)}_{I_{xy}'}$$

$$= -0.222 - 0.444 = -0.67\ [kg\,m^2]$$

$$I_{xz} = I_{yz} = 0$$

10

8 continued

Step 2: use rotational motion equation:

$$\underset{\sim}{I}_c \underset{\sim}{\alpha} + \cancel{\underset{\sim}{\omega} \times (\underset{\sim}{I}_c \underset{\sim}{\omega})}^{\;\underset{\sim}{0}} = \underset{\sim}{M}_c$$

$$\underset{\sim}{M}_c = \underset{\sim}{CA} \times \underset{\sim}{F} = \times \begin{array}{l} \underset{\sim}{CA}\,(1.333, -0.167, 0) \\ \underset{\sim}{F}\,(\;\;0, \;\;0, 12) \\ \hline (-2, -16, 0) \;\; [Nm] \end{array}$$

$$\begin{pmatrix} 0.5 & 0.67 & 0 \\ 0.67 & 2.67 & 0 \\ 0 & 0 & 3.17 \end{pmatrix} \begin{pmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{pmatrix} = \begin{pmatrix} -2 \\ -16 \\ 0 \end{pmatrix}$$

10

$\therefore \quad \alpha_z = 0$

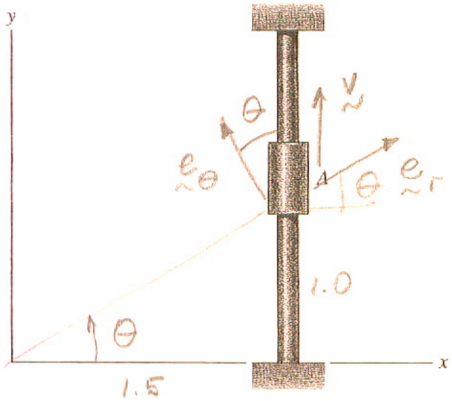
$$\alpha_x = \frac{\begin{vmatrix} -2 & 0.67 \\ -16 & 2.67 \end{vmatrix}}{\begin{vmatrix} 0.5 & 0.67 \\ 0.67 & 2.67 \end{vmatrix}} = \frac{5.38}{0.886} = 6.07 \left[\frac{rad}{s^2}\right]$$

$$\alpha_y = \frac{\begin{vmatrix} 0.5 & -2 \\ 0.67 & -16 \end{vmatrix}}{0.886} = \frac{-6.66}{0.886} = -7.52 \left[\frac{rad}{s^2}\right]$$

5

EM311M - Dynamics
Final Exam
CPE 2.214, 7-10pm, Saturday, Dec 11, 2004

1. At the instant shown, the coordinates of the slider A are $x = 1.5, y = 1.0$ ft, and its velocity represented in Cartesian coordinates is $\boldsymbol{v} = (0, 5)$ ft/s. Determine the slider's velocity components in polar coordinates. (5 points)



$$\sin\theta = \frac{1}{\sqrt{1+1.5^2}} = 0.5547$$

$$\cos\theta = \frac{1.5}{\sqrt{1+1.5^2}} = 0.8321$$

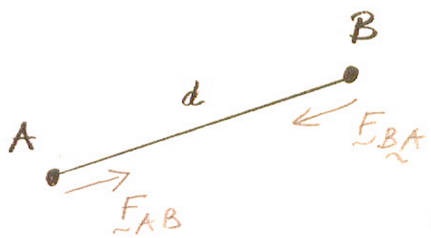
5

$$v_r = 5\sin\theta = 2.77 \ [ft/s]$$

$$v_\theta = 5\cos\theta = 4.16 \ [ft/s]$$

Check : $v_r^2 + v_\theta^2 = 24.98 \sim 5^2$ ✓

2. Particles A and B are connected with a rigid link. Explain why the work of reactive forces representing the interaction between the particles is zero. (5 points)



$$\underset{\sim}{AB}^2 = (\underset{\sim}{r}_B - \underset{\sim}{r}_A)^2 = \text{const}$$

$$\therefore \quad 2\, \underset{\sim}{AB} \cdot (d\underset{\sim}{r}_B - d\underset{\sim}{r}_A) = 0$$

$$\underset{\sim}{F}_{AB} = F\frac{\underset{\sim}{AB}}{|\underset{\sim}{AB}|}$$

$$\underset{\sim}{F}_{BA} = -F\frac{\underset{\sim}{AB}}{|\underset{\sim}{AB}|}$$

Incremental work =

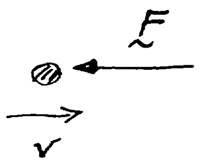
5

$$\underset{\sim}{F}_{BA} \cdot d\underset{\sim}{r}_B + \underset{\sim}{F}_{AB} \cdot d\underset{\sim}{r}_A$$

$$= -F\frac{\underset{\sim}{AB}}{|\underset{\sim}{AB}|} \cdot d\underset{\sim}{r}_B + F\frac{\underset{\sim}{AB}}{|\underset{\sim}{AB}|} \cdot d\underset{\sim}{r}_A$$

$$= -\frac{F}{|\underset{\sim}{AB}|}\, \underset{\sim}{AB} \cdot (d\underset{\sim}{r}_B - d\underset{\sim}{r}_A) = 0 \ !$$

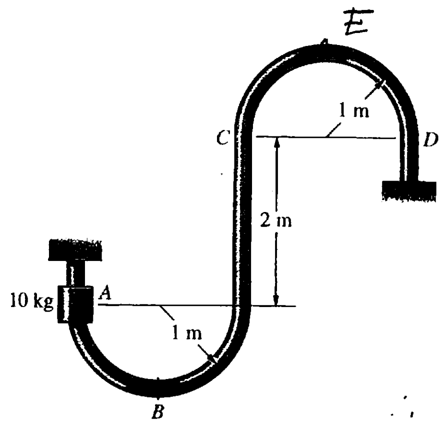
3. Is the drag force $\boldsymbol{F} = (-cv^2, 0, 0), (c > 0)$ conservative? Explain. (5 points)



No! Conservative force is expressed in terms of a potential $\underset{\sim}{F} = -\nabla \underset{\sim}{V}$, where $V = V(x, y, z)$ and, therefore, it may only depend upon the position, not velocity

5

4. The bar is smooth. Determine the minimum velocity the 10-kg slider must have at A to reach point D. (5 points)



the point: the slider must make it through point E with a velocity at E $v_E \geq 0$,

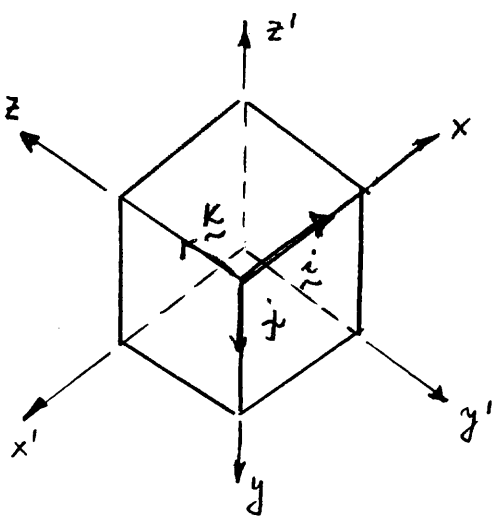
$$\frac{\cancel{m} v_A^2}{2} - \cancel{m} g \underset{3m}{h} = \frac{\cancel{m} v_E^2}{2} \geq 0$$

$$\therefore \quad v_A^2 \geq 2gh = 58.86$$

$$v_A \geq \sqrt{2gh} = \underline{\underline{7.672}} \left[\frac{m}{s}\right]$$

5

5. Compute transformation matrix from system $0xyz$ to system $0x'y'z'$. (5 points)

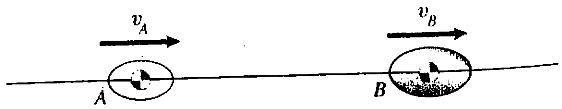


$$\underset{\sim}{\alpha} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

5

$v_B' = v_B^{after}$, $v_A' = v_A^{after}$

6. Objects A and B have masses m_A and m_B and velocities $\underbrace{v_A^{before}}_{v_A}, \underbrace{v_B^{before}}_{v_B}$. Show that, if coefficient of restitution $e = 1$, the total kinetic energy is conserved during the impact. (5 points)



$$e = \frac{v_B' - v_A'}{v_A - v_B} = 1$$

$$\therefore \quad v_B' - v_A' = v_A - v_B$$

$$\therefore \quad v_B' + v_B = v_A' + v_A$$

Conservation of linear momentum $\Rightarrow$

$$m_A v_A + m_B v_B = m_A v_A' + m_B v_B'$$

$$\therefore \quad m_A (v_A - v_A') = m_B (v_B' - v_B) \quad / \cdot \; v_B' + v_B \; (= v_A' + v_A)$$

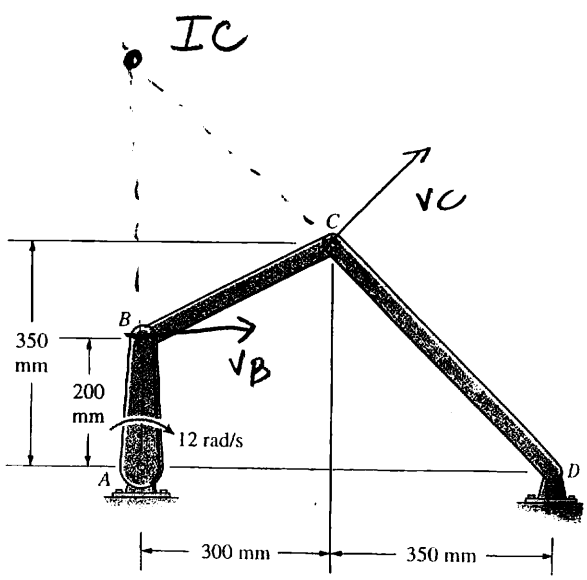
$$\therefore \quad m_A (v_A^2 - v_A'^2) = m_B (v_B'^2 - v_B^2)$$

$$\therefore \quad m_A v_A^2 + m_B v_B^2 = m_A v_A'^2 + m_B v_B'^2$$

Divide by 2 sidewise to finish ...

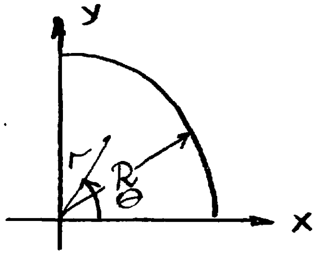
5

7. Determine the instantenous center (IC) of zero velocity for member BC. (5 points)



5

8. Compute the product of inertia I_{xy} for a quadrant of a homogeneous circle with radius R and mass m. (5 points)



Parametrization :

$$\begin{cases} x = r\cos\theta & 0 < r < R \\ y = r\sin\theta & 0 < \theta < \frac{\pi}{2} \end{cases}$$

5

jacobian

$$I_{xy} = \rho \int_A xy\, dA = \rho \int_0^{\frac{\pi}{2}} \int_0^R r\cos\theta \; r\sin\theta \; r\, dr\, d\theta$$

$$= \rho \int_0^R r^3 dr \int_0^{\frac{\pi}{2}} \underbrace{\cos\theta \sin\theta}_{\frac{1}{2}\sin 2\theta} d\theta = \rho \frac{r^4}{4}\Big|_0^R \left(-\frac{1}{4}\cos 2\theta\right)\Big|_0^{\frac{\pi}{2}}$$

$$= \rho \frac{R^4}{4} \frac{1}{4}(1+1) = \rho \frac{R^4}{8} = \underbrace{\rho \frac{\pi R^2}{4}}_{m} \frac{4}{\pi R^2} \frac{R^4}{8} = \underline{\underline{\frac{1}{2\pi} m R^2}}$$

9. Derive the principle of linear momentum for a system of particles (5 points)

m_i $\underset{\sim}{F_i}$ $\underset{\sim}{R_i}$

$$m_i \frac{d\underset{\sim}{v_i}}{dt} = \underset{\sim}{F_i} + \underset{\sim}{R_i} \quad \Big/ \sum_i$$

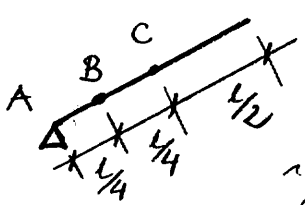
$$\sum_i m_i \frac{d\underset{\sim}{v_i}}{dt} = \sum_i \underset{\sim}{F_i} + \cancel{\sum_i \underset{\sim}{R_i}}^{\,\underset{\sim}{0}} \quad \Big/ \int$$

$$\sum_i m_i\left(\underset{\sim}{v_i}(t_2) - \underset{\sim}{v_i}(t_1)\right) = \sum_i \int_{t_1}^{t_2} \underset{\sim}{F_i}\, dt$$

5

$$\underbrace{\sum_i m_i \underset{\sim}{v_i}(t_1)}_{\text{momentum at } t_1} + \underbrace{\sum_i \int_{t_1}^{t_2} \underset{\sim}{F_i}\, dt}_{\text{impulse}} = \underbrace{\sum_i m_i \underset{\sim}{v_i}(t_2)}_{\text{momentum at } t_2}$$

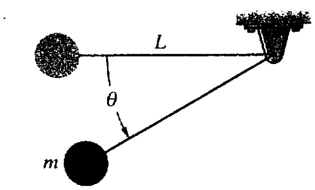
10. A slender bar with mass m and length l is rotating about its endpoint A with constant angular velocity $\omega = (0, 0, \omega)$. Compute the angular velocity of the bar with respect to point B (half way between point A and center of mass C). (5 points)

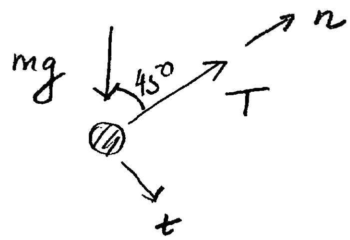


Nothing to compute ! Angular velocity is an attibute of the rigid body, i.e. it has nothing to do with any point of computation !

5

11. The 2-kg mass m is released from rest with the string horizontal. The length of the string is $L = 0.5$ m. Determine the velocity of the mass and the tension in the spring when $\theta = 45^o$. (25 points)





Frenét coordinates: $\quad m a_n = T - mg \cos 45^\circ$

$$a_n = \frac{v^2}{l} \Rightarrow T = \frac{m v^2}{l} + mg \cos 45^\circ \quad 7$$

Principle of work and energy:

$$\cancel{T_1}^{\,0} + U_{12} = T_2 \quad 8$$

$$\cancel{m} g \, l \sin\theta = \frac{1}{2} \cancel{m} v^2$$

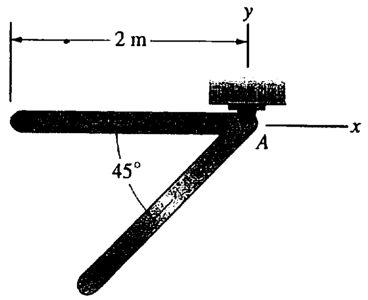
$$\therefore v^2 = 2 g \, l \sin\theta$$

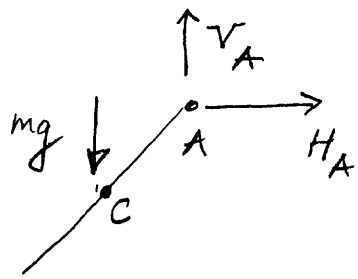
$$v = 2.63 \, [\tfrac{m}{s}] \quad 5$$

$$T = \frac{m v^2}{l} + mg \cos 45^\circ$$

$$= 27.67 + 13.87 = 41.54 \, [N] \quad 5$$

12. The 10-kg slender bar is released from rest in the horizontal position. When it has fallen to the position shown, what are the x and y components of force exerted on the bar by the pin support A ? (25 points)





2

$I_A = \frac{1}{3} m l^2$
$= 13.33 \ [kg\, m^2]$

Principle of work and energy $\Rightarrow$

$T_1 + U_{12} = T_2$

$U_{12} = 10 \cdot 9.81 \ 1 \cdot \sin 45° = 69.37 \ [N \cdot m]$

$T_2 = \frac{1}{2} I_A \omega^2 = 6.67 \omega^2$

$69.37 = 6.67 \omega^2 \Rightarrow \omega = 3.226 \ [\frac{rad}{s}]$

5

Equation of rotational motion

$I_A \alpha = M_A$

$13.33 \ \alpha = 10 \cdot 9.81 \cdot 1 \cos 45°$

$\therefore \ \alpha = 5.204 \ [\frac{rad}{s^2}]$

5

Kinematics

$a_C = \cancel{a_A}^{0} + \alpha \times AC - 3.226^2 (-0.707, -0.707, 0)$

$\alpha = (0, 0, 5.204)$, $AC = (-0.707, -0.707, 0)$: $\alpha \times AC = (3.68, -3.68, 0)$

$= (11.04; 3.68, 0) \ [\frac{m}{s^2}]$

5

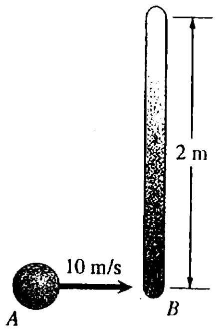
Equations of translational motion :

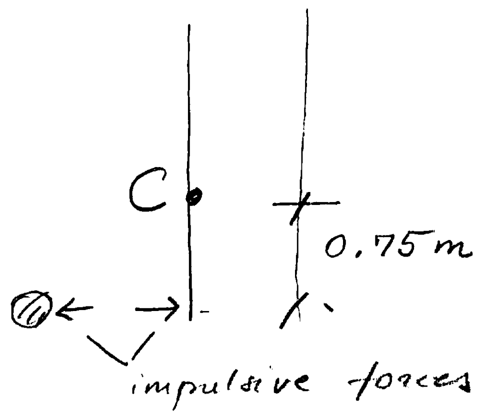
$m \ a_{cx} = H_A \Rightarrow H_A = 110.4 \ [N]$

$m \ a_{cy} = V_A - 10 \cdot 9.81 \Rightarrow V_A = 134.9 \ [N]$

8

13. The 1-kg sphere A is moving at 10 m/s when it strikes the end of the 3-kg stationary slender bar B. If the sphere adheres to the bar, what is the bar's angular velocity after the impact ? (25 points)





Step 1 : Center of mass of the combined system :

$$y = \frac{3 \cdot 1}{1+3} = 0.75 \; [m]$$

(5)

Step 2 : Moment of inertia wrt center of mass C of the system

$$I_c = 1 \cdot 0.75^2 + \left(\frac{1}{12} 3 \, 2^2 + 3 \; 0.25^2\right)$$
$$= 1.75 \; [kg \, m^2]$$

(5)

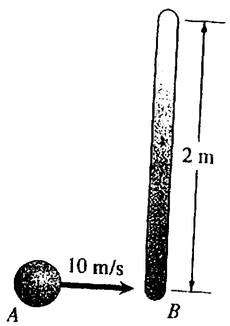
Step 3 ; Conservation of angular momentum of the whole system wrt center of mass C

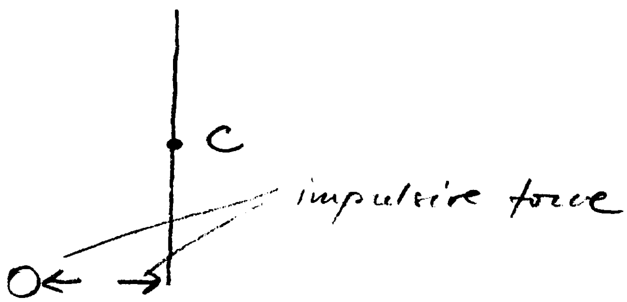
$$1 \cdot 10 \cdot 0.75 = I_c \, \omega^{after}$$

$$\therefore \; \omega^{after} = 4.29 \; \left[\frac{rad}{s}\right]$$

(15)

13. The 1-kg sphere A is moving at 10 m/s when it strikes the end of the 3-kg stationary slender bar B. If the sphere adheres to the bar, what is the bar's angular velocity after the impact ? (25 points)





An alternative solution (there are many more ..)

- Conservation of linear momentum for the bar in y-direction $\Rightarrow v'_{C_y} = 0$

- Conservation of linear momentum for the system in x-direction

$$1\cdot 10 = 1\cdot v'_A + 3\, v'_{Cx} = 1\cdot(v'_{Cx}+\omega) + 3v'_{Cx}$$

$$\underset{\sim}{v}'_A = \underset{\sim}{v}'_C + \underset{\sim}{\omega} \times \underset{\sim}{CA} = (v'_{Cx},0,0) + \underbrace{(0,0,\omega)\times(0,-1,0)}_{(\omega,0,0)}$$

$$= v'_{Cx} + \omega$$

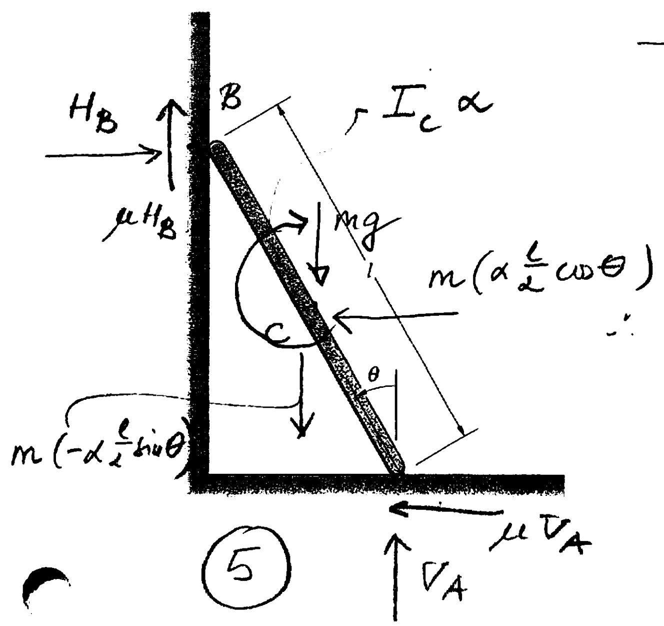
- Conservation of angular momentum for the bar wrt stationary point that coincides with B

$$0 = \underbrace{\underset{\sim}{BO}(0,1,0) \times m\underset{\sim}{v}'_C(3v'_{Cx},0,0)}_{(0,0,-3v'_{Cx})} + (0,0,\underbrace{I_C\omega}_{\frac{1}{12}\cdot 3\cdot 4\cdot\omega}) = (0,0,\omega - 3v'_{Cx})$$

$$\therefore \begin{cases} \omega - 3v'_{Cx} = 0 \\ 4v'_{Cx} + \omega = 10 \end{cases} \Rightarrow \omega = \frac{30}{7} \sim 4.29 \left[\frac{rad}{s}\right]$$

$$I_c = \frac{1}{12} m l^2$$

14. A slender bar of mass m is released from rest in the position shown. The static and kinetic coefficients of friction at the floor and wall have the same value μ. If the bar slips, what is its angular acceleration at the instant of release? (25 points)



5

Kinematics :

$$\underset{\sim}{a}_C = \underset{\sim}{a}_A + \overset{\alpha\ (0,0,\alpha)}{\underset{\sim}{\alpha}} \times \underset{\sim}{AC}\left(-\frac{l}{2}\sin\theta,\ \frac{l}{2}\cos\theta,\ 0\right)$$

$$(a_{Ax}, 0, 0) \quad \left(-\alpha \frac{l}{2}\cos\theta,\ -\alpha\frac{l}{2}\sin\theta,\ 0\right)$$

$$\therefore \quad a_{C_y} = -\alpha \frac{l}{2} \sin\theta$$

$$\underset{\sim}{a}_C = \underset{\sim}{a}_B + \overset{\alpha\ (0,0,\alpha)}{\underset{\sim}{\alpha}} \times \underset{\sim}{BC}\left(\frac{l}{2}\sin\theta,\ -\frac{l}{2}\cos\theta,\ 0\right)$$

$$(0, a_{By}, 0) \quad \left(\alpha\frac{l}{2}\cos\theta,\ \alpha\frac{l}{2}\sin\theta,\ 0\right)$$

$$\therefore \quad a_{C_X} = \alpha \frac{l}{2} \cos\theta$$

10

$\Sigma F_x = 0$

$$-\mu V_A + H_B - m\alpha \frac{l}{2}\cos\theta = 0$$

$\Sigma F_y = 0$

$$V_A + \mu H_B - m\left(-\alpha\frac{l}{2}\sin\theta\right) - mg = 0$$

$$\begin{pmatrix} -\mu & 1 \\ 1 & \mu \end{pmatrix} \begin{pmatrix} V_A \\ H_B \end{pmatrix} = \begin{pmatrix} m\alpha\frac{l}{2}\cos\theta \\ mg - m\alpha\frac{l}{2}\sin\theta \end{pmatrix}$$

$$V_A = \frac{\begin{vmatrix} m\alpha\frac{l}{2}\cos\theta & 1 \\ mg - m\alpha\frac{l}{2}\sin\theta & \mu \end{vmatrix}}{\begin{vmatrix} -\mu & 1 \\ 1 & \mu \end{vmatrix}}$$

$$= \frac{m\alpha\frac{l}{2}\cos\theta\,\mu - mg + m\alpha\frac{l}{2}\sin\theta}{-\mu^2 - 1}$$

$$= \frac{mg - m\alpha\frac{l}{2}(\cos\theta\,\mu + \sin\theta)}{1+\mu^2}$$

10

$\underline{\sum M_B = 0}$

$$V_A \cancel{l}\sin\theta - \mu V_A \cancel{l}\cos\theta$$
$$- mg\frac{\cancel{l}}{2}\sin\theta - m\left(-\alpha\frac{l}{2}\sin\theta\right)\frac{\cancel{l}}{2}\sin\theta$$
$$- m\alpha\frac{l}{2}\cos\theta\frac{\cancel{l}}{2}\cos\theta - \frac{1}{12}ml^{\cancel{2}}\alpha = 0$$

(one l cancels out)

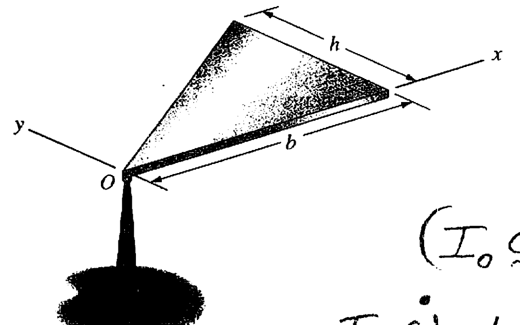
$$\frac{mg - m\alpha\frac{l}{2}(\cos\theta\,\mu + \sin\theta)}{1+\mu^2}(\sin\theta - \mu\cos\theta)$$

$$-\frac{1}{2}mg\sin\theta + \frac{1}{4}ml\alpha\sin^2\theta - \frac{1}{4}ml\alpha\cos^2\theta - \frac{1}{12}ml\,\alpha = 0$$

$$\frac{mg(\sin\theta - \mu\cos\theta)}{1+\mu^2} - \frac{1}{2}mg\sin\theta$$

$$+\alpha\left(-\frac{m\frac{l}{2}(\cos\theta\,\mu + \sin\theta)}{1+\mu^2}(\sin\theta - \mu\cos\theta) + \frac{1}{4}ml(\sin^2\theta - \cos^2\theta)\right.$$
$$\left. - \frac{1}{12}ml\right) = 0 \qquad \Rightarrow \alpha = \text{(some horrible formula ...)}$$

15. The dimensions of the 6-kg plate are $b = 0.9$m and $h = 0.6$m. The plane is supported with a ball-and-socket support at O. If the plate is held in the horizontal position and then released from rest, what are the components of its angular acceleration at that instant ? (25 points)



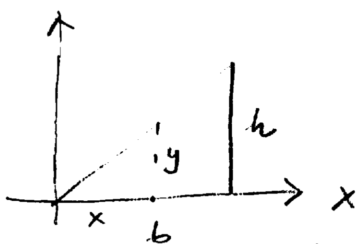
Rotation about point O

$$\dot{\underset{\sim}{H}}_O = \underset{\sim}{M}_O$$

$$(I_O \underset{\sim}{\omega})^\circ = \underset{\sim}{M}_O$$

$$I_O \underbrace{\dot{\underset{\sim}{\omega}}}_{=\underset{\sim}{\alpha}} + \overset{0}{\cancel{\underset{\sim}{\omega} \times I_O \underset{\sim}{\omega}}} = \underset{\sim}{M}_O$$

5



$$\begin{cases} 0 < x < b \\ 0 < y < \frac{h}{b} x \end{cases} \qquad \frac{y}{x} = \frac{h}{b}$$

$$I_x = \rho \int_0^b \int_0^{\frac{h}{b}x} y^2 \, dy \, dx = \rho \int_0^b \frac{y^3}{3} \Big|_0^{\frac{h}{b}x} dx$$

$$= \frac{\rho}{3} \int_0^b \frac{h^3 x^3}{b^3} = \frac{\rho h^3}{3 b^3} \frac{x^4}{4} \Big|_0^b = \frac{1}{12} \rho b h^3$$

$$= \frac{1}{6} m h^2 = 0.36 \ [kg\, m^2]$$

3

$$I_y = \rho \int_0^b \int_0^{\frac{h}{b}x} x^2 \, dy \, dx = \rho \int_0^b x^2 \, y \Big|_0^{\frac{h}{b}x} dx$$

$$= \frac{\rho h}{b} \int_0^b x^3 \, dx = \frac{\rho h}{b} \frac{x^4}{4} \Big|_0^b = \frac{\rho h}{4 b} b^4$$

$$= \frac{1}{2} m b^2 = 2.43 \ [kg\, m^2]$$

3

$$I_{xy} = \rho \int_0^b \int_0^{\frac{h}{b}x} xy \, dy \, dx$$

$$= \rho \int_0^b x \left. \frac{y^2}{2} \right|_0^{\frac{h}{b}x} dx$$

$$= \frac{\rho}{2} \int_0^b x \frac{h^2}{b^2} x^2 \, dx = \frac{\rho h^2}{2b^2} \left. \frac{x^4}{4} \right|_0^b$$ 4

$$= \frac{1}{8} \rho \frac{h^2 b^4}{b^2} = \frac{1}{4} m h b = 0.81 \ [kg\,m^2]$$

$$I_z = I_x + I_y = 2.79 \ [kg\,m^2] \qquad \underline{I_{yz} = I_{xz} = 0}$$

$$\begin{pmatrix} 0.36 & -0.81 & 0 \\ -0.81 & 2.43 & 0 \\ 0 & 0 & 2.79 \end{pmatrix} \begin{pmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{pmatrix} = \begin{pmatrix} -11.77 \\ 17.66 \\ 0 \end{pmatrix}$$

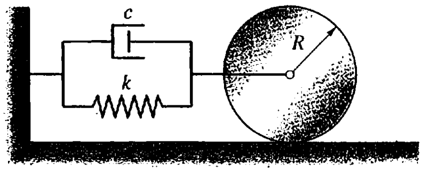
$$\underset{\sim}{M}_o = \begin{array}{l} \underset{\sim}{oc}\,(0.6, 0.2, 0) \\ \times \underset{\sim}{W}\,(0, 0, -58.86) \\ \hline (-11.77, 35.32, 0) \ [N.m] \end{array}$$ 10

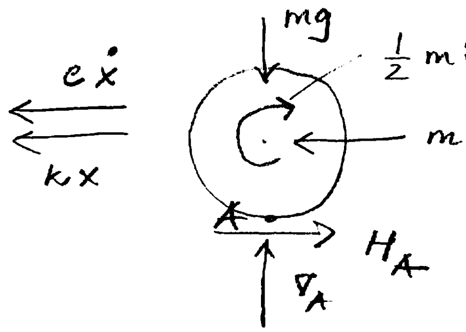
$$\alpha_z = 0$$

$$\alpha_x = \frac{\begin{vmatrix} -11.77 & -0.81 \\ 35.32 & 2.43 \end{vmatrix}}{\begin{vmatrix} 0.36 & -0.81 \\ -0.81 & 2.43 \end{vmatrix}} = \frac{0}{0.219} = 0 \ \left[\frac{rad}{s^2}\right]$$

$$\alpha_y = \frac{\begin{vmatrix} 0.36 & -11.77 \\ -0.81 & 35.32 \end{vmatrix}}{0.219} = \frac{-3.181}{0.219} = 14.52 \ \left[\frac{rad}{s^2}\right]$$

16. The homogeneous disk weighs 80 lb and its radius is $R = 1$ ft. It rolls on the plane surface. The spring constant is $k = 100$ lb/ft and the damping constant is $c = 3$ lb-s/ft. Determine the frequency of small vibrations of the disk relative to its equilibrium position. (25 points)





$\frac{1}{2} m R^2 \alpha = -\frac{1}{2} m R \ddot{x}$

10

Kinematics:

$$-\alpha R = \ddot{x}$$

5

$$\alpha = -\frac{1}{R}\ddot{x}$$

$\Sigma M_A = 0$

$$m\ddot{x}\cancel{R} - \left(-\tfrac{1}{2} m \cancel{R}\ddot{x}\right) + c\dot{x}\cancel{R} + kx\cancel{R} = 0$$

$$\frac{3}{2} m \ddot{x} + c\dot{x} + kx = 0$$

$$\ddot{x} + \frac{2c}{3m}\dot{x} + \frac{2k}{3m}x = 0$$

5

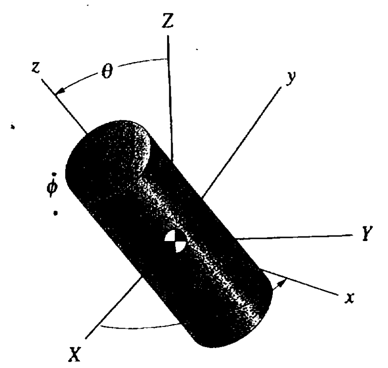
$$\gamma = \frac{c}{3m} = \frac{3}{3\,\frac{80}{32.2}} = 0.4025 \left[\frac{\cancel{lb}\cdot s}{\cancel{ft}\ \cancel{lb}\ \frac{s^2}{\cancel{ft}}} = \frac{1}{s}\right]$$

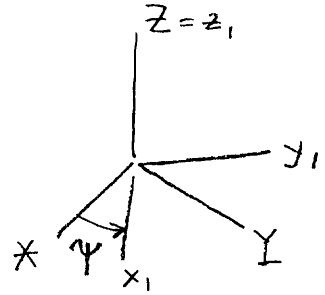
$$\omega_o^2 = \frac{2k}{3m} = 26.83 \left[\frac{\cancel{lb}}{\cancel{ft}\ \cancel{lb}\ \frac{s^2}{\cancel{ft}}} = \frac{1}{s^2}\right]$$

$$\omega = \sqrt{\omega_o^2 - \gamma^2} = \underline{\underline{5.16}} \left[\frac{1}{s}\right]$$

5

17. (bonus) A satellite can be modeled as an 1000-kg cylinder 4m in length and 2m in diameter. If the nutation angle is $\theta = 20°$, and the spin rate $\dot{\phi}$ is one revolution per second, what is the satellite's precession rate $\dot{\psi}$ in revolutions per second? (25 points)

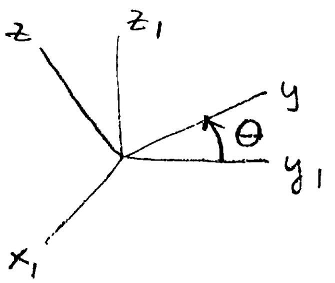




ang. velocity of $x_1 y_1 z_1$ wrt XYZ

$$\underset{\sim}{\omega}_1 = (0, 0, \dot{\psi})$$

(in both XYZ, $x_1 y_1 z_1$)



ang velocity of xyz wrt $x_1 y_1 z_1$

$$\underset{\sim}{\omega}_2 = \underset{\sim}{0} \quad (\theta = \text{const})$$

Transformation matrix from $x_1 y_1 z_1$ to xyz

$$\underset{\sim}{\alpha} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

$\underset{\sim}{\omega}_1$ in system xyz

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 0 \\ \sin\theta\,\dot{\psi} \\ \cos\theta\,\dot{\psi} \end{pmatrix}$$

<u>Calculations done in system xyz</u>

5

$$\underset{\sim}{\Omega} = \underset{\sim}{\omega}_1 = (0, \sin\theta\,\dot{\psi}, \cos\theta\,\dot{\psi})$$

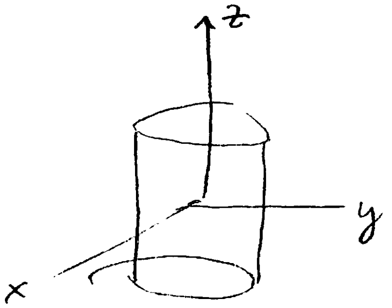
ang. velocity of the satellite wrt system xyz

$$\underset{\sim}{\omega}_3 = (0, 0, \dot{\varphi}) \quad (\text{in } xyz)$$

ang velocity of the satellite wrt XYZ (in xyz!)

$$\underset{\sim}{\omega} = \underset{\sim}{\Omega} + \underset{\sim}{\omega}_3 = (0, \sin\theta\,\dot{\psi}, \cos\theta\,\dot{\psi} + \dot{\varphi})$$

5



$$0 < r < R \qquad \left\{ \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{array} \right.$$
$$0 < \theta < 2\pi$$
$$-\frac{h}{2} < z < \frac{h}{2}$$

$$I_x = \rho \int_0^R \int_0^{2\pi} \int_{-\frac{h}{2}}^{\frac{h}{2}} \underbrace{\left(r^2\sin^2\theta + z^2\right)}_{y^2+z^2} \, dz \, d\theta \, r dr$$

$$= \rho\left(\frac{R^4}{4} h \cdot \pi + \frac{R^2}{\cancel{2}} \cancel{2}\pi \frac{h^3}{24}\right) = \rho \pi R^2 h \left(\frac{R^2}{4} + \frac{h^2}{24}\right)$$

$$= \frac{1}{24} m \left(6R^2 + h^2\right)$$

$$\cos^2\theta - \sin^2\theta = \cos 2\theta$$
$$1 - 2\sin^2\theta = \cos 2\theta$$
$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\int_0^{2\pi} \sin^2\theta \, d\theta = \frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi} = \pi$$

$$I_z = \rho \int_0^R \int_0^{2\pi} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(r^2\sin^2\theta + r^2\cos^2\theta\right) dz \, d\theta \, r dr$$

$$= \rho \frac{R^4}{4} 2\pi h = \frac{1}{2} m R^2$$

$$I_z = \frac{1}{2} \cdot 1000 \cdot 1^2 = 500 \ [kg\, m^2]$$

$$I_x = \frac{1}{24} 1000 \left(6 + 16\right) = 916.7 \ [kg\, m^2]$$

5

Equation of rotational motion

$$\dot{\underset{\sim}{H}}_c = \underset{\sim}{M}_c = \underset{\sim}{0}$$

$$(\underset{\sim}{I}_c \underset{\sim}{\omega})^{\circ} = \underset{\sim}{0}$$

$$I_c \cancel{\dot{\underset{\sim}{\omega}}}^{\,0} + \underset{\sim}{\Omega} \times (I_c \underset{\sim}{\omega}) = \underset{\sim}{0}$$ 5

Tensor of inertia at C

$$\underset{\sim}{I}_c = \begin{pmatrix} I_x & 0 & 0 \\ 0 & I_x & 0 \\ 0 & 0 & I_z \end{pmatrix}$$

$$\times \quad \underset{\sim}{\Omega} \quad (0, \ \sin\theta\,\dot{\psi}, \ \cos\theta\,\dot{\psi})$$

$$\underset{\sim}{I}_c \underset{\sim}{\omega} \quad (0, \ I_x \sin\theta\,\dot{\psi}, \ I_z(\cos\theta\,\dot{\psi} + \dot{\varphi}))$$

$$(I_z(\cos\theta\,\dot{\psi} + \dot{\varphi})\sin\theta\,\dot{\psi} - I_x \cos\theta \sin\theta\,\dot{\psi}^2, \ 0, \ 0)$$

$$(I_z - I_x)\cos\theta\,\cancel{\sin\theta}\,\dot{\psi}^{\cancel{2}} + I_z\,\dot{\varphi}\,\cancel{\sin\theta}\,\cancel{\dot{\psi}} = 0$$

$$\dot{\psi} = \frac{I_z}{(I_x - I_z)\cos\theta}\,\dot{\varphi}$$

$$\dot{\psi} = \frac{500}{416.7 \cos 20^\circ} \cdot 1 = 1.28 \left[\frac{rev}{s}\right]$$ 5